This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

ALFRED S. NILES

PRINCIPLES

OF

THE MECHANICS

OF

MACHINERY AND ENGINEERING.

BY JULIUS WEISBACH,

PROFESSOR OF MECHANICS AND APPLIED MATHEMATICS IN THE ROYAL MINING
ACADEMY OF FREIBURG.

FIRST AMERICAN EDITION.

EDITED
BY WALTER R. JOHNSON, A.M., CIV. & MIN. ENG.

FORMERLY PROFESSOR OF MECHANICS AND NATURAL PHILOSOPHY IN THE FRANKLIN INSTITUTE, AND OF CHEMISTRY AND NATURAL PHILOSOPHY IN THE MEDICAL
DEPARTMENT OF PENNSYLVANIA COLLEGE. AUTHOR OF A
REPORT TO THE UNITED STATES NAVY DEPARTMENT ON AMERICAN COALS, &C. &C.

IN TWO VOLUMES.

ILLUSTRATED WITH ONE THOUSAND ENGRAVINGS ON WOOD.

VOL. I.

THEORETICAL MECHANICS.

PHILADELPHIA:
LEA AND BLANCHARD.
1848.

TA 350
W 442
v. 1

Entered according to the Act of Congress, in the year 1848, by
LEA AND BLANCHARD,
in the Clerk's Office of the District Court of the Eastern District of Pennsylvania.

PHILADELPHIA:
T. K. AND P. G. COLLINS, PRINTERS.

SECTION III.

STATICS OF RIGID BODIES.

CHAPTER I.

PAGE
§ 83—97. General laws of the statics of rigid bodies 76

CHAPTER II.

§ 98—120. Centre of gravity 88

CHAPTER III.

§ 121—137. Equilibrium of bodies rigidly connected and supported 109

CHAPTER IV.

§ 138—153. Equilibrium in funicular machines 127

CHAPTER V.

§ 154—180. On the resistances of friction and rigidity . . 145

CHAPTER VI.

§ 181—211. Elasticity and rigidity 182

SECTION IV.

DYNAMICS OF RIGID BODIES.

CHAPTER I.

§ 212—229. Doctrine of the moment of inertia 225

CHAPTER II.

§ 230—238. Centrifugal force 245

CHAPTER III.

§ 239—252. Of the action of gravity on motions along constrained paths 259

CHAPTER IV.

PAGE
§ 253—271. The doctrine of impact 228

SECTION V.

STATICS OF FLUID BODIES.

CHAPTER I.

§ 272—283. On the equilibrium and pressure of water in vessels . 309

CHAPTER II.

§ 284—294. On the equilibrium of water with other bodies . 322

CHAPTER III.

§ 295—302. On the equilibrium and pressure of air . . . 338

SECTION VI.

DYNAMICS OF FLUID BODIES.

CHAPTER I.

§ 303—311. The general laws of the efflux of water from vessels 350

CHAPTER II.

§ 312—322. On the contraction of the fluid vein by the efflux of water through orifices in a thin plate . . . 364

CHAPTER III.

§ 323—336. On the efflux of water through tubes . . . 382

CHAPTER IV.

§ 337—344. On the resistances of water in passing through contraction 401

CHAPTER V.

PAGE
§ 345—353 .On the efflux of water under variable pressure . 415

CHAPTER VI.

§ 354—359. On the efflux of air from vessels and tubes . . 428

CHAPTER VII.

§ 360—371. On the motion of water in canals and rivers . . 438

CHAPTER VIII.

§ 372—382. Hydrometry or the doctrine of the measurement of water 453

CHAPTER IX.

§ 383—394. On the impulse and resistance of fluids . . . 466

PROPERTY OF
ALFRED S. NILES

PREFACE BY THE AMERICAN EDITOR.

THE want of a good standard work on those branches of Mechanics which pertain to Machinery and Engineering, has long been felt as a deficiency in our means of instruction in the higher institutions, as well as in the office of the practical engineer. Any one who has had occasion to direct the study of young men preparing for the duties of the profession, must, in the course of his practice, have encountered the difficulty, on the one hand, of inducing students to extend their researches after the principles of their profession into the intricacies of the higher calculus, and, on the other hand, of contenting himself with the very limited discussion of principles, found in many works which have been hitherto employed to give the desired basis for professional knowledge. The difficulty has, of necessity, been met by the expedient of recommending detached parts of several distinct works.

The treatises on Natural Philosophy were a very inadequate substitute for a practical work on the Mechanics of Engineering. Many of the topics requisite to be handled in a work of this nature, are, in the books now in use, either wholly omitted, or so slightly passed over, as to be of little service beyond the general statement of certain laws, and perhaps a few illustrations, indifferently applied, and furnishing but feeble helps to the exact understanding of the laws applicable to the subject.

In presenting the claims of Prof. Weisbach's treatise to the consideration of Engineers and Machinists of the United States, we cannot do better than adopt the remarks of the English translator.

"From the well earned reputation of Professor Weisbach as a teacher and original investigator, from the interest which attaches itself, especially at the present moment, to all that pertains to Mechanics or Engineering, from the able manner in which he has treated both the theoretical and practical portion of his subject, and from the variety and abundance of examples which illustrate the principles and formulæ, it is presumed that a translation of his excellent work may not be unacceptable to the English reader. 'The Mechanics of Machinery and Engineering' has met with deserved favor in Germany, and is not unknown to many of the profession in our own country. The aim and objects of the undertaking are fully explained by the Author in his Preface; and it is hoped that the same end may be obtained here, by affording, through the medium of a translation, valuable aid to the acquirement of professional knowledge."

In the course of this volume, six great divisions of the science will be found to be embraced. Of these, the *pure mathematical science of motion*, including simple and compound, rectilinear and curvilinear motions, and the laws and expressions relating to the free descent of bodies, have, of course, claimed the first attention.

The *Physical Science of Motion* has next been treated; then the *division of forces*, and their *measure*,—the *density* and the *state of aggregation* of bodies have been duly presented. In the third section, the *statics of rigid bodies* claim a careful investigation, and are followed, in the fourth, by the *dynamics* of the same class of bodies; while, in the fifth, the *statics*, and in the sixth, the *dynamics of fluid bodies* are treated with marked ability; and in several parts with an originality and freshness which indicate that the author is here in his peculiar domain—the scene of his chosen labors, and that on which his energies, as an original inquirer, have been very successfully employed. In this branch of his subject, Prof. Weisbach has included the discussion of the mechanics of elastic as well as of non-elastic fluids. We are indebted to him for several new formulæ and valuable tables of constants.

As students of this work are, from the wide diffusion of French and German science, likely to meet, in other treatises, frequent references to French and German weights and measures, it has been deemed useful not only to present, as in the English edition, the annexed comparative table of weights and measures, but also to retain from the *original* a few examples completely worked out as exercises in computation under the different systems. The same has been done in respect to temperatures measured by the Centigrade thermometer. The facility afforded by the decimal divisions of all the French weights and measures strongly recommend them both for scientific and practical purposes.

The additions of the American Editor are generally thrown into notes suitably designated, or, when embraced in the text, are usually enclosed in brackets. A number of new illustrations have been added, particularly such as relate to the principles of the "toggle joint," of rolling and dragging friction, of carriage wheels, and of floating docks. A table of co-efficients for the flow of water through wiers, the result of some recent experiments, not heretofore published, will be found in the sixth section.

It has been found necessary to make a very considerable number of corrections of the English copy, especially in the working of examples by the translator, where a departure from the original had been made, on account of a change of weights and measures. Should some few errors still have escaped notice, it will be no matter of surprise, and of the less importance, perhaps, since the principle of computation is in all cases pointed out, the steps severally indicated, and the general law, therefore, readily very applied to each particular illustration.

A comparison with the original work in German has been made, and the great care exercised by the learned author in its preparation and publication has been rendered abundantly evident.

Justice to the artisans engaged in getting out the present edition requires us to say, that the illustrations here presented will not suffer in comparison with those of either the German or the English edition. In some instances they are decidedly superior to either.

PHILADELPHIA, *March*, 1848.

NOTE.—In regard to a matter of minor importance, that of the notation used in this work, a few words may be proper, in order to apprize the reader, in advance, that the use of the simple point (.) as the sign of multiplication has required the adoption of the comma

(,) to mark the decimal division of numbers. Should any departures from this application be observed, they will be such as are readily understood.

In using the Italian (a), and the Greek alpha (α), the latter is most commonly applied to arcs and angles, and the former to lines or other quantities. This distinction, which is important to avoid confusion, and which is strictly observed in the original, has been sometimes lost sight of in the English edition.

Vide p. 217.

AUTHOR'S PREFACE.

In giving to the public the First Volume of my Elementary Treatise on Mechanics, for Engineers and Machinists, I feel some degree of diffidence. Although I am conscious that, in composing the book, I have proceeded with the greatest care and circumspection; I am, nevertheless, apprehensive that it cannot satisfy all, since the views, wishes and requirements of different individuals differ so widely. One reader may probably consider this or that chapter too long and too minutely treated, which another may find too short. Some will miss higher science in the treatment of certain subjects, which others would have wished to see treated in a still more popular manner. But many years of study, much experience in teaching, and continued observation, have led me to the method, according to which I have composed the present work, and I consider it the most appropriate for the intended purpose.

My chief aim in writing this work was the attainment of the greatest simplicity in enunciation and proof; and with this to give the demonstration of all problems, important in their practical application, by the lower mathematics only. If we consider the great variety of knowledge which the Engineer and Machinist have to acquire, who wish to do credit to their profession, we, their instructors, should make it our duty to render well-grounded study of science easy by simplicity of explanation, by the use of only the best known and easiest auxiliary sciences, and by eschewing everything that is unnecessary.

I have therefore avoided, in the present work, the use of the differential and integral calculus; for although there exist now more frequent opportunities, than formerly, for learning these methods, it is still unquestionable, that without constant practice, the readiness of using them is very soon lost; and that there are, consequently, many excellent practical men who have entirely forgotten how to apply them. Some popular authors give the results of the more difficult problems without proofs. I cannot approve of this, and have preferred to give the proofs of practically important problems in an elementary way, although this may sometimes appear rather long and tedious. There are, therefore, but few formulæ in the work unaccompanied by their derivation. A general acquaintance with some doctrines of natural philosophy, but especially an intimate knowledge of pure elementary mathematics, are of course necessary for the understanding of the present work.

I have taken especial pains to preserve the right medium between *generalizing* and *individualizing;* for, although I am fully aware of the advantages of generalization, I must still adhere to the opinion, that in an elementary work, too much generalizing is to be avoided. Simple cases are, in practice, of more frequent occurrence than complicated. It is also undeniable, that in treating a general case, the knowledge which might be gained by the treatment of a specific case, is frequently lost, and that it is not unfrequently easier to deduce the compound from the simple, than to eliminate the special from the general.

The "Mechanics of Engineering and Machinery" must not be mistaken for a work on the construction of machines, but it is to be considered merely as an introduction to or preparatory science for this. This Treatise of Mechanics is to stand in the same relation to the construction of machines, as descriptive geometry stands to the drawing of machines. After mechanics and descriptive geometry have been learned, the instructions on the construction and the drawing of machines may with advantage be united in one course.

The propriety of dividing the Mechanics of Engineering into a theoretical and practical part, may perhaps be doubted. If it be borne in mind, that this work is to furnish instructions on all mechanical relations, in architecture and the science of machines, the utility, or rather necessity of this division must be apparent. In order to form a complete opinion of a building or machine, the most various doctrines of mechanics (*i. e.* the doctrines of friction, strength of materials, inertia, impact, efflux, &c.), must be taken into consideration; the material for the study of the mechanics of a building or machine must, therefore, be collected from all parts of mechanics. Now, as it is much more useful practically to be able to study the doctrines relative to every individual machine in connection, than to have to collect them from all departments of mechanical science, the utility of the adopted division seems to be beyond all doubt.

Having practical application always in view, I have endeavored to illustrate each doctrine, as much as possible, with appropriate examples, taken from practice; and I can truly assert, that this book excels most works of a similar nature, in the great number and appropriate choice of worked out examples. The large number of carefully executed figures will, no doubt, likewise assist the attainment of the above-mentioned object; and I cannot omit here to express how greatly satisfied I feel with the manner in which the publishers have performed their part in the getting up of the book.

Finally, it is necessary to point out to the reader of the work, that he will find much that is new and peculiar to the author. Passing over smaller matters, which occur in almost every chapter, I will mention only the following more comprehensive subjects. A general ? and easy method of determining the centre of gravity of plane surfaces and even-sided polyhedra, will be found in the paragraphs, 107, 112, and 113; an approximate formula for the catenary in the paragraphs 147 and 148; supplements to the theory of the friction of axes in the paragraphs 167, 168, 169, 172, and 173. The doctrine of impact especially has received essential additions by the paragraphs 262 and 263, since the impact of imperfectly elastic bodies has been hitherto insufficiently treated; and the case, where a perfectly elastic body comes in contact with a partially elastic body, has been passed over entirely. The largest number of additions, and some entirely new laws will be found in the hydraulics, the author having made this branch of the sciences his particular study during a number of years. The laws of the incomplete contraction of the fluid vein, discovered by the author, appear here for the first time in a course of mechanics. The chief results of the author's experiments on the flow of water through slides, cocks, clacks and valves, are likewise given, as well as his principal observations made during his latest experiments on the flow of water through prolonged oblique tubes, and through angular, curved, and long straight tubes, though he has not yet been able to publish the third number of his "Untersuchungen im Gebiete der Mechanik und Hydraulik," which is to contain the results of these experiments. The chapters on flowing water and the gauging and impulse of water have likewise received additional matter from the author.

But now, after completion of the first volume, I cannot refrain from wishing that several subjects had been treated in a different manner, although I have not been able to discover any essential errors or imperfections in it. Should the reader here and there find omissions, I must refer him to the second volume, which will contain supplements, most of which have been intentionally reserved for that volume, a has been mentioned in several passages of that now published.

It will give me great satisfaction and pleasure, if the purpose which I had in view in writing this work has been attained in some measure I wished to supply the practical man with useful advice, the instructor in mechanics with a guide for teaching, and the young engineer and machinist with a welcome auxiliary for the acquiremen of the science of mechanics.

JULIUS WEISBACH.

FREIBERG, *March*, 1846.

COMPARATIVE TABLES OF ENGLISH, FRENCH, AND GERMAN MEASURES AND WEIGHTS.

I. Measures of length.

English (and Russian) foot.	French metre.	Paris foot.	Prussian, Danish, and Rhenish foot.
1	.3047945	.9382923	.9711361
3.280899	1	3.078444	3.186199
1.065765	.3248394	1	1.035003
1.029722	.3138535	0.9661806	1

II. Square measure.

Square foot.	Sq. metre.	Square foot.	Square foot.
1	.09289969	0.8803934	0.9431053
10.76430	1	9.476817	10.15187
1.135856	.1055207	1	1.071232
1.060327	.09850405	0.9335049	1

III. Cubic measure.

Cubic foot.	Cubic metre.	Cubic foot.	Cubic foot.
1	.02831531	0.8260668	.9158836
35.31658	1	29.17385	32.34587
1.210556	.03427727	1	1.108728
1.091842	0.9019342	.03091584	1

IV. Measures of capacity.

English gallon, =277.27384 cubic inches.	French litre, =.001 cubic metre.	Prussian quart, =64 cubic inches.
1	4.543458	3.967977
.2200967	1	0.8733386
.2520176	1.145031	1

Bushel, = 8 gallons.	Hectolitre, =100 litres.	Scheffel, = 3072 cubic inches.
1	0.3634767	.6613296
2.751208	1	1.819455
1.512105	0.5496150	1

V. Weights.

English pound, avoirdupois.	French.		Prussian, Hanoverian, Brunswick, and Hessian pound.
	Kilogram.	Livre, poids de marc.	
1	.4535976	.9266439	.9698245
2.204597	1	2.042879	2.138072
1.079163	.4895058	1	1.046599
1.031114	.4677110	.9554758	1

Hundred weight. = 112 lbs. avd.	Quintal métrique. = 100 kilo.	Quintal, =100 livres (old measure.)	Centner, = 110 lbs.
1	0.5080293	1.037841	0.9874577
1.968390	1	2.042877	1.943702
0.9635386	0.4095058	1	0.9514536
1.012702	0.5144821	1.051023	1

L. G.

The United States standard gallon has a capacity of 231 cubic inches, and contains 8.3388822 avoirdupois pounds, or 58372.1754 Troy grains of distilled water at 39°.83 Fah., the barometer being at 30 inches. The U. S. standard bushel contains 2150.42 cubic inches, and 77.627413 pounds avoirdupois of distilled water at 39°.83 Fah.—Am. Ed.

PRINCIPLES

OF

THE MECHANICS

OF

MACHINERY AND ENGINEERING.

SECTION I.

PHORONOMY; OR, THE PURE MATHEMATICAL SCIENCE OF MOTION.

CHAPTER I.

SIMPLE MOTION.

§ 1. *Rest and Motion.*—Every body occupies a certain position in space; a body is at rest when it does not change its position; and is in motion, on the other hand, when it successively passes from one position into others. The rest and motion of a body are either absolute or relative, according as we refer its position to a space which itself is at rest or in motion, or considered to be in either state.

Upon the earth there is no rest, for all bodies upon the earth share in its motion about the sun, and about its own axis; but if we suppose the earth to be at rest, then all those terrestrial bodies are at rest which, with reference to the earth, do not change their position.

§ 2. *Kinds of Motion.*—The continual succession of positions which a body in its motion gradually occupies, forms a space which is called the trajectory, or path of the moving body. The path of a moving point is a line; that of a geometrical body, is another body; but by this latter is generally understood that line which a certain point of the body, viz. the centre, describes in its motion.

When the path is a straight line, the motion is rectilinear; when a curved line, the motion is curvilinear.

With reference to time, motion is uniform or variable.

§ 3. A motion is uniform, when equal spaces are described by it in equal and arbitrarily small times; and variable, when this equality does not take place. When the spaces, described in equal times, increase continuously with the time, a variable motion is called—accelerated; and when the spaces decrease,—retarded. {negatively ac-celerated}

Periodic differs from uniform motion in this, that equal spaces are described within certain intervals only, which are called periods.

The apparent diurnal revolution of the fixed stars, and the progressive motion of the hands of a watch are instances of uniform motion. Falling and upwardly projected bodies, the sinking of the surface of water by its flow from vessels are instances of variable motion. The oscillations of a pendulum, the play of the piston of a steam engine, &c., are illustrations of periodic motion.

§ 4. *Uniform Motion.*—Velocity is the ~~rapidity~~ rate or magnitude of a motion. The greater the space is, which a body describes in a given time, the more rapid is its motion, and the greater is its velocity. In uniform motion, the velocity is invariable; and in a variable one, it changes at every instant. The measure of the velocity at any determinate point of time, is the path which a body actually describes, or would describe in a unit of time or second, if from that moment the motion were to become uniform, and the velocity to remain invariable. In general this measure is called simply—velocity.

§ 5. When a body describes the path σ at each particle of time, and a second consists of very many (n) such particles; the path during one second is the velocity, or rather the measure of the velocity:

$$c = n . \sigma.$$

After a time, t (seconds) $n . t$ particles have elapsed, but in each a space σ has been described; the whole space, therefore, which corresponds to the time t is:

$$s = n . t . \sigma = n . \sigma . t, \text{ i. e.}$$

$$\text{I. } s = ct.$$

so that in uniform motion, the space (s) is a product of the velocity (c) and the time (t).

Inversely

$$\text{II. } c = \frac{s}{t} \text{ and III. } t = \frac{s}{c}.$$

Example 1. A locomotive, going with a velocity of 30 feet per second, passes over in 2 hours = 120 min. = 7200 seconds, a space (s) $30 \times 7200 = 216000$ feet.—2. If a time, $3\frac{1}{2}$ minutes = 210 seconds, be required to raise up a ton weight from a shaft 1200 feet deep, its mean velocity (v) must be taken $= \frac{1200}{210} = \frac{40}{7} = 5\frac{5}{7} = 5{,}714 \ldots$ feet.—3. A horse, moving with a velocity of 6 feet per second, requires to perform 24,000 feet, a time $\frac{24000}{6} = 4000$ seconds.

§ 6. If two different uniform motions be compared with each other, the following is arrived at:

The spaces are $s = c\,t$ and $s_1 = c_1 t_1$, therefore their ratio is $\frac{s}{s_1}$

$= \frac{c\,t}{c_1 t_1}$. If $t_1 = t$, then $\frac{s}{s_1} = \frac{c}{c_1}$, or if $c_1 = c$, then $\frac{s}{s_1} = \frac{t}{t_1}$, lastly if $s_1 = s$, $\frac{c}{c_1} = \frac{t_1}{t}$.

The spaces described in different uniform motions in equal times are to each other as the velocities; the spaces described with equal velocities are as the times; and lastly, the velocities corresponding to equal spaces are inversely proportional to the times.

§ 7. *Uniformly variable Motion.*—A motion is uniformly variable when its velocity either increases or diminishes by a ~~certain~~ equal amount in equal and arbitrarily small times. It is either uniformly accelerated or uniformly retarded, according as in the first a gradual increase, or in the second a gradual diminution of velocity takes place.

In *vacuo*, the motion of a falling body is uniformly accelerated; were the air to exert no influence upon it, the motion of a body vertically projected would be uniformly retarded.

§ 8. Any change in the strength or magnitude of the velocity of a body is called acceleration; it is either positive or negative, according as there is increase or diminution of the velocity. The greater this increase or diminution within a given time, the greater is the acceleration. In uniformly variable motion, the acceleration is invariable, and may be measured by the increase or diminution of velocity which takes place in a ~~second~~ unit of time. In every other motion, the measure of the acceleration is the increase or diminution which a body would acquire if, from the moment in which the acceleration begins, it lose its variability and the motion pass into a uniformly variable one. Diminution of velocity is termed ***retardation.***

The measure is very commonly called the ***velocity.***

§ 9. If the velocity of a uniformly accelerated motion increase ($\varkappa$) in infinitely small particles of time, and a second of time is made up of such particles, the increment of velocity, or the acceleration, in one second is:

$$p = n\,\varkappa,$$

and the increment after t seconds $= n\,t\,.\,\varkappa = n\,\varkappa\,.\,t = pt$.

If the initial velocity (the moment from which the time is counted) $= c$, the terminal velocity, *i. e.* the velocity acquired after the time (t) is:

$$v = c + pt.$$

For motion, commencing without velocity, $c = 0$, therefore $v = p\,t$, and for uniformly retarded motion, having a negative acceleration (p), $v = c - p\,t$.

Example 1. The acceleration of a body falling freely in vacuo $= 32\,.\,2$ feet; it acquires, therefore, after 3 seconds, a velocity $v = pt = 32\,.\,2 \times 3 = 96\,.\,6$ feet.—2. A sphere rolling down an inclined plane, with an initial velocity $c = 25$ feet, acquires in its course, at each second, 5 feet additional velocity; its velocity, therefore, after $2\frac{1}{2}$ seconds: $v = 25 + 5 \times 2.5 = 25 + 12.5 = 37.5$ feet, &c.; proceeding from the last point uniformly, it will pass over 37.5 feet in every second.—3. A locomotive going with a 30 feet velocity is so retarded, that in each second it loses 3.5 feet of velocity; its acceleration is $-\,3.5$; its velocity, therefore, after 6 seconds is $v = 30 - 3.5 \times 6 = 30 - 21 = 9$ feet.

§ 10. *Uniformly accelerated Motion.*—The velocity of every motion may be regarded as invariable within a small particle of time τ. We may, therefore, put the space described in such time $\sigma = v \, . \, \tau$, and we obtain the whole space in the finite time t, by the measurement of these small spaces. Now for all these small spaces, the time τ is one and the same; this sum, therefore, may be put equal to the product of these particles of time, and the sum of the velocities corresponding to equal intervals.

In uniformly accelerated motion, the sum $(0 + v)$ of the velocities, in the first and last moment, is as great as the sum $p\tau + (v - p\,\tau)$ of the velocities in the second and last but one; also, equal to the sum $2\,p\,\tau + (v - 2\,p\,\tau)$ of the velocities in the third and last but two; and this sum is equal to the terminal velocity v. Here, therefore, the sum of all the velocities is equal to the product $\left(v\,.\frac{n}{2}\right)$ of the terminal velocity v, and half the number of all the particles of time. The space described is the product $\left(v\,.\frac{n}{2}\,.\,\tau\right)$ of the terminal velocity v, and half the number and magnitude of the particles. Now the magnitude (τ) of such a particle, multiplied by the number, gives the time t; the space, therefore, described in the time t with a uniformly accelerated motion is $s = \frac{v\,t}{2}$.

The space, therefore, described in uniformly accelerated motion is as in uniform motion when, in the latter case, its velocity is half as great as the terminal velocity of the former.

Example 1. If a body in 10 seconds has acquired a velocity v by uniformly accelerated motion of 26 feet, the space described in that time is $s = \frac{26 \times 10}{2} = 130$ feet.—2. A carriage which, in its accelerated motion, goes over 25 feet in 2¼ seconds, proceeds at the end with a velocity $v = \frac{2 \times 25}{2.25} = \frac{50 \times 4}{9} = 22.22$.

§ 11. The two fundamental formulæ of uniformly accelerated motion:

$$\text{I. } v = p\,t \text{ and II. } s = \frac{v\,t}{2},$$

which express that the velocity is a product of the acceleration and the time; and the space, half the velocity and the time; include two other principal formulæ which are obtained, if from both equations v be eliminated once, and t twice. It follows that:

$$\text{III. } s = \frac{p t^2}{2} \text{ and IV. } s = \frac{v^2}{2p}.$$

From this, the space described is a product of half the acceleration, and the square of the time; and it is also the quotient of the square of the velocity by twice the acceleration.

These four formulæ give, by inversion, after one or other of the magnitudes contained have been separated, eight other formulæ.

Example 1. A body moving with an acceleration of 15,625 feet, describes in 1½ second a space $\frac{15.625 \times (1.5)^2}{2} = 15.625 \times \frac{9}{8} = 17.578$ feet.—2. A body transported with an acceleration $p = 4.5$ into a velocity $v = 16.5$ feet, has described a space $s = \frac{(16.5)^2}{2.4,5} = 30.25$ feet.

§ 12. By a comparison of two uniformly accelerated motions, we arrive at the following:

The velocities are $v = pt$ and $v_1 = p_1 t_1$, the spaces on the other hand are $s = \frac{pt^2}{2}$ and $s_1 = \frac{p_1 t_1^2}{2}$; from this it follows:

$$\frac{v}{v_1} = \frac{pt}{p_1 t_1} \text{ and } \frac{s}{s_1} = \frac{pt^2}{p_1 t_1^2} = \frac{vt}{v_1 t_1} = \frac{v^2 p_1}{v_1^2 p}.$$

If we put $t_1 = t$, we have $\frac{s}{s_1} = \frac{v}{v_1} = \frac{p}{p_1}$; the spaces described are to each other as the terminal velocities; or, as the accelerations.

If further, we take $p_1 = p$, it gives $\frac{v}{v_1} = \frac{t}{t_1}$ and $\frac{s}{s_1} = \frac{t^2}{t_1^2} = \frac{v^2}{v_1^2}$; so that, in like accelerations, and also in one and the same uniformly accelerated motion, the terminal velocities are proportional to the times and the spaces described to the squares of the times, as also to the squares of the terminal velocities.

Further, if $v_1 = v$ gives $\frac{p}{p_1} = \frac{t_1}{t}$, and $\frac{s}{s_1} = \frac{t}{t_1}$; in equal velocities the accelerations are inversely, and the spaces directly proportional to the times.

Lastly, $s_1 = s$ gives $\frac{p}{p_1} = \frac{t_1^2}{t^2} = \frac{v^2}{v_1^2}$; with equal spaces, the accelerations are inversely as the squares of the times, and directly as the squares of the terminal velocities.

§ 13. For a uniformly accelerated motion commencing with a velocity (c) we have § 9:

$$\text{I. } v = c + pt,$$

and as the space ct belongs to the invariable velocity (c), and the space $\frac{pt^2}{2}$ to the acceleration p:

$$\text{II. } s = ct + \frac{pt^2}{2}.$$

If we eliminate p from both equations, we have:

$$\text{III. } s = \frac{c+v}{2} t,$$

and substituting the value of t,

$$\text{IV. } s = \frac{v^2 - c^2}{2p}.$$

Example 1. A body propelled with an initial velocity $c = 3$ feet, and with an acceleration $p = 5$ feet, describes in 7 seconds, a space $s = 3 . 7 + 5 . \frac{7^2}{2} = 21 + 122.5 =$

143.5 feet.—2. Another body which in 3 minutes = 180 seconds, changes its velocity from 2½ feet into 7½ feet, performs in this time a distance $\frac{2.5 + 7.5}{2} . 180 = 900$ feet.

§ 14. For a uniformly retarded motion with an initial velocity c, these formulæ are applicable:

$$\text{I. } v = c - pt,$$

$$\text{II. } s = ct - \frac{pt^2}{2},$$

$\{s = vt + \frac{1}{2} at^2\}$

$$\text{III. } s = \frac{c+v}{2} . t,$$

$$\text{IV. } s = \frac{c^2 - v^2}{2p},$$

they are derived from the former §, when p is made negative. Whilst in uniformly accelerated motion, the velocity increases without limit, in a uniformly retarded one, the velocity at a certain point of time becomes null, and afterwards negative, *i. e.* it goes on in an inverse direction.

If in the first formula we put $v = 0, pt = c$, the time at which the velocity becomes null is, $t = \frac{c}{p}$; if we substitute this value of t in the second equation, we have the space which the body has described at the point of time $= \frac{c^2}{2p}$.

If the time be greater than $\frac{c}{p}$, the space is less than $\frac{c^2}{2p}$; if it be $= \frac{2c}{p}$, the space becomes null, and the body returns to the point from which it set out. If the time be greater than $\frac{2c}{p}$, then s becomes negative, and the body is on the opposite side of its initial point.

Example. A body which rolls up an inclined plane with an initial velocity of 40 ft., by which it suffers a retardation of 8 feet per second, ascends only $\frac{40}{8} = 5$ seconds and $\frac{40^2}{2.8} = 100$ feet in height, then rolls back and returns after 10 seconds with a velocity of 40 feet to its initial point; and after 12 seconds, arrives at a distance $40 \times 12 - 4 \times (12)^2 = 96$ feet below this point if the plane extend itself backwards.

§ 15. *Free Descent of Bodies.*—The free or vertical descent of bodies in vacuo, offers the most important example of uniformly accelerated motion. The acceleration of this motion brought about by gravity is designated by the letter g, and has the mean value of:

9,81	metres
30,20	Paris feet.
32,22	English feet.
31,03	Vienna feet.
31,25	Prussian feet.

If either of these values of g be substituted in the formula:

$$v = gt, s = g\frac{t^2}{2} \text{ and } s = \frac{v^2}{2g}, v = \sqrt{2gs},$$

all questions, with reference to the free descent of bodies, may be answered. For the English measure:

$$v = 32{,}2 \,.\, t = 8{,}02 \sqrt{s};\; s = 16{,}1 \,.\, t^2 = .0155\, v^2$$
$$\text{and } t = 0{,}031\, v = 0{,}249 \sqrt{s}.$$

Example 1. A body acquires in its free descent of 4 seconds a velocity $v = 32.2 \times 4 = 128.8$ feet, and describes in this time a space $s = 15.625 \times 4^2 = 250$ feet.—2. A body falling from a height $s = 9$ feet, has a velocity $v = 8.02 \sqrt{9} = 24.06$ feet.—3. A body projected vertically with a velocity of 10 feet ascends to a height $s = 0.016 \times 10^2 = 1.6$ feet, and requires for it a time $t = 0.031 \times 10 = 0.3$, or about one-third of a second.

§ 16. The following table will show the relations of the motion to the time in the free descent of bodies.

Time in seconds.	0	1	2	3	4	5	6	7	8	9	10
Velocity.	0	$1g$	$2g$	$3g$	$4g$	$5g$	$6g$	$7g$	$8g$	$9g$	$10g$
Space.	0	$1\frac{g}{2}$	$4\frac{g}{2}$	$9\frac{g}{2}$	$16\frac{g}{2}$	$25\frac{g}{2}$	$36\frac{g}{2}$	$49\frac{g}{2}$	$64\frac{g}{2}$	$81\frac{g}{2}$	$100\frac{g}{2}$
Difference.	0	$1\frac{g}{2}$	$3\frac{g}{2}$	$5\frac{g}{2}$	$7\frac{g}{2}$	$9\frac{g}{2}$	$11\frac{g}{2}$	$13\frac{g}{2}$	$15\frac{g}{2}$	$17\frac{g}{2}$	$19\frac{g}{2}$

The last horizontal column of this table gives the spaces which the freely falling body describes in the single seconds. We see that these spaces are to each other as the odd numbers 1, 3, 5, 7, &c., whilst the times and velocities are as the natural numbers 1, 2, 3, 4, &c., and the spaces fallen through as their squares 1, 4, 9, 16, &c. For example, the velocity after six seconds, is $6g = 193{,}2$ feet, that is, the body would, if it proceeded from this time uniformly upon an horizontal plane, offering no impediment, pass over in each second a space $6g = 193{,}2$ feet. This space it describes in the course of the following and seventh second, but not in reality, for according to the last column it amounts to $13 . \frac{g}{2} = 13 \times 16{,}1 = 209{,}3$ feet, in the eighth second it is $15 . \frac{g}{2} = 15 . 16{,}1 = 241$ feet, &c.

Remark.—Many writers designate the space of 16 feet, which a body freely descending will describe in one second, by g, and term it properly the acceleration of gravity. They have then for the free descent of bodies, the following formula:

$$v = 2gt = 2\sqrt{gs},$$
$$s = gt^2 = \frac{v^2}{4g},$$
$$t = \frac{v}{2g} = \sqrt{\frac{s}{g}}.$$

This custom, which is met with in Germany only, is disappearing by degrees, and in consequence of its being frequently misunderstood, and the many mistakes which arise therefrom, this is much to be desired.

§ 17. If the free descent of a body go on with a certain initial velocity (c) the formulæ are of the following kind:

$$v = c + gt = c + 32{,}2\,t;\ \text{also}\ v = \sqrt{c^2 + 2\,g\,s} = \sqrt{c^2 + 64{,}4.s};$$

$$s = ct + g\frac{t^2}{2} = ct + 16.1\,t^2,\text{also}\ s = \frac{v^2 - c^2}{2g} = 0{,}0155\,(v^2 - c^2).$$

If, on the other hand, the body be projected vertically to a height with the velocity c, then:

$$v = c - gt = c - 32{,}2\,t;\ \text{also}\ v = \sqrt{c^2 - 2\,g\,s} = \sqrt{c^2 - 64{,}4s};$$

$$s = ct - g\frac{t^2}{2} = ct - 16{,}1\,t^2;\ \text{also}\ s = \frac{c^2 - v^2}{2g} = 0{,}0155\,(c^2 - v^2).$$

If we consider a given velocity c as the terminal velocity acquired by a free descent, then the corresponding space fallen through $\frac{c^2}{2g}$ $= 0{,}0155\,.\,c^2$ is called the height due to the velocity. By the introduction of this quantity, some of the foregoing formulæ may be expressed more simply. If the height $\left(\frac{c^2}{2g}\right)$ due to the initial velocity c be put $= h$, and that due to the terminal velocity $\frac{v^2}{2g} = h_1$, we have the following for falling bodies:

$$h_1 = h + s,\ s = h_1 - h;$$

and for ascending: $h_1 = h - s,\ s = h - h_1$.

The space of fall or ascent is, therefore, equal to the difference of the heights due to velocity.

Example. The velocities are 5 and 11 feet, the heights due to velocity $= 0{,}0155 \cdot (5)^2 =$ 0,3875 feet, and $0{,}0155\,.\,11^2 = 1{,}875$ feet; the space which is described during the passage from one velocity to the other: $s = 1{,}8755 - 0{,}3875 = 1{,}4880$ feet.

§ 18. From the formula $s = h - h_1$ it also follows that a body vertically projected has at each point that velocity which it would have, but in an inverse direction, were it to have fallen from the height still remaining to that point, and which it then actually possesses in its following descent.

Example. A body is thrown up with a 15 feet velocity, and strikes in its rise against an elastic impediment, which for the moment throws it back with the same velocity with which it struck. How great then is this velocity, and the time of ascending and descending? To the velocity ($c = 15$ ft.) corresponds the height of ascent $h = 3{,}49$ ft.; the height due to velocity at the moment of impact is $h_1 = 3{,}49 - 2{,}00 = 1{,}49$, and consequently this velocity $= 8{,}02\sqrt{1{,}49} = 9{,}652$ ft. The time to attain the whole height (3,49 ft.) is $t = 0{,}032.\,v = 0{,}032\,.\,15 = 0{,}480''$, for the height 1,49 ft. $t_1 = 0{,}032\,.\,10 = 0{,}320''$; there remains then for the time required to rise to the height of 2 ft., or the time from the commencement to impact: $t - t_1 = 0{,}480 - 0{,}320 = 0{,}160''$, and the whole time of rising and falling $= 2\,.\,0{,}160 = 0{,}320''$. This is also but the $\frac{0{,}320}{0{,}960}$th $=$ 3d part of the time, which would be necessary for rising and falling if the body were to rise and fall unimpeded. This fall finds its application in the forging of hot iron, because in the gradual cooling of that metal it is desirable that the blows of the hammer follow as quickly as possible in a short time. When the hammer is thrown back by an elastic arrangement, it will give in the same time, in the proportions above laid down, thrice the number of blows to what it would give were its rise umimpeded.

Remark 1. The transformation of the velocity into height due to velocity, and the reverse is very often required in practical mechanics, and especially in hydraulics. A table where this is set down is of great use to the practical man.

Remark 2. The foregoing formulæ are only strictly correct for a free descent in vacuo; they may be used with tolerable accuracy for a fall in air, if the falling bodies have a weight great in proportion to their volume, and if the velocities do not come out very great. For the rest, they are also used under other circumstances and relations in many other descents, as will hereafter be shown.

§ 19. *Variable Motions in Particular.*—For variable motions especially, in which the periodic are also included, the formulæ

$$1.\ \kappa = p\,\tau, \text{ and}$$

$$2.\ \sigma = v\,\tau$$

hold good:—the increment of velocity (κ) acquired in a very small time τ (element of time), is a product of the acceleration p and this time; and the space σ described in the element of time τ is a product of the velocity (v) and the time τ. By inversion:

$$3.\ p = \frac{\kappa}{\tau} \text{ and}$$

$$4.\ v = \frac{\sigma}{\tau}$$

Acceleration is the quotient of the increment of velocity by the element of the time τ in which it is acquired. Velocity is the ratio of the element of space to that of the time.

The two last formulæ may be used for the measurement of the acceleration and velocity. *Ex.* From the motion given by the formula $s = at^2$ when a is the space described after the first second, it follows: if t increase by τ and s by σ, $s + \sigma = a\,(t + \tau)^2$. Now $(t + \tau)^2 = t^2 + 2\,t\,\tau + \tau^2$, or because τ is small $= t^2 + 2\,t\,\tau$, it therefore follows $s + \sigma = at^2 + 2\,at\,\tau$, or $\sigma = 2\,at\,\tau$; lastly, $v = \frac{\sigma}{\tau} = 2\,at$. By the same hypotheses, we learn from the last formula $v + \kappa = 2\,a\,(t + \tau) = 2\,a\,t + 2\,a\,\tau$, so that $\kappa = 2\,a\,\tau$ and the acceleration $p = \frac{\kappa}{\tau} = 2\,a$. We have, therefore, in this way found from the formulæ for the spaces, formulæ for the velocity and acceleration.

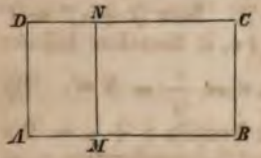
§ 20. The velocity $c = \frac{s}{t}$ differs from the velocity $v = \frac{\sigma}{\tau}$ of an element of time, and is given when the space, which in a certain time or period of a periodic motion is described, is divided by the time itself. This is called the mean velocity, and may be also regarded as that velocity which a body must have in order to describe uniformly in a given time (t) a given space (s), which, in reality, is described variably. So, for example, in uniformly variable motion, the velocity is equal to half the sum $\left(\frac{c + v}{2}\right)$ of the initial and terminal velocities: for, according to § 13, the space is equal to this sum $\left(\frac{c + v}{2}\right)$ multiplied by the time (t).

While a handle turns uniformly in a circle, the load attached to it, the piston of an air or water-pump, for instance, moves variably up and down; the velocity of this at its lowest and highest point is at a minimum, viz., null; at half the height a maximum, viz., equal to the velocity of the handle. In half a revolution, the mean velocity equals the whole height of ascent, *i. e.*, the diameter of the circle which the handle describes, divided by the time of half a revolution. The diameter $= 2\,r$ and the time $= t$, then the mean velocity of the load $= \frac{2\,r}{t}$. The handle in this time describes the semicircle $\pi\,r$; its velocity, therefore, $= \frac{\pi\,r}{t}$, and consequently the mean velocity of the load $= \frac{2}{\pi} = \frac{2}{3.141} = 0{,}6366$ times as great as the invariable velocity of the handle.

§ 21. *Graphical Representation.*—The laws of motion found above may be expressed by geometrical figures, or, as it is said, graphically represented. Graphical representations especially facilitate the conception, sustain the thoughts, prevent mistakes, and serve not unfrequently for the discovery of a quantity, and on that account are of great use in mechanics.

In uniform motion the space (s) is the product (ct) of the velocity and the time, and in geometry the area of a rectangular figure is the product of the height and base. We can, therefore, represent the space (s) uniformly described by a rectangle $A\,B\,C\,D$, Fig. 1, whose base $A\,B$ is the time (t) and whose height ($A\,D = B\,C$) is the velocity (c), provided that the time be expressed in the same unit of length as the velocity, and that the second of time and the foot be represented by one and the same line.

Fig. 1.



§ 22. Whilst, in uniform motion, the velocity ($M\,N$) at any other time ($A\,M$) of the motion is one and the same, it differs at every instant in a variable one; this motion, therefore, can only be represented by a quadrilateral figure $A\,B\,C\,D$, Fig. 2, which has $A\,B$ the time for base, and for the other limits, three lines $A\,D$, $B\,C$, $C\,D$, of which the first two are equal to the initial and terminal velocities, and the last is determined by the extremity (N) of the different velocities at the intervals (M). The line $C\,D$ is either straight or curved, according to the different kinds of variable motion from the commencement, ascending or descending, or lastly concave or convex towards the base. But in every case the area of this figure must be put equal to the variably described space (s); for each area of space $A\,B\,C\,D$, Fig. 3, may be considered as decom-

Fig. 2.

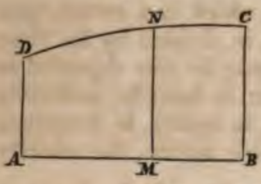
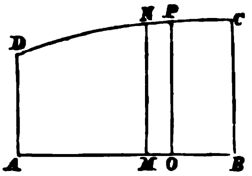


Fig. 3.



posable into many small rectangular strips, like $M\ O\ P\ N$, of which each is a product of a part ($M\ O$) of the base, and its corresponding height ($M\ N$ or $O\ P$), and the spaces described in a certain time composed of particles of which each is a product of that particle and its corresponding velocity.

§ 23. In uniformly variable motion; the increase or diminution ($v-c$) of the velocity ($=pt$, § 13) is proportional to the time. If in the Figures 4 and 5, the line $D\ E$ be drawn parallel to the base $A\ B$, and $B\ E$ and $M\ O$ = to the initial velocity $A\ D$ be cut off from the lines $M\ N$ and $B\ C$, there remain the lines $C\ E$ and $N\ O$ for the increase or diminution of the velocity, for which from the above we have the proportion

Fig. 4.

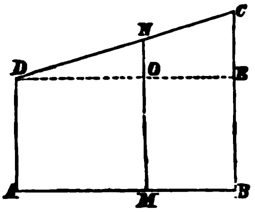
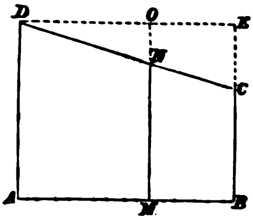


Fig. 5.



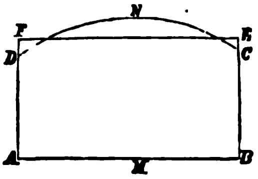
$$N\,O : C\,E = D\,O : D\,E.$$

Such a proportion requires that N as well as each point of the line $C\ D$ lie in the straight line connecting C and D, and also that the line $C\ D$ limiting the different velocities ($M\ N$) be a right line.

In consequence of this, the uniformly accelerated and uniformly retarded space described may be represented by the area of a trapezium $A\ B\ C\ D$, which has for the height $A\ B$, the time (t), and for the parallel bases the initial and terminal velocities $A\ D$ and $B\ C$. The formula of § 13, $s = \frac{c+v}{2} \cdot t$ is in perfect accordance with this. In uniformly accelerated motion, the fourth side $D\ C$ ascends from its initial point, and in uniformly retarded motion descends. In a uniformly accelerated motion beginning with a velocity null, the trapezium becomes a triangle whose area is $\frac{1}{2}\ B\ C \times A\ B = \frac{1}{2}\ ct$.

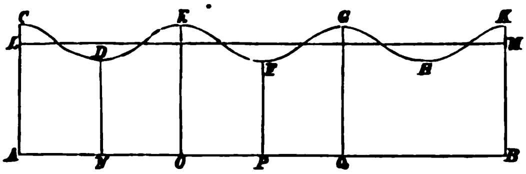
§ 24. The *mean velocity* of a variable motion is the quotient of the *space divided by the time;* multiplied, therefore by the time, it gives as a product the trajectory, and consequently may be also considered as the height $A\ F = B\ E$ of the parallelogram $A\ B\ E\ F$ Fig. 6, which has the time (t) for the base $A\ B$, and an area equal to the four-sided figure $A\ B\ C\ N\ D$ which measures the trajectory or space

Fig. 6.



passed through. The mean velocity is, therefore, likewise obtained by transforming the four-sided figure $A B C N D$ into a parallelogram $A B E F$ of the same length. Its determination is of importance, particularly in periodic motions, which occur in nearly all machines. The law for these motions is represented by a curved line $C D E F G H K$, Fig. 7. If the line $L M$ running parallel with $A B$ cuts off the same space as the curved line, and is, as it were, the axis round which $C D E F$ . . . . coils itself, then the distance $A L = B M$ between the two parallel lines $A B$ and $L M$ is the *mean* velocity of the periodic motion, whilst $A C$, $O E$, $B K$, &c., is the *maximum*, and $N D$, $P F$, &c., the *minimum* velocity of a period $A O$, $O Q$, $Q B$, &c.

Fig. 7.



§ 25. The acceleration also, or the increase of velocity during a second of time may be easily shown in the figure. In the case of uniformly variable motion it remains unchangeable; it is hence the difference $P Q$, Figs. 8 and 9, between two velocities $O P$ and $M N$, the one of which appertains to a longer time by one second ($M O$) than the other. If the motion is not uniformly variable, and the line of velocity $C D$ therefore a curve, then for each second of time (M) the acceleration varies, and is, consequently, not the real difference $P Q$ between the two velocities $O P$ and $M N = O Q$ Figs. 10 and 11; but it is the increase $R Q$ of the velocity $M N$, which would occur if commencing at the moment M the motion became uniformly accelerated, and the curved line of velocity $N C$ passed into the straight line $N E$. Now the tangent or line of contact $N E$, is that straight line in the direction of which a curve $D N$ proceeds, when from a certain point (N) it ceases to change its direction; hence the new line of velocity coincides with the tangent, and the perpendicular line $O R$ which cuts it, is accordingly the velocity which would take place

Fig. 8.

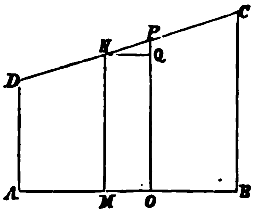
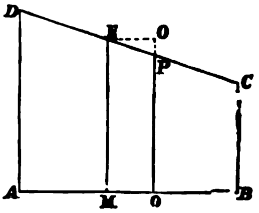


Fig. 9.



after the lapse of a second, supposing the motion to have become uniformly accelerated from the commencement of that period, and lastly,

Fig. 10.

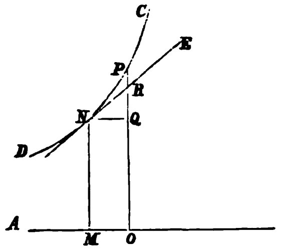
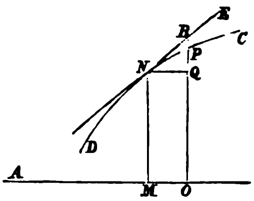


Fig. 11.



the difference *R Q* between this velocity and the primary velocity (*M N*) is the acceleration for that moment which is determined by the point *M* in the time line *A B*.

CHAPTER II.

COMPOUND MOTION.

§ 26. *Compound Motion.*—One and the same body may at the same time have two or more motions; every (relative) motion consists of the motion within a certain space, and of the motion of this space within, or in relation to, a second space. Each point upon the surface of the earth has thus two motions, for it revolves daily once round the axis of the earth, and simultaneously with the earth once yearly round the sun. A person walking on board a ship has two motions in relation to the shore, his own motion and that of the water; water flowing from a hole in the bottom or side of a vessel, whilst the latter is moving along in a carriage, has two motions, the motion from the vessel and the motion with the vessel, &c.

Hence we distinguish *simple* and *compound* motion. Those rectilinear motions are called *simple*, of which other rectilinear or curvilinear motions, consequently called compound, are made up, or may be imagined to be made up.

The combination of several simple motions to form one single motion, and the resolution of a compound motion into several simple motions, will be treated of in the sequel.

§ 27. When simple motions occur in the direction of one and the same straight line, their sum or difference gives the resulting compound motion, the former, when the motions take place in the same direction; the latter, when their directions are opposite. The truth of this axiom becomes directly obvious, when the contemporary spaces of the simple motions are united into one. The contemporary spaces $c_1\,t$ and $c_2\,t$ correspond with the uniform motions and their velocities c_1 and c_2; if these motions go on in the same direction, then after t seconds the space becomes $s = c_1\,t + c_2\,t = (c_1 + c_2)\,t$, and consequently the resulting velocity with which the compound motion proceeds is the sum of the velocities of the simple motions. When the directions of both motions are opposite, then $s = c_1\,t - c_2\,t = (c_1 - c_2)\,t$, here, therefore, the resulting velocity is equal to the difference of the simple velocities.

Example 1. To a person moving with a velocity of four feet upon the deck of a ship, in the same direction with the motion of the ship itself, which has a velocity of six feet, the objects on the shore appear to pass by with a velocity of $4 + 6 = 10$ feet.—2. The water which flows from the lateral opening of a vessel with a velocity of 25 feet, whilst the vessel containing it is moved in an opposite direction with a velocity of 10 feet, has, in relation to the other objects at rest, only a velocity of $25 - 10 = 15$ feet.

§ 28. The same relations obtain with variable motions. If one and the same body have, in addition to the primary velocities, c_1 and c_2 the constant accelerations p_1 and p_2, then the corresponding spaces are $c_1\,t$, $c_2\,t$, $p_1 \frac{t^2}{2}$, $p_2 \frac{t^2}{2}$, if the velocities and the accelerations are in the same direction, the whole space corresponding to these simple motions, will be:

$$s = (c_1 + c_2)\,t + (p_1 + p_2)\,\frac{t^2}{2}.$$

If $c_1 + c_2 = c$ and $p_1 + p_2 = p$, we then obtain $s = ct + p\,\frac{t^2}{2}$, and it follows, consequently, that not only the velocity of the resulting or compound motion is made up of the sum of the simple velocities, but that also the sum of the accelerations of the simple motions gives the resulting acceleration.

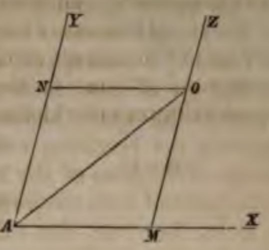
Example. A magnet falls more quickly to the earth than another body, when a mass of iron is immediately below it. The acceleration which the magnet experiences, in consequence of this iron, may be considered invariable when the height from which it falls is small and the mass of iron very considerable, ~~viz.~~, an extensive layer of magnetic iron ore. If this acceleration were 5 feet, then the magnet would fall with an increased velocity of $31{,}25 + 5 = 36{,}25$ feet in the first second, therefore it would fall $18\frac{1}{8}$ feet instead of $15\frac{5}{8}$ feet. e.g.

§ 29. *Parallelogram of the Velocities.*—If a body has at the same time two motions differing from each other in direction, it will assume a medium direction between them; and if these motions are of different kinds, viz., the one, uniform, and the other ~~uniformly increasing~~, var. then the direction will vary in every part of the motion, and the motion itself become curvilinear.

The place O, Fig. 12, which a body moving simultaneously in the directions $A\,X$ and $A\,Y$ will occupy after a certain time (t), is found amount

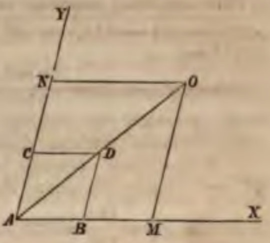
when the fourth corner of the parallelogram $A\,M\,O\,N$, determined by the contemporaneous trajectories $A\,M = x$ and $A\,N = y$, as well as by the angle $X\,A\,Y$, or the ~~distance~~ by which the directions of motion deviate from each other, is known. The correctness of this mode of procedure becomes evident when the trajectories x and y are supposed described one after the other, and not at the same time. In compliance with the one motion, the body describes the trajectory $A\,M = x$; and in compliance with the other, the trajectory proceeding from M in the direction of $A\,Y$, therefore in a line $M\,Z$ parallel to $A\,Y$, or the trajectory $A\,N = y$. If $M\,O = A\,N$, then O is the position of the body corresponding to both motions x and y simultaneously, which, in accordance with the construction, is the fourth corner of the parallelogram $A\,M\,O\,N$. We may likewise imagine that the space $A\,M = x$ is passed over in a line $A\,X$, which with all its points proceeds at the same time in the direction $A\,Y$, and therefore carries with it M in a parallel direction to $A\,Y$, and causes this point to perform the trajectory $M\,O = A\,N = y$.

Fig. 12.



§ 30. If both the motions in the directions $A\,X$ and $A\,Y$ take place uniformly and with the velocities c_1 and c_2, then the spaces will become after a certain time (t) : $x = c_1\,t$ and $y = c_2\,t$; their relationship $\frac{y}{x} = \frac{c_2}{c_1}$ is, therefore, the same at all times, a peculiarity which is only proper to the straight line $A\,O$, Fig. 13. Hence it follows that the compound motion proceeds in a straight line. If, with the velocities $A\,B = c_1$ and $A\,C = c_2$, the parallelogram $A\,B\,C\,D$ is constructed, its fourth corner gives the position D, in which the body will be placed after the lapse of one unit of time ~~second~~. But as the resulting motion is rectilinear, it follows that it must always occur in the direction of the diagonal of that parallelogram which is constructed by the velocities. If the trajectory $A\,O$ which is actually passed through in the time t be $= s$, then, on account of the similarity of the triangles $A\,M\,O$ and $A\,B\,D$, we have: $\frac{s}{x} = \frac{A\,D}{A\,B}$, and it consequently follows that this trajectory $s = \frac{x\,.\,A\,D}{A\,D}$ B $= \frac{c_1\,t\,.\,A\,D}{c_1} = A\,D\,.\,t$. In accordance with the last equation, the trajectory in the diagonal is proportional to the time (t), the motion itself consequently uniform, and the diagonal $A\,D$ its velocity.

Fig. 13.

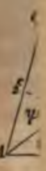


The diagonal, therefore, of a parallelogram formed by two velocities,

[illegible]

and the angle which they make with each other, gives the direction and magnitude of the actually resulting motion. This parallelogram is called the *parallelogram of velocities*, the simple velocities are called the *components*, and the compound velocity the *resultant*.

§ 31. By the use of trigonometrical formulæ, the direction and magnitude of the mean velocity may be ascertained by calculation. The resolution of one of the equal triangles, viz., *A B D*, of which the parallelogram of velocities *A B D C* (Fig. 14) is composed, gives the



quer

sequ

Lastly, if the velocities enclose a right angle, then likewise we obtain more simply:

$$c = \sqrt{c_1^2 + c_2^2} \text{ and } tang.\ \phi = \frac{c_2}{c_1}.$$

Example 1. The water flowing from a vessel or a machine has a velocity $c_1 = 25$ ft., whilst the vessel is moved with a velocity $c_2 = 19$ feet in a direction forming an angle $a^\circ = 130^\circ$ with the direction of the flowing water. What is the direction and magnitude of the resultant, or as it is also called, the absolute velocity?

The required resulting velocity is $c = \sqrt{25^2 + 19^2 + 2\,.\,25\,.\,19\ cos.\ 130^\circ} = \sqrt{625 + 361 - 50.19\ cos.\ 50^\circ} = \sqrt{986 - 950\ cos.\ 50^\circ} = \sqrt{986 - 610{,}7} = \sqrt{375{,}3} =$ 19,37 feet.

Moreover, $sin.\ \phi = \frac{19\ sin.\ 130^\circ}{19{,}37} = 0{,}9808\ sin.\ 50^\circ = 0{,}7513$, and consequently the angle by which the resultant differs from the velocity c_1 $\phi = 48^\circ, 42'$ and the angle which it makes with the direction of motion of the vessel: $a - \phi = 81^\circ 18'$.

2. If the former velocities were acting at right angles to each other, then $cos.\ a = cos.\ 90^\circ = 0$, thence the mean velocity $c = \sqrt{986} = 31{,}40$ feet; for its direction we should have $tang.\ \phi = \frac{19}{25} = 0.76$, and consequently, its deviation from the first velocity: $\phi = 37^\circ, 14'$.

§ 32. Any given velocity may be supposed to consist of two components, and can consequently be resolved into them, in accordance with certain conditions. If, for instance, the angles $D\,A\,B = \phi$ and $D\,A\,C = \psi$, Fig. 14, are given, and enclose the velocities required together with the mean velocity $A\,D = c$, then draw through the terminal point D other lines which represent the degrees corresponding to c, parallel to the directions $A\,X$ and $A\,Y$: the points of section will then cut off the required velocities $A\,B = c_1$ and $A\,C = c_2$.

[illegible] Town

2

Trigonometry expresses these velocities by the formula $c_1 = \frac{c \, sin. \, \psi}{sin. \, (\phi + \psi)}$, $c_2 = \frac{c \, sin. \, \phi}{sin. \, (\phi + \psi)}$. In the usual practical cases, the two velocities are at right angles to each other, and then $\phi + \psi = 90°$, $sin. \, (\phi + \psi) = 1$, and it follows:

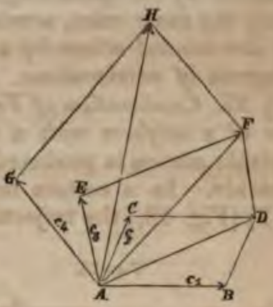
$$c_1 = c \, cos. \, \phi \text{ and } c_2 = c \, sin. \, \phi.$$

Therefore, with one component (c_1) and its angle of direction (ϕ), the direction and magnitude of the other component may be estimated. Lastly, from the velocities c, c_1 and c_2 alone their angles of direction may be determined, as the three angles of a triangle may be computed by the three sides.

Example. Suppose velocity $c = 10$ feet is to be resolved into two components which deviate from its direction by the angle $\phi = 65°$ and $\psi = 70°$. These velocities will be: $c_1 = \frac{10 \, sin. \, 70°}{sin. \, 135°} = \frac{9{,}397}{sin. \, 45°} = 13{,}29$ feet and $c_2 = \frac{10 \, sin. \, 65°}{sin. \, 135°} = \frac{9{,}063}{0{,}7071} = 12{,}81$ ft.

§ 33. *Composition and Resolution of Velocities.*—By repeated application of the parallelogram of velocities, any number of velocities may be reduced to one. By constructing the parallelogram $A\,B\,D\,C$, Fig. 15, the ~~mean~~ resultant velocity $A\,D$ ~~to~~ of c_1 and c_2 is obtained; by constructing the parallelogram $A\,D\,F\,E$, we get the mean velocity $A\,F$ to $A\,D$ and $A\,E = c_3$; and in like manner by constructing the parallelogram $A\,F\,H\,G$, the mean velocity $A\,H = c$ to $A\,F$ and $A\,G = c_4$ is obtained, and thus the mean of c_1, c_2, c_3, and c_4.

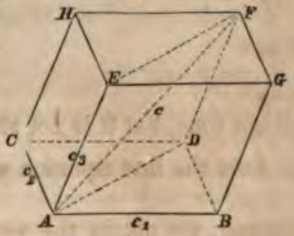
Fig. 15.



The simplest method of obtaining the mean velocity in question, is by the construction of a polygon $A\,B\,D\,F\,H$, the sides of which $A\,B$, $B\,D$, $D\,F$, and $F\,H$, are drawn parallel and equal to the given velocities c_1, c_2, c_3, and c_4; the last side $A\,H$ is then always the resultant velocity.

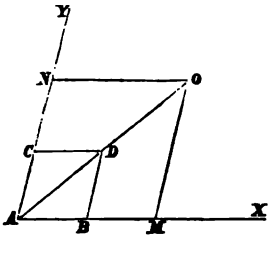
In the case, also, in which the velocities are not in the same plane, the mean velocity may be ascertained by repeated application of the parallelogram of velocities. The mean velocity $A\,F = c$ (Fig. 16) of three velocities $A\,B = c_1$, $A\,C = c_2$ and $A\,E = c_3$, which are not in the same plane, is the diagonal of a parallelepipedon $B\,C\,G\,H$, the sides of which are equal to these velocities. The *parallelepipedon of velocities* is, therefore, also a term in general use.

Fig. 16.



§ 34. *Composition of the Accelerations.*—Two uniformly accelerated motions, beginning with null velocity, produce, when combined, a uniformly accelerated motion in a straight line. If the accelerations of these motions, proceeding

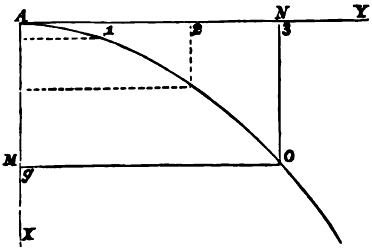
Fig. 17.



in the directions *A X* and *A Y* (Fig. 17) are p_1 and p_2, then, at the close of the time t, the spaces will be $A\,M = x = \frac{p_1\,t^2}{2}$, and $A\,N = y = \frac{p_2\,t^2}{2}$, and their relation $\frac{x}{y} = \frac{p_1\,t^2}{p_2\,t^2} = \frac{p_1}{p_2}$ is in no way dependent upon the time; and, consequently, the trajectory *A O* of the compound motion is rectilinear. If *A B* is made $= p_1$ and $B\,D = A\,C = p_2$, we obtain a parallelogram *A B D C*, which is similar to the parallelogram *AMON*, and from which $\frac{A\,O}{A\,D} = \frac{A\,M}{A\,B} = \frac{\frac{1}{2} p_1 t^2}{p_1} = \frac{1}{2}.\,t^2$; therefore, $A\,O = \frac{1}{2}\,A\,D\,.\,t^2$. From this equation it appears that the trajectory *A O* of the compound motion is proportional to the square of the time; the motion itself, therefore, uniformly accelerated, and its acceleration is the diagonal *A D* of the parallelogram constructed by the simple accelerations p_1 and p_2.

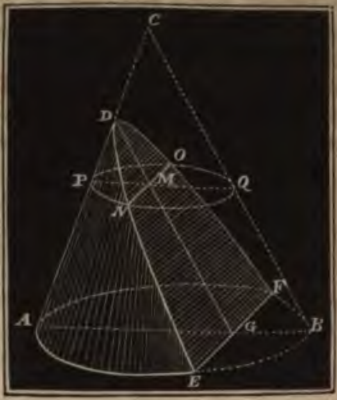
In the same manner, therefore, as velocities can be composed or resolved by the parallelogram of velocities, and, according to precisely the same rules, accelerations may be united into one, or broken up into several others by a parallelogram, which is called the *parallelogram of accelerations.*

Fig. 18.



§ 35. *Combination of Velocity and Acceleration.*—By the *combination* of a *uniform* with a *uniformly accelerated motion*, an entirely variable motion is produced when the directions of the motions do not coincide. In a certain time t, with the velocity c in the direction *A Y*, (Fig. 18,) the trajectory $A\,N = y = c\,t$ will be described, and in the same time with an unchangeable acceleration, and a direction *A X* at right angles to the former, the trajectory $A\,M = x = \frac{p\,t^2}{2}$ will be described, and the body arrives at the terminal point *O* of the parallelogram composed of $y = c\,t$ and $x = \frac{p\,t^2}{2}$. With the aid of these formulæ, the position of the body can be determined for any given time, but it is not always in one and the same straight line, for if from the first equation we take $t = \frac{y}{c}$, and place this value in the second, we obtain the equation $x = \frac{p\,y^2}{2\,c^2}$. In accordance with this, the trajectories (x) in the direction of the second motion do not correspond with those in the first, but with the squares (y^2) of those in the first; and, consequently, the trajectory of the body is not rectilinear, but is a certain curved line, known in geometry by the name of *parabola*.

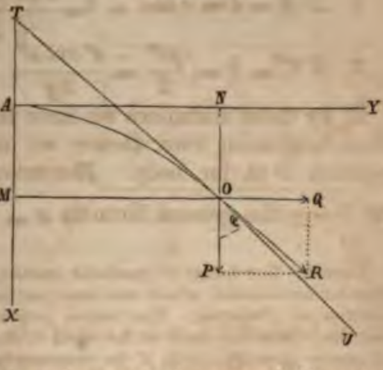
Fig. 19.



Remark. Let ABC, Fig. 19, be a cone with a circular base $AEBF$, let DEF be a section of it parallel to the side BC and at right angles to the section ABC, and let $OPNQ$ be a second section parallel to the base, and, consequently, also circular. Then let EF be the line of section between the base and the first section, and ON that between both sections; imagine, then, in the triangular section ABC, the parallel diameters AB and PQ, and in the section DEF, the axis DG. Then, for the half chord of the circle, $MN = MO$, the equation applies $MN^2 = PM \times MQ$; but $MQ = BG$ and for PM we have the proportion $PM : MD = AG : DG$; hence, it follows $MN^2 = BG \times \frac{DM \times AG}{DG}$. But, in like manner, $GE^2 = BG \times AG$; if one equation is divided by the other, we obtain, therefore, $\frac{DM}{DG} = \frac{MN^2}{GE^2}$; the portions cut off from the axis (*abscissæ*) bear, therefore, the same proportion to each other as the squares of the corresponding perpendiculars (*ordinates*). This law agrees completely with the law for motion found above; this motion, therefore, takes place in a curved line DNE, which can be shown to be a section of the cone (Conic Section).

§ 36. *Parabolic Motion.*—In order thoroughly to comprehend motion produced by the combination of velocity and acceleration, we must be able also to indicate the *direction*, *velocity* and *the space passed through* ~~during~~ at the end of any length of time (t). The velocity parallel to AY is invariable and $= c$, that which is parallel to AX is variable and $= pt$. If with these velocities, $OQ = c$ and $OP = pt$, the parallelogram $OPRQ$ is constructed, Fig. 20, we obtain in its diagonal OR the resulting mean, or that velocity with which the body at O follows the parabolic curve AOU. This velocity itself is $v = \sqrt{c^2 + pt^2}$.

Fig. 20.



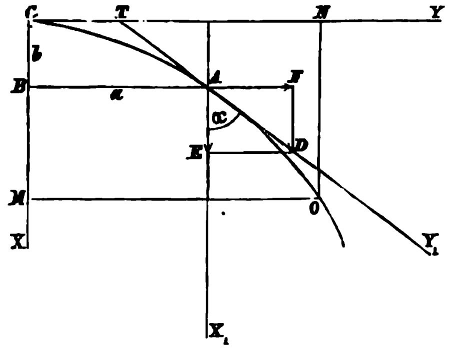
In like manner, OR is the tangent or direction in which the body at O proceeds for a single instant, and we obtain for the angle $POR = XTO = \phi$ which it makes with the second direction (axis) AX, the formula *tang.* $\phi = \frac{OQ}{OP} = \frac{c}{pt}$.

In order, lastly, to find the space passed through, or the curve AO

$= s$ we can apply the equation $\sigma = v\tau$ (§ 19); according to which we can calculate minute portions of it, which may be considered as its elements. The higher branches of geometry supply us with a complicated formula for calculating the parabolic curve.

§ 37. As yet we have assumed that the primary directions of motion formed a right angle with each other, and we must now study more closely that case in which the direction of the acceleration makes a certain angle with that of the velocity. If the body (Fig. 21) has in the direction $A\,Y_1$ the velocity c, and in the direction $A\,X_1$, which makes with the first the angle $X_1\,A\,Y_1 = \alpha$ the velocity p, then A is no longer the vertex, and $A\,X_1$ no longer the axis, but only the direction of the axis of the parabola. The vertex C is ~~much more~~ dependent upon the co-ordinates $A\,B = a$ and $B\,C = b$, the latter of which coincides with the axis, and the former is at right angles to it, beginning at the commencing point of the motion A. The velocity $A\,D = c$ is made up of the components $A\,F = c\ \sin.\ \alpha$ and $AE = c.\ \cos.\ \alpha$. The former of these remains always the same, but the latter must be made equal to the variable velocity $p\,t$, supposing that the body has required the time t to move from the vertex C to the real commencing point A.

Fig. 21.



We have, therefore, $c.\ \cos.\ \alpha = p\,t$, consequently $t = \frac{c\ \cos.\ \alpha}{p}$, and

1. $A\,B = a = c\ \sin.\ \alpha.\ t = \frac{c^2 \sin.\alpha \cos.\alpha}{p} = \frac{c^2 \sin.2\alpha}{2p}$, and

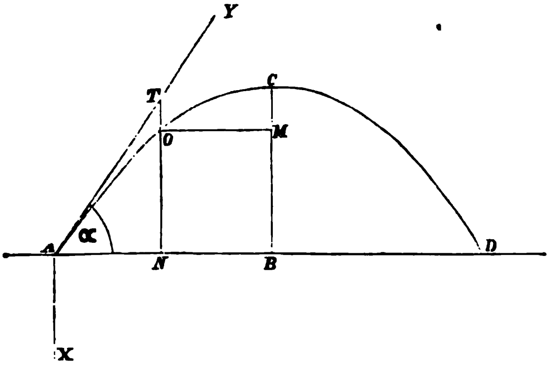
2. $B\,C = b = \frac{p t^2}{2} = \frac{c^2 \cos.\alpha^2}{2p}$.

If by these distances we have found the vertex of the parabola C, then, beginning from thence we can find for any required time the position O of the body. Moreover we have: making $C\,M = x$ and $M\,O = y$ the general formula $x = \frac{p y^2}{2\,c^2 \sin.\alpha^2}$, also $y = c\ \sin.\ \alpha \sqrt{\frac{2x}{p}}$.

Remarks. The theory of parabolic motion produced by an invariable velocity and a constant acceleration, which we have just been considering, finds its application in the doctrine of *Projectiles*. The bodies projected either upwards or downwards would describe a parabolic curve as the result of their primary velocity (c) and the acceleration of gravity ($g = 32.2$ feet), if the resistance of the air could be prevented, or the motion could take place in a vacuum. If the projectile velocity is not great, and the body is very heavy as compared with its volume, then the deviation from the parabola is small enough to be altogether neglected. The most perfect instance of the parabolic course is witnessed in columns of water flowing from vessels or from jets, &c. Bodies shot off, viz., bullets, describe curves which deviate considerably from the parabola in consequence of the great resistance of the air.

§ 38. A body projected at an angle of elevation $Y A D = \alpha$ (Fig. 22), rises to a certain height $B C$, which is called the *height of projection*, and it attains the horizontal plane, from which it departed at A, at a distance AD, which is called the *range of projection*. It follows, according to § 37, from the velocity c, the acceleration g, and the angle of elevation, that when p is replaced by g and α° by $90^\circ + \alpha^\circ$, therefore *cos.* α by *sin.* α:

Fig. 22.



the height of projection is $B C = b = \frac{c^2 \sin.\alpha^2}{2g}$, and,

the half of the range of projection $A B = a = \frac{c^2 \sin.2\alpha}{2g}$.

On the contrary, the height corresponding to any horizontal distance $A N = A B - N B = a - y$ becomes $N O = B M = C B - C M =$

$$b - x = b - \frac{g y^2}{2 c^2 \cos.\alpha^2} = \frac{c^2 \sin.\alpha^2}{2 g} - \frac{g y^2}{2 c^2 \cos.\alpha^2} = h \sin.\alpha^2 - \frac{y^2}{4 h \cos.\alpha^2},$$

when h represents the height due to velocity $\frac{c^2}{2g}$.

It is evident from the formula for the range of projection, that this will be greatest when $\sin.\ 2\alpha = 1$, therefore $2\alpha = 90^\circ$, *i e* $\alpha = 45^\circ$. A body ascending, therefore, at an angle of elevation of 45° attains the greatest range of projection.

Example 1. A jet of water ascending at an angle of elevation of 66° with a velocity of 20 feet, which has therefore a height due to velocity $h = 0{,}016.20^2 = 6{,}4$ feet, attains the height $b = h \sin.\ \alpha^2 = 6{,}4\ (\sin.\ 66^\circ)^2 = 5{,}34$ feet, and has a range of projection $a = 2.\ 6{,}4 \sin.\ 132^\circ = 2.6{,}4 \sin.\ 48^\circ = 9{,}51$ feet. The time which each particle of water requires to perform the whole parabolic curve $A\ C\ D$ is $= \frac{2\ c.\ \sin\ \alpha}{g} = \frac{2\ .\ 20\ .\ \sin. 66^\circ}{31{,}25} = 1.17$ seconds. The height corresponding to the horizontal distance $A N = 3$ feet, is, as $y = \frac{9{,}51}{2} - 3 = 1{,}755$, $NO = 5{,}34 - \frac{1{,}755^2}{4.6{,}4\ (\cos.\ 66^\circ)^2} = 5{,}34 - 0{,}73 = 4{,}61$ feet.

2. The jet of water flowing from a horizontal tube has at a height of 1¾ feet a range (half a range of projection) of 5¼ feet, what is the velocity of the water?

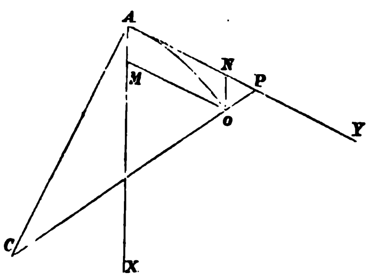
From the formula $x = \frac{g\,y^2}{2\,c^2} = \frac{y^2}{4\,h}$, it follows $h = \frac{y^2}{4x}$, if x in this case $= 1{,}75$ and $y = 5{,}25$, then $h = \frac{5{,}25^2}{4\,.\,1{,}75} = 3{,}937$ feet, and the velocity corresponding to this height is $c = 15{,}68$ feet.

§ 39. *Curved Motions in General.*—By the combination of several velocities and several invariable accelerations, a parabolic motion is likewise produced, for not only the velocities but the accelerations also may be united into a single one; the result is, therefore, the same as if there were only one velocity and one acceleration, *i. e.* only one uniform and one uniformly accelerated motion.

If the accelerations are variable, they can just as well be united into a mean as if they were constant, for it is admissible to consider them invariable within the limits of an infinitely small space of time (τ); and the corresponding motions, therefore, during that space of time, as uniformly accelerated. Of course the resulting acceleration is variable, as are its components themselves. If this resulting acceleration be combined with the given velocity, it is possible to deduce a small parabolic curve, according to which the motion is effected during the small portion of time. If again the velocity and mean acceleration are determined in the same manner for the next small portion of time, we are enabled to obtain a new curve belonging to another parabola; and if this be farther repeated, we at last obtain the whole course.

§ 40. Any minute portion of any curve may be considered as the arc of a circle. The circle to which this arc belongs is called the *circle of curvature*, the radius pertaining to it is the *radius of curvature*. The course of a moving body may, in the same manner, be composed of the arcs of circles, and thus a formula for its radius established.

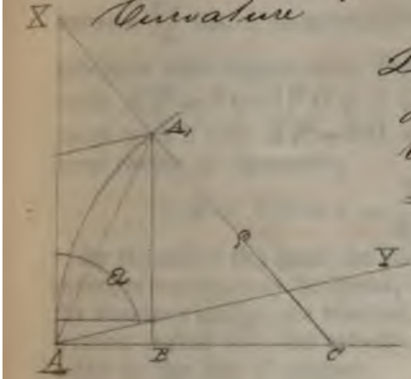
Fig. 23.



Let $A\,M$ (Fig. 23) be a very small trajectory described with a uniformly accelerated motion $x = \frac{p\tau^2}{2}$ in the direction $A\,X$, and let $A\,N$ be a very small uniformly described trajectory $y = c\tau$, and O the fourth terminating point of the parallelogram constructed from x and y, *i. e.* the point which a body proceeding from A would occupy at the end of the short time (τ). Let $A\,C$ be drawn at right angles to $A\,Y$, and let us observe from what point C in this line, a small arc of a circle through A and O can be drawn. On account of the smallness of the arc $A\,O$, we may assume that not only $C\,A$, but also $C\,O\,P$ is at right angles to $A\,Y$; that, therefore, in the small triangle $N\,O\,P$ the angle $N\,P\,O$

Cursed Motion in general

Formula for the Radius of Curvature



Let the velocity in the direction AX be equal to (c), that in the direction of ~~y~~ AY be equal to pdt. The space passed over in the direction AX = cdt in a space of time = dt. The component of pdt in a normal direction is $pdt^2 \sin a$

The space passed over in the direction AY = pdt² –

Let ρ = Radius of Curvature, Since for an infinitely small arc the tangent does not differ essentially from the arc nor from the chord, if the chord AA' be taken for the diagonal at the point A then the angle AA'C may be considered as a right angle and from the properties of the right angled triangle we have

AC : AA' :: AA' : AB, $\rho : cdt :: cdt : pdt^2 \sin \alpha$

$\rho = \frac{c^2dt^2}{pdt^2 \sin \alpha} = \frac{c^2}{p \sin \alpha}$ = Radius of Curvature

In the value of ρ in Art 41 the axes of X and Y are at right angles to each other whence $\rho = \frac{c^2}{p}$

is a right angle. The solution of this triangle gives us $OP = ON$ $\sin.\ ONP = AM \sin.\ XAY = \frac{p\tau^2}{2} \sin.\ \alpha$, and the tangent $AP =$ $AN + NP = c\tau + \frac{p\tau^2}{2} \cos.\ \alpha = (c + \frac{p\tau}{2} \cos.\ \alpha)\tau$, may be made $= c\tau$, because $\frac{p\tau}{2} \cos.\ \alpha$, on account of the infinitely small factor τ, is inappreciable with respect to c. But now according to the property of the circle $\overline{AP}^2 = PO \times (PO + 2\ CO)$, or, as PO vanishes when compared with $2\ CO$, $\overline{AP}^2 = PO \times 2\ CO$; we have, therefore, the desired *radius of curvature*,

$$CA = CO = r = \frac{\overline{AP}^2}{2PO} = \frac{c^2\tau^2}{p\tau^2 \sin.\ \alpha} = \frac{c^2}{p \sin.\ \alpha}.$$

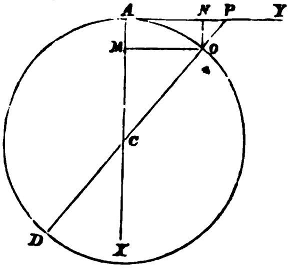
By the aid of the same formula, the radii of curvature of all the elements of curves may be found, when the respective velocities (c) and the acceleration (p) are inserted, and also the angle α which the acceleration makes with the velocity, or with the direction of motion indicated by the line of contact.

Example. For the parabolic path caused by the acceleration of gravity, we have $r = 0{,}031 \frac{c^2}{\sin.\ \alpha}$, and in the vertex of these curves, where $\alpha = 90°$, therefore, $\sin.\ \alpha = 1$, it results that $r = 0{,}031\, c^2$. With a velocity of 20 feet, it would therefore be found that $r = 12{,}4$ feet; the further, however, the body is removed from the vertex, so much the smaller α becomes, and so much the greater, therefore, the radius of curvature.

§ 41. Proceeding from a point A (Fig. 24), where the acceleration is effected at right angles to the direction of motion AY, if, therefore, $\alpha = 90°$, we obtain the radius of curvature $CA = r = \frac{c^2}{p}$, and the velocity at the following point O is composed of c and of $p\tau$, hence $v = \sqrt{c^2 + p^2\tau^2} = c + \frac{p^2\tau^2}{2c}$, because τ is infinitely small compared with c. If we make $v = c + \frac{p^2}{2c}\tau . \tau$, we may then consider $\frac{p^2\tau}{2c}$ as the acceleration, and $\frac{p^2\tau}{2c} . \tau$ as the corresponding increase of velocity. But as τ is infinitely small, the acceleration $\frac{p^2\tau}{2c}$ becomes also infinitely small, and in one unit ~~second~~ of time we have an infinitely small increase of velocity, and may therefore consider the motion uniform, and consequently make $v = c$.

If, with the direction of motion, the direction of acceleration also changes, and if these remain constantly at right angles to each other, then we shall always have $v = c$; the velocity of motion, therefore, remains invariably the same as it was at the commencement, namely $= c$. An acceleration such as this, which is always at right angles to the motion, or causes the body to deviate at right angles from the motional direction, is called *normal acceleration*, and we hence know that it alone never causes a change of velocity, but only a deviation from the straight direction. According to the formula above, $r = \frac{c^2}{p}$ we must make the normal acceleration $p = \frac{c^2}{r}$ = *the square of the velocity divided by the ~~respective~~ radius of curvature.*

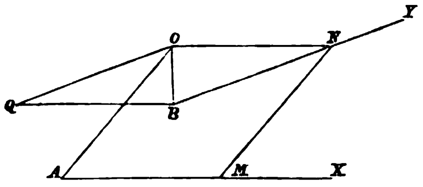
Fig. 24.



In the circle $A\,O\,D$ (Fig. 24) the radius of curvature (r) is the radius of the circle $C A = C O$ itself; hence, when motion occurs in it, the acceleration $p = \frac{c^2}{r}$ is invariable. An invariable acceleration, therefore, which constantly causes the body to deviate at right angles from its motional direction, obliges it to revolve in a circle.

Example. A body which rotates in a circle of 5 feet diameter, in such a manner, that, for each revolution, it requires 5 seconds of time, has a velocity $c = \frac{2\pi r}{t} = \frac{2\pi . 5}{5} = 2 . \pi = 6{,}283$ feet, and a normal acceleration $p = \frac{(6{,}283)^2}{5} = 7{,}896$ feet; viz., in every second it will deviate from a straight line by $\frac{1}{2}\,p = \frac{1}{2} \times 7{,}896 = 3{,}948$ feet.

Fig. 25.



§ 42. In the *simultaneous motions of two bodies*, a constant change is taking place in their relative position, distance, &c., but with the aid of the foregoing formulæ it may be found for any given moment of time.

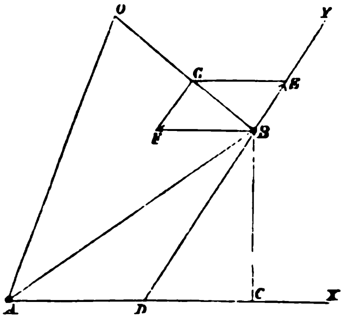
In Fig. 25, let A be the point of application of the one body, B that of the other; the first advances in the direction $A\,X$ in a certain time (t) to M, the second in the direction BY in the same time to N; we then have in this line the relative position and distance of the bodies A and B at the end of this time. If we draw AO parallel with MN, and also make $AO = MN$, then will the line AO likewise give the opposite position of the bodies A and B.

If further we draw ON, we obtain a parallelogram in which ON is also $= AM$. If finally we make BQ parallel and equal to NO, and draw OQ, we have then another parallelogram $BNOQ$, in which one side BN is the absolute path (y) of the second body, and the other side BQ the path (x) of the first body, described in the opposite direction. The fourth corner O is the relative position of the second body, in so far as it is referred to the position of the first body, which is considered as invariable.

The relative position O of a body (B) in motion is also found if we add to the body, besides its own proper motion (BN), that AM of the body (A) to which we refer its position BQ, but in an inverse direction, and then resolve these motions by the parallelogram *BNOQ* in the usual manner.

§ 43. If the motions of the bodies *A* and *B* are uniform, we may substitute for *AM* and *BN* the velocities c and c_1, *i. e.* the spaces described in one second. We obtain, therefore, the relative velocity of the one body, when we add to the same in an opposite direction, besides its own absolute velocity, that of the body to which the first velocity is referred. The same relation takes place with the accelerations.

Fig. 26.



Example. A locomotive train sets out upon the line *A X*, Fig. 26, from *A* with a velocity of 35 feet; another simultaneously from *B* upon the line *B Y*, which makes with the former the angle $B\,D\,X = 56°$ with a velocity of 20 feet. If now the initial distances $A\,C = 30{,}000$ feet, and $C\,B = 24{,}000$ feet, how great is the distance *A O* of the two trains at the end of a quarter of an hour? From the absolute velocity $B\,E = c_1 = 20$ feet of the second train, the inverse velocity $B\,F = c = 35$ feet of the first, and the included angle $E\,B\,F = \alpha = 180° - B\,D\,C = 180° - 56° = 124°$. The relative velocity of the second train $B\,G = \sqrt{c^2 + c_1^2 + 2\,cc_1 \cos. \alpha} = \sqrt{35^2 + 20^2 - 2.35.20. \cos. 56°} = \sqrt{1225 + 400 - 1400 \cos. 56°} = \sqrt{1625 - 782{,}9} = \sqrt{842{,}2} = 29{,}02$ feet. For the angle $G\,B\,F = \phi$, which this makes with the first direction of motion: $\sin. \phi = \frac{c_1 \sin. 56°}{29{,}02} = \frac{20.0{,}8290}{29{,}02}$; $Log. \sin. \phi = 0{,}75690 - 1$, hence $\phi = 34°, 50'$. Therefore in 15′ the relative space described is $B\,O = 29{,}02.900 = 26118$ feet, the distance $A\,B = \sqrt{(30000)^2 + (24000)^2} = 38419$ feet. The angle $B\,A\,C = A\,B\,F$, whose tangent $= \frac{24000}{30000} = 0{,}8 = 38°, 40'$, therefore the angle $A\,B\,O = 38°, 40' + \phi = 38°\,40' + 34°, 50' = 73°, 30'$, and the distance of the two trains after 15′:

$$
\begin{aligned}
A\,O &= \sqrt{\overline{A B}^2 + \overline{B O}^2 - 2\,A\,B\,.\,B\,O \cos. A\,B\,O} \\
&= \sqrt{38419^2 + 26118^2 - 2\,.\,38419\,.\,26118 \cos. 73°, 30'} \\
&= \sqrt{1588190000} = 39852 \text{ feet.}
\end{aligned}
$$

SECTION II.

MECHANICS, OR THE PHYSICAL SCIENCE OF MOTION IN GENERAL.

CHAPTER I.

FUNDAMENTAL PRINCIPLES OF MECHANICS.

§ 44. *Mechanics.*—Mechanics is the science which treats of the laws of the motion of material bodies. It is an application of phoronomics to the bodies of the external world, in so far as the latter is concerned with the motion only of geometrical bodies.

Mechanics is a part of natural philosophy, or of the doctrine of laws according to which changes take place in the material world, viz., that part which considers the changes in bodies resulting from measureable motions.

§ 45. *Force.*—Force is the cause of motion or change of motion in material bodies. Every change of motion, viz., every change in the velocity of a body must be regarded as the effect of a force. For this reason we measure the force called gravity by a body falling freely, because the same incessantly changes its velocity. On the other hand, rest, or the invariability of the state of motion of a body, must not be attributed to the absence of forces, for opposite forces destroy each other and produce no effect. The gravity with which a body falls to the ground still acts, though the body rest upon a table; but this action is counteracted by the solidity of the table or of the support.

§ 46. A body is in equilibrium, or the forces acting upon a body are in equilibrium, when there is no residuary effect, no motion produced or changed, or when each neutralizes the other. In a body suspended by a thread, the strength of the thread is in equilibrium with gravity. In forces, equilibrium is destroyed, and motion arises if one of the forces be removed, or in any way counteracted; for instance, a steel spring, bent by a weight, enters into motion when the weight is taken away, because the force of the spring, called elasticity, then comes into action.

Statics is that part of mechanics which treats of the equilibrium of forces. Dynamics, on the other hand, treats of forces in so far as they produce motion.

§ 47. *Division of Forces.*—According to their effects, forces are either *moving forces* or *resistances;* that is, as motion is brought about or impeded. Gravity, the elasticity of a steel spring, &c., belong to motive forces. Friction, the solidity of bodies, &c., are resisting forces or resistances, because by them motion is either diminished or destroyed, and can by no means be brought about. Moving forces are divided into accelerating and retarding; the first produces a positive, the second a negative acceleration; by the one an accelerating, by the other a retarding motion is produced. Resistances are retarding forces, but a retarding force is not always a resistance. Gravity, for example, acts upon a body projected vertically upwards to retard it; but gravity, on this account, is no resisting force; for, by the consequent falling down of the body, it then again becomes a motive one.

There is a distinction between constant and variable forces. While constant forces always act in the same way, and, therefore, produce like effects in like particles of time, *i. e.* equal increments or decrements of velocity, the effects of variable forces are different at different times; while the former bring about a uniformly variable motion, to the latter corresponds a variably accelerated or a variably retarded one.

§ 48. *Pressure.*—*Pressure* and *traction* are the first effects of forces upon material bodies. By means of them, bodies are compressed and extended, and especially changed in their form. The pressure in traction brought about by gravity, acting vertically downwards, which the support of a heavy body, or the string to which a body is attached has to sustain, is called the weight of the body.

Pressure and traction, and weight also, are magnitudes of a particular kind, which can only virtually be compared with each other, as the action of forces serves for their measurement. The simplest, and on that account the most general, means of measuring forces is by weights.

§ 49. *Equality of Forces.*—Two weights, or two pressures, or tractions, and also the forces which correspond to these last, are equal, when one may be replaced by the other, without producing different effects. If, for example, a steel spring be bent by a weight G, as by another G_1, then are these weights, and therefore the gravities in both bodies, equal. If a loaded balance be made to vibrate as much by a weight G as by another G_1, substituted for G, these two weights G, G_1 are equal; in this case, the arms of the balance may be equal or unequal, and the remaining load great or small.

A pressure or weight (force) is 2, 3, 4, &c., times as great as another pressure, &c., if it produces the same effect as 2, 3, 4 . . . n pressures together of the second kind. If a balance, otherwise loaded at will, is brought into the same vibration by a weight (G) as by the addition of 2, 3, 4, equal weights (G_1), the weight (G) is 2, 3, 4, &c., times as great as the weight (G_1).

§ 50. *Matter.*—Matter is that by means of which bodies belonging to the external world, which in contradistinction to geometical bodies we term material or physical, act upon our senses. Mass is the quantity of matter composing a body.

Bodies of equal volume, or equal geometrical contents, have generally different weights when they consist of different kinds of matter. We cannot, therefore, infer the weight of a body from its volume until we first know the weight of a unit of volume, for instance, a cubic foot or cubic centimetre of the matter of the body.

§ 51. *Unit of Weight.*—The measurement of weights and forces consists in a comparison of them with some given invariable weight, taken as unity. The choice of this unit of weight or force is perfectly arbitrary; it is nevertheless advantageous in practice, that the weight of a volume of some universally diffused body, equivalent to that of the unit, should be chosen.

The units of weight or pressure are different in different countries. In England, the unit of pressure from which all the rest are derived is the weight of 22,185 cubic inches of distilled water (at a temp. 62° Fahr. taken in air, and the height of barometer at 30 inches). This weight is equal to 5760 grains; which again is equal to one pound troy, and 7000 such grains constitute the pound avoirdupois. The gramme is the weight of a cubic centimetre of pure water in a state of maximum density (at a temperature of 4° C.). The Prussian pound is also a unit referred to a weight of water. A Prussian cubic foot of distilled water in vacuo, and at a temperature 15° R. weighs 66 Prussian pounds. Now a Prussian foot = 139,13 Paris lines = 0,3137946 metres = 1,029722 English feet: hence it follows that a Prussian pound = 467,711 grammes = 1,031114 pounds English.*

§ 52. *Inertia.*—Inertia is that property of matter, in consequence of which it can of itself alone neither acquire nor change motion. Every material body remains at rest so long as no force acts upon it, and every material body once set into motion maintains a *uniform* unaccelerated *rectilinear motion*, so long as it is not subjected to the action of a force. Hence, when a change takes place in the condition of motion of a body, when it changes its direction of motion, or when it acquires a greater or less velocity, this is not to be attributed to the body as a certain quantum of matter, but to the agency of some foreign cause or force. In as much as a development of force takes place at every change in the motion of a material body, in so far inertia may be ranked amongst forces.

If we could entirely remove the forces acting upon a mass in motion, it would move on uniformly without ceasing, but we find nowhere such a uniform motion, because it is not possible for us to withdraw a mass from the action of every force. When a body moves upon an horizontal table, gravity, which is then counteracted by the table, exerts upon the body no immediate action, except that from the pressure of the body against the table there arises a resistance, which we shall consider more closely in the sequel under the name of friction, which incessantly abstracts velocity from the moving body, imparts to it a retarded motion, and brings it finally to rest.

---

* In the United States, the standard weight is the pound troy, the original of which is the mint pound, constructed by Capt. Kater at the request of Mr. Gallatin.—Am. Ed.

The air likewise opposes resistance to a moving body, and from this resistance, if the friction of the body were entirely put aside, a gradual diminution of velocity would ensue. But we find that the loss of velocity becomes the less, and that the motion also approximates more and more to a uniform one, the more we diminish the number and strength of these resistances; and hence we may conclude, that, by the removal of all moving forces and resistances, an entirely uniform motion must take place.

§ 53. *Measure of Forces.*—The force (P) which accelerates an inert mass (M) is proportional to the acceleration (p), and to the mass itself (M): it increases in equal masses as the increment of velocity in infinitely small times, and increases by equal increments of velocity in the same ratio as the masses become greater. The *m*tuple acceleration of one and the same mass, or of equal masses requires an *m*tuple force, and an *n*tuple mass for the same acceleration, an *n*tuple force. ?

As we have not yet chosen a measure of the mass, we may, therefore, at once, put $P = Mp$, *i. e.* the force equal to the product of the mass and the acceleration, and, at the same time, in place of the ~~power~~ force, its effect, *i. e.* the pressure produced by it.

The correctness of this general law of motion may be readily proved by direct experiment: for example, by letting equal and differently movable masses be impelled upon an horizontal table by means of bent springs; and, it is obvious, from this, too, that all the consequences deduced, and all the laws developed from them for compound motions, fully correspond with observation and the phenomena of nature.

§ 54. *Mass.*—All bodies fall at one and the same place of the earth, and in vacuo equally fast, viz., with an invariable acceleration $g = 9{,}81$ metres $= 32{,}2$ feet (§ 15); if, therefore, the mass of a body $= M$, and the weight measuring its gravity $= G$, we have from the last formula $G = Mg$, *i. e.*
the weight of a body is a product of its mass and the acceleration of gravity, and inversely: $M = \frac{G}{g}$, *i. e.*
the mass of a body is its weight divided by the acceleration of gravity, or the mass is that weight which a body would otherwise have if the acceleration of gravity were = to unity, as a metre, a foot, &c. At a point upon, or in the vicinity of the earth, or of any other heavenly body, where bodies do not fall with 9,81 metres = 32,2 feet, but with a velocity (after the first second) of one metre = 3¼ ft., the mass, or rather its measure, is from hence immediately given by the weight of the body.

According as we express the acceleration of gravity in metres or in feet, we have, therefore, the mass

$$M=\frac{G}{9,81}=0,1019\ G, \text{ or}$$

$$M=\frac{G}{32,2}=0,031\ G.$$

The mass of a 20 lb. heavy body, $M=0,031\times 20=0,62$ lb., and inversely the weight of a mass of 20 lbs. $G=32,2\times 20=644$ lbs.

§ 55. In so far as we assume the acceleration (g) of gravity as invariable, it follows that the mass of a body is exactly proportional to its weight, and that also for the masses M and M_1, with the weights G and G_1:

$$\frac{M}{M_1}=\frac{G}{G_1}.$$

We hence obtain the weight as a measure of the mass of a body; the greater the mass which a body measures, the greater is its weight.

The acceleration of gravity is, in fact, somewhat variable, it becomes greater the nearer we approach the poles of the earth, and diminishes the more we advance towards the earth's equator; it is greatest at the poles, and least at the equator. It also diminishes the more a body is above or below the level of the sea; and attains its greatest value at the level of the sea. But, since a mass, so long as nothing is added to, or taken from it, is invariable, so that at all points of the earth, as well as those beyond it, at the moon, for instance, it is still the same; it hence follows that the weights also of bodies are variable and dependent upon the place of the bodies, and must be altogether proportional to the acceleration of gravity, corresponding with the place, or

$$\frac{G}{G_1}=\frac{g}{g_1}.$$

One and the same steel spring is differently bent by one and the same weight at different places of the earth; it is least at the equator, on high mountains, and in deep mines; greatest in the vicinity of the poles, and at the level of the sea.

§ 56. *Density* is the intensity with which space is filled by matter. A body is so much the denser the more matter there is in its space. The natural measure of density is that quantity of matter (that mass) which fills a unit of volume, because matter can only be measured by weight, so that the weight of a unit of volume, a cubic metre, or cubic foot of some matter, serves as a measure of its density.

For example: the density of a cubic foot of water = 62,38 lb., and that of cast iron = 452,13 lb., because a cubic foot of water weighs 62,38 lb. = 998,08 oz. avd., and a cubic foot of cast iron weighs 452,13 lb.

From the volume V of a body and its density γ, its weight $G = V\gamma$. The volume multiplied by the density gives the weight of a body.

The density of bodies is either uniform or variable, according as equal volumes of the same body are of equal or of unequal weight. The density of metals, for instance, is uniform, or they are homogeneous, because equal and very small parts of them are of the same weight: on the other hand, granite is a body of variable density, because made up of parts of different densities.

Example.—1. If the density of lead be 708 lbs., 3,2 cubic feet of lead weigh $= 708 \times 3,2 = 2265$ lbs.—2. If the density of bar iron $= 485,8$ lbs.; a mass of it of 205 lbs. has a volume $V = \frac{G}{\gamma} = \frac{205}{485,8} = 0,4219$ cubic ft. $= 0.4219 \times 1728 = 729,04$ cubic inches.—3. 10,4 cubic feet of deal, perfectly saturated with water, weigh 577 lbs.; the density of this wood is therefore: $\gamma = \frac{G}{V} = \frac{577}{10,4} = 55,5$ lbs.

§ 57. *Specific Gravity.*—Specific gravity or specific weight is the relation of the density of a body to that of the density of some other, generally water, taken for unity. Now the density is equal to the weight of a unit of volume: hence the specific gravity is also the relation of the weight of one body to that of another, viz. water, under the same volume.

In order not to confound the specific weight with that which belongs to a body of a certain magnitude, the last is usually called the absolute weight.

If γ be the density of matter (of water) to which we refer the density of other matter, and γ_1 the density of any one kind of matter, whose specific gravity we will designate by ε, then the formula

$$\varepsilon = \frac{\gamma_1}{\gamma} \text{ and } \gamma_1 = \varepsilon \cdot \gamma.$$

holds good, and the density of a substance is equal to its specific gravity into the density of water.

The absolute weight G of a mass of volume V and specific gravity ε is: $G = V\gamma_1 = V\varepsilon\gamma$.

Example.—1. The density of pure silver is 653,368 lbs. and that of water $= 62,38$ lbs., consequently the specific gravity of the former $= \frac{653,368}{62,38} = 10,474$; i. e. each mass of silver is $10\frac{1}{2}$ times as heavy as a mass of water filling the same space.—2. The specific gravity of quicksilver $= 13,598$; its density, therefore, is $= 13,598 \times 62,38 = 848,24$ lbs.; a mass of 35 cubic inches, therefore, weighs:

$$G = 848,24 \,.\, V = \frac{848 \times 35}{1728} = 17,18 \text{ lbs.}$$

Remark. In these calculations the use of the French measure and weight has this advantage, that in order to effect the multiplication of ε and γ, it is merely requisite to advance the decimal point; because a cubic centimetre of water weighs one gramme, and a cubic metre a million, or one thousand kilogrammes. The density of quicksilver, according to the French measure and weight $= 13,598 \times 1000 = 13598$ kilog.; i. e. a cubic metre of quicksilver weighs 13598 kilogrammes.

§ 58. The following table contains the specific gravities of certain bodies constantly coming into application in mechanics:

Mean specific gravity of dry laurel wood . .	= 0,659
" saturated with water . .	= 1,110
Mean specific gravity of dry pine wood . .	= 0,453
" saturated with water . .	= 0,839*
Quicksilver	= 13,598
Lead	= 11,33

---

* See "On the Absorption of Water by Wood."—*Polytechnisches Mittheilungen*, Part iv. 1844.

Copper, cast and compact		= 8,75
“ forged		= 8,97
Brass		= 8,55
Iron, cast, white		= 7,50
“ “ gray		= 7,10
“ “ medium		= 7,06
“ bar iron		= 7,60*
Zinc, fused		= 7,05
“ rolled		= 7,54
Granite		2,50 to 3,05
Gneiss		2,39 to 2,71
Limestone		2,40 to 2,86
Sandstone		1,90 to 2,70
Brick		1,40 to 2,22
Masonry, with lime mortar of quarry stone	fresh	= 2,46
	dry	= 2,40
“ “ “ of sandstone	fresh	= 2,12
	dry	= 2,05
“ “ “ of brick	fresh	= 1,55 to 1,70
	dry	= 1,47 to 1,59
Earth, loamy, hard stamped	fresh	= 2,06
	dry	= 1,93
Garden earth	fresh	= 2,05
	dry	= 1,63
Dry, poor earth		= 1,34

§ 59. *State of Aggregation.*—Bodies appear to us, according to the different cohesion of their parts, under three principal conditions, which we term states of aggregation. They are either *solid* or *fluid*, and in the latter case, either liquid or gaseous. Solid bodies are those whose parts adhere so strongly together that a certain force is required to change the form of these bodies, or to effect their division. Fluid bodies, on the other hand, are those whose parts may be displaced about each other by the smallest force. Elastic fluid bodies, whose representant is atmospheric air, are distinguished from the liquid represented by water, in as much as there is inherent in them an endeavor to dilate themselves more and more, which is not the case with water, &c.

While solid bodies have a proper form and determinate volume, liquid or aqueous bodies possess only a determinate volume without any proper form, and the elastic extensible fluid bodies have neither one nor the other.

§ 60. *Division of Forces.*—Forces are different according to their nature; we will here mention the principal:

1. *Gravity*, by means of which all bodies tend to approach towards the centre of the earth.

---

* Rolled *boiler plate* iron has a sp. gr. from 7,6013 to 7,7922, or a mean of 7,7344, the amount of variation being $\frac{1}{40.5}$th part of the mean density. By seventeen trials of *hammered bar* iron, its mean sp. gr. was found to be 7,7254. *See* “*Report on Strength of Materials for Steam Boilers*,” p. 232. Also *Journal Franklin Inst.*, 1837.—Am. Ed.

2. The *force of inertia*, which manifests itself when changes in the velocity of inert masses occur.

3. The *muscular force* of animated beings; the force exerted by the muscles of men and animals.

4. *Elasticity* or *spring-force*, which bodies exhibit in a change of their form or volume.

5. The *force of heat* or *caloric*, in consequence of which bodies expand or contract by a change of temperature.

6. The *magnetic force*, or the attraction and repulsion of magnets.

7. The *cohesive force*, the force by which the parts of a body are kept together, and resist separation.

8. *Adhesion*, the force with which bodies brought into close contact attract each other.

The resistances of friction, rigidity, solidity, &c., arise mainly from the force of adhesion.

§ 61. *In reference to forces we have to distinguish:*

1. Its *point of application*, that point of a body on which the force immediately acts.

2. Its *direction*, the straight line in which a force moves forward its point of application, or strives to move it forward, or to impede its motion. The *direction of a force*, like every direction of motion, has *two senses*, it can take place from left to right, or from right to left, from above to below, and from below to above. The one is termed *positive*, *the other negative*. As we write from left to right, and from above to below, it would be most convenient were we to call these motions positive, and those in the opposite direction, negative.

3. The *absolute magnitude* or *intensity* of a force, which, as above stated, is measured by weights, as pounds, kilogrammes, &c.

§ 62. *Action and re-action.*—The first effect which a force produces in a body, is a change of form or volume combined with extension or contraction, which begins at the point of application, and from thence diffuses itself further and further. By this inward change of the body, its inherent elasticity is called into action, puts itself into equilibrium with the force, and, therefore, is equal and opposed to the force. Action and re-action are equal and opposed to each other. This law not only prevails in reference to forces produced by contact, but also in the so-called forces of attraction and repulsion amongst which the magnetic force and gravity itself may be ranked. The more strongly a magnet attracts a bar of iron, the more strongly is the magnet itself attracted by the iron. The force with which the moon is attracted towards the earth (gravitation) is equal to that with which the moon reacts upon the earth. The force with which a weight presses upon its support is given back in an opposite direction; the force with which a workman draws or pushes at a machine, &c., reacts upon the workman and strives to move him in the opposite direction. When a body impinges against another, the pressures are reciprocally equal on each of the bodies.

§ 63. *Division of Mechanics.*—The whole subject of mechanics may be included under two principal divisions, according to the state of aggregation of bodies.

1. The *mechanics of solid bodies*, which is also well named *geomechanics.*

2. The *mechanics of fluid bodies*, hydromechanics or hydraulics; the last is subdivided into:

1. Into the mechanics of water and liquid bodies especially, *hydromechanics* or *hydraulics.*

2. Into the *mechanics* of air, and *other aëriform* bodies, especially, aëromechanics, the mechanics of elastic fluids.

If we now have regard to the division of mechanics into statics and dynamics, we have the following parts:

1. *Statics of solid bodies*, or *geostatics.*
2. *Dynamics of solid bodies*, or *geodynamics.*
3. *Statics of fluids*, or *hydrostatics.*
4. *Dynamics of fluids*, or *hydrodynamics.*
5. *Statics of aëriform bodies*, or *aërostatics.*
6. *Dynamics of aëriform*, *aërodynamics*, or *pneumatics.*

CHAPTER II.

THE MECHANICS OF A MATERIAL POINT.

§ 64. A material point is a material body, whose dimensions are indefinitely small in comparison with the space occupied by it. In order to simplify the representation, we will in the following consider only the motion and equilibrium of a material point. A finite body is a continuous union of an infinite number of material points. If the single points or elements are all perfectly equal, *i. e.* move equally quick, in parallel straight lines, we may then apply the theory of the motion of a material point to that of the whole body, because, in this case, we may assume that equal parts of the mass of the body are impelled by equal parts of the force.

§ 65. *Simple constant Force.*—If (p) be the acceleration with which a mass (M) is impelled by a force, we have, from § 53, the force:

$P = Mp$, and inversely, the acceleration, $p = \frac{P}{M}$.

If, further, we put the mass $M = \frac{G}{g}$, where G is the weight of the body, and g the acceleration of gravity, we have the force:

1. $P = \frac{p}{g} G$, and the acceleration:

2. $p = \frac{P}{G} g$.

We find, therefore, the force (P) which impels a body with a certain acceleration (p) when we multiply the weight of the body (G) by the ratio $\left(\frac{p}{g}\right)$ of its acceleration, to that of gravity.

Inversely, the acceleration (p), with which a body is moved forward by a force (P) is given, when the acceleration (g) of gravity is multiplied by the ratio $\left(\frac{P}{G}\right)$ of the force and weight of the body.

Example. Let us suppose a body lying on an horizontal and perfectly smooth table, which presents no impediment to the body in its course, but counteracts the effect of gravity upon it. If this body be pressed upon by a force acting horizontally, the body will give way to this influence, and move forward in the direction of this force. If the weight of this body be $G = 50$ lbs., and if $P = 10$ lbs. presses uninterruptedly upon it, it will enter into a uniformly accelerated motion with the acceleration $p = \frac{P}{G} \cdot g = \frac{10}{50} \times 32,2 = 6,44$ feet. On the other hand, if the acceleration with which a 42 lb. heavy body becomes accelerated by a force (P) = 9 feet, then will this force $P = \frac{p}{g} \cdot G = \frac{9}{32,25} \times 42 = 0,031 \times 378 = 11,7$ lbs.

§ 66. If the force which acts upon a body is constant, there arises a uniformly variable motion, and indeed a uniformly accelerated one, if the direction of the force corresponds with the initial direction of the motion; and, on the other hand, a uniformly retarded one, if the direction of the force is opposite to that of the initial direction of motion. If we substitute in the formulæ (§ 13 and § 14) for p, the value $\frac{P}{M} = \frac{P}{G} g$, we obtain the following:

I. For uniformly accelerated motions:

$$1.\quad v = c + \frac{P}{G} gt, \text{ or } v = c + 32,2\,\frac{P}{G} t.$$

$$2.\quad s = ct + \frac{P}{G}\frac{gt^2}{2}, \text{ or } s = ct + 16,1\,\frac{P}{G} t^2.$$

II. For uniformly retarded motions:

$$1.\quad v = c - \frac{P}{G} gt = c - 32,2\,\frac{P}{G} t.$$

$$2.\quad s = ct - \frac{P}{G}\frac{gt^2}{2} = ct - 16,1\,\frac{P}{G} t^2.$$

With the help of these formulæ all those questions may be answered which can be proposed relative to the rectilinear motions of bodies by a constant force.

Example.—1. A carriage weighing 2000 lbs. goes with a 4 feet velocity upon a horizontal line, offering no impediments to it, and pushed forward by an invariable force of 25 lbs. during 15 seconds, with what velocity will it proceed after the action of this force? This velocity $v = c + 32,2\,\frac{P}{G}\,t$, but $c = 4$, $P = 25$ lbs., $G = 2000$, and $t = 15$; hence it follows, $v = 4 + 32,2 \cdot \frac{25}{2000} \cdot 15 = 10,03$ feet.—2. Under similar circumstances a carriage, weighing 5500 lbs., which, setting out with a uniform velocity, has traversed 950 feet in 3 minutes, is so impelled forward by a force acting continuously for 30 seconds, that it afterwards passes over 1650 feet in 3 minutes; what is this force? Here the initial velocity $c = \frac{950}{3,60} = 5,277$ feet per second, and the terminal velocity, $v = \frac{1650}{3,60} = 9,166$ feet; therefore $\frac{P}{G}\, gt = v - c = 3,889$, and the force $P = \frac{3,889 \,.\, G}{gt} = 0,031 \times 3,889 \times \frac{5500}{30} = 0,12056 \times \frac{550}{3} = 22,10$ lbs.—3. A sledge, weighing 1500 lbs., sliding forward with a 15 ft. velocity, loses, through friction, upon its horizontal support, its whole motion in 25 seconds; how great is this friction? Here the motion is uniformly retarded, and the terminal velocity $v = 0$; hence $c = 32,2\, \frac{Pt}{G}$, and $P = 0,031\, \frac{Gc}{t} = 0,031 \times \frac{1500 \times 15}{25} = 0,031 \times 900 = 27,9$ lbs. the friction demanded.—4. Another sledge, of 1200 lbs. and 12 feet initial velocity, has to overcome by its motion a friction of 45 lbs.; what velocity has it after 8 seconds, and how great is the distance described? The terminal velocity is $v = 12 - 32,2 \times \frac{45 \times 8}{1200} = 12 - 9,66 = 2.34$ feet, and the distance described $s = \left(\frac{c+v}{2}\right) t = \left(\frac{12 + 2.34}{2}\right) \times 8 = 57.36$ feet.

§ 67. *Mechanical Effect.*—The *work done*, or mechanical effect, is that effect of a force which it produces in overcoming a *resistance:* as that of inertia, friction, gravity, &c. Work is performed when loads are lifted, a great velocity imparted to masses, bodies changed in their form or divided, &c. The work done, or the mechanical effect produced depends not only on the force, but also on the distance through which it is made to act or to overcome the resistance; it increases, of course, simultaneously with the force and the distance. If we lift a body slowly enough to allow of our neglecting its inertia, the labor expended is then proportional to its weight; for 1, the effect is the same whether m (3) times the weight ($m\,G$) is lifted to a certain height, or whether m (3) bodies of the single weight (G) are lifted to the same height; it is, namely, m times as great as the effort necessary for the lifting of a single weight to that height; and, again, 2, the work is the same, whether one and the same weight be raised to n (5) times the height ($n\,h$), or n (5) times through the height, and it is of course n (5) times as great as if the same weight were raised to a single height (h). The work again done by a slowly falling weight is proportional to the magnitude of this weight and the height from which it has descended. This proportionality also holds in every other kind of work done. In order to make a saw-cut of a given depth of double the length, there are twice as many particles to separate as from a cut of a single length; the work, therefore, is twice as great. The double length requires double the distance to be described by the force, consequently the work is proportional to the distance. In like manner the work of a pair of mill stones increases with the quantity of grains of a certain kind of corn, which they grind to a certain degree. This quantity, under otherwise similar circumstances, is proportional to the number of revolutions, or rather to the distance which the upper mill-stone, during the grinding of this quantity of corn, has gone through; consequently the work increases in proportion to the distance.

§ 68. The dependence above shown of the work produced by a force upon the magnitude of the force and distance described by it, allows us to take that amount of work which is expended in overcoming a resistance of the magnitude of the unit of weight (as a kilogramme, pound, &c.), along a path of the magnitude of the unit of length (metre or foot,) as a unit of the mechanical effect, or the dynamical unit, and then we may put the measure of this equal to the product of the force or resistance, and the distance described in the direction of the force whilst overcoming this resistance.

If we put the amount of the resistance itself $= P$, and the distance described by the force, or rather by its point of application, in overcoming this $= s$, the labor expended is:

$$L = P\,s \text{ units of work.}$$

In order to define more clearly the unit of work, for which the single name, *dynam*, may be used, both factors P and s are generally given; and, therefore, instead of units of work, we say kilogram-metres, pounds-feet; and inversely, metrekilo. and feet-pounds according as the weight and distance are expressed in kilogrammes and metres, or in pounds and feet. These terms are usually expressed for simplicity by the abbreviations *mk*, or *km*, *lb. ft.*, or *ft. lb.*

Example.—1. In order to raise a stamper 210 lbs. 15 inches high, the mechanical effect $L = 210 \times \frac{15}{12} = 262{,}5$ ft. lbs. is necessary.—2. By a mechanical effect of 1500 ft. lbs., a sledge, which in its motion has to overcome a friction of 75 lbs., is driven forward a space $s = \frac{L}{P} = \frac{1500}{75} = 20$ feet.

§ 69. Not only in an invariable force or constant resistance is the labor a product of the force and distance, but also the labor may be expressed as a product of the distance and force, when the resistance whilst being overcome is variable, if a mean value of the continuous succession of forces be taken as *the force*. The relation is here the same as that of the time, the velocity, and the space; for the last may be regarded as a product of the time by the mean value of the velocities. The same graphical representations are here also applicable. The mechanical effect produced or expended may be considered as the area of a rectangular figure, *ABCD*, Fig. 27, whose base *AB* is the space described (s), and whose height is either the invariable force (P) itself, or the mean of the different values of the forces. In general, the work may be represented by the area of a figure *ABCD*, Fig. 28, which has for its base the space (s), and whose height above each point of the base is equal to the force corresponding

Fig. 27.

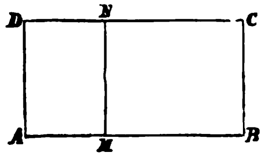
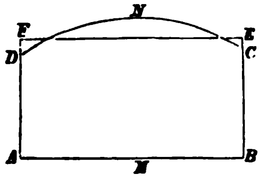


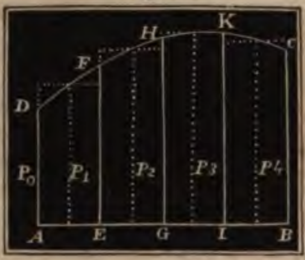
Fig. 28.



with each point of the path described. If the figure $ABCD$ be transformed into a rectangular one $ABEF$ of like area, we have the height $AF = BE$ for the mean value of the force—*the mean effort*.

§ 70. Arithmetic and geometry give different methods for finding a mean value from a constant succession of magnitudes. Amongst these, Simpson's rule is that which is the most frequently applied in practice, and it combines a high degree of accuracy with great simplicity.

Fig. 29.



In every case it is necessary to divide the space $AB = s$ (Fig. 29) into n (the more the better) equal parts, as $AE = EG = GI$, &c., and to measure the forces $EF = P_1$, $GH = P_2$, $IK = P_3$, &c., at the ends of these parts of the distance. If, then, we put the initial force $AD = P_0$ and the force at the other end $BC = P_n$, we have for the mean force:

$$P = (\tfrac{1}{2} P_0 + P_1 + P_2 + P_3 + \ldots\ldots + P_{n-1} + \tfrac{1}{2} P_n) \div n,$$

and, therefore, its work is:

$$Ps = (\tfrac{1}{2} P_0 + P_1 + P_2 + \ldots + P_{n-1} + \tfrac{1}{2} P_n) \frac{s}{n}.$$

If the number of parts (n) be even, viz., 2, 4, 6, 8, &c., Simpson's rule gives still more accurately the mean force:

$$P = (P_0 + 4\,P_1 + 2\,P_2 + 4\,P_3 + \ldots\ldots + 4\,P_{n-1} + P_n) \div 3\,n,$$

and, therefore, the corresponding work:

$$Ps = (P_0 + 4\,P_1 + 2\,P_2 + 4\,P_3 + \ldots\ldots + 4\,P_{n-1} + P_n) \frac{s}{3n}.$$

Example. In order to find the mechanical work which a draught horse performs in drawing a carriage over a certain way, we make use of a dynamometer, or measurer of force, which is put into communication on one side with the carriage, and on the other with the traces of the horse, and the force is observed from time to time. If the initial force $P_0 = 110$ lbs., the force, after describing 25 feet $= 122$ lbs.; after 50 feet $= 127$ lbs.; after 75 feet $= 120$ lbs., and at the end of the whole distance of 100 feet $= 114$ lbs.; then the mean value, according to the first formula: $P = (\tfrac{1}{2} \,.\, 110 + 122 + 127 + 120 + \tfrac{1}{2} \,.\, 114) \div 4 = 120{,}25$ lbs., and the mechanical work: $P\,s = 120{,}25 \times 100 = 12025$ ft. lbs.

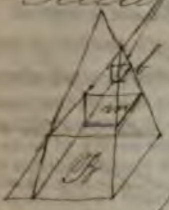
from the second formula: $P = (110 + 4 \,.\, 122 + 2 \,.\, 127 + 4 \,.\, 120 + 114) \div 3 \times 4$
$= \frac{1446}{12} = 120{,}5$ lbs., and the mechanical work
$P\,s = 120{,}5 \times 100 = 12050$ ft. lbs.

§ 71. *Principle of the Vis Viva, or Living Forces.*—If, in the formula of (§ 13) $s = \frac{v^2 - c^2}{2p}$ or $ps = \frac{v^2 - c^2}{2}$ we substitute for the acceleration p, its value $\frac{P}{G} g$, we thus obtain $Ps = \left(\frac{v^2 - c^2}{2g}\right) G$, or if we designate the heights due to the velocities $\frac{v^2}{2g}$ and $\frac{c^2}{2g}$ by h and h_1:

$$P\,s = (h - h_1)\,G.$$

If we interpret this equation, so useful in practical mechanics, we

Prismoidal Formula and its application to Simpson's Rule



~~Area~~ Volume of the figure = $(b + 4m + B)\frac{h}{3}$

$[f(x-h) + 4f(x) + f(x+h)]\frac{h}{3} = \int_{h-x}^{h+x} f(x)\,dx = \int f(x+h)\,dx$

$\int f(x-h)\,dx,\ f x - D_x f(x) h + \frac{D_x^2 f(x) h^2}{2} - \frac{D_x^3 f(x) h^3}{1.2.3} + \frac{D_x^4 f(x) h^4}{1.2.3.4} + \frac{D_x^5 f(x) h^5}{1.2.3.4.5} + \&c,\ + 4f(x),\ + f(x) + D_x f(x) h + \frac{D_x^2 f(x) h^2}{2} + \frac{D_x^3 f(x) h^3}{1.2.3} + \frac{D_x^4 f(x) h^4}{1.2.3.4} + \frac{D_x^5 f(x) h^5}{1.2.3.4.5} + \&c =$

$[6f(x) + D_x^2 f(x) h^2 + \frac{D_x^4 f(x) h^4}{1.3.4} + \frac{D_x^6 f(x) h^6}{1.3.4.5.6} + \&c]\frac{h}{3} =$

$2f(x)h + \frac{2D_x^2 f(x) h^3}{3} + \frac{D_x^4 f(x) h^5}{36} + \&c$ Equation 1

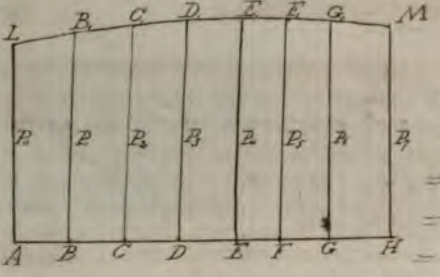
$\int[f(x) + D_x f(x) h + \frac{D_x^2 f(x) h^2}{1.2} + \frac{D_x^3 f(x) h^3}{1.2.3} + \&c]\,dx$

$\int[f(x) - D_x f(x) h + \frac{D_x^2 f(x) h^2}{1.2} - \frac{D_x^3 f(x) h^3}{1.2.3} + \&c]\,dx$

$\int_{-h}^{h}[D_x f(x)\,dx\,h + \frac{D_x^3 f(x) h^3}{1.3}\,dx] =$

$2f(x)h + \frac{2D_x^2 f(x) h^3}{2.3} + \frac{D_x^4 f(x) h^5}{3.4.5} + \&c$ Equation 2

Equation 1 being compared with Equation 2 Shows a difference of $\frac{D_x^4}{90}$ which is the limit of absolute correctness of Simpson's Rule, which is therefore applicable perfectly to such quantities as have no fourth differential, the general expression for which quantities is $x^3 + x^2 + x + c$



Prove that the Prismoidal Formula is applicable to Simpson's Rule.

$(B+4M+b)\frac{h}{6}$ i.e. the lower base plus four times the middle section plus the upper base multiplied by one sixth of the height.— By Simpson's Rule we have

$P_S=(P_0+4P_1+2P_2 \ldots\ldots\ldots + 4P_{n-1}+P_n)\frac{S}{3n}$

In the above figure we may consider ACCL as a prismoid and by the application of the prismoidal formula we have $(P_0+4P_1+P_2)\frac{h}{6}$ and taking the next two divisions we have $(P_2+4P_3+P_4)\frac{h}{6}$ and so on for any number of divisions adding together these two results ~~and substituting in Simpson's Rule~~ we shall have $(P_0+4P_1+2P_2 \ldots + P_n)\frac{h}{6}$ But the line AH is divided into n equal parts and the line $AC=\frac{2S}{n}=h$ substituting which value we have $\frac{S}{3n}=\frac{h}{6}$ which occurs in Simpson's Rule find that the work* ($P s$) which a mass either acquires when it passes from a lesser velocity (c) into a greater (v), or produces, when it is compelled to pass from a greater velocity into a less, is constantly equal to the product of the weight of this mass, and the difference of the heights due to the velocities $\left(\frac{v^2}{2g} - \frac{c^2}{2g}\right)$.

Example 1. In order to impart to a carriage of 4000 lbs. weight, upon a perfectly smooth railroad, a velocity of 30 feet, a mechanical work $P s = \frac{v^2}{2g} G = 0{,}0155\, v^2 G = 0{,}0155 \times 900 \times 4000 = 55800$ ft. lbs. is required; and just so much work will this carriage perform if a resistance be opposed to it, and it be gradually brought to rest.—2. Another carriage of 6000 lbs. goes forward with a velocity of 15 feet, which is transformed by a force acting upon it into a velocity of 24 feet, how great is the work acquired by this carriage, or done by the force? To the velocities 15 and 24 feet correspond the heights due to velocity $h_1 = \frac{c}{2g} = 3{,}49$ ft., and $h = \frac{v^2}{2g} = 8{,}928$ ft.; from this the mechanical work $P s = (h - h_1)\, G = 5{,}441 \times 6000 = 32646$ ft. lbs. If, now, the distance be known in which this change of velocity goes on, the force may be found; and when this is known, the distance may be determined. In this last case, for example, let the distance of the carriage amount to 100 feet, and, whilst describing this, the velocity passes from 15 into 24 feet; we have the force $P = (h - h_1)\frac{G}{s} = \frac{32646}{100} = 326{,}46$ lbs. Were the force itself 2000 lbs., the space s would be $= (h - h_1)\, \frac{G}{P} = \frac{32646}{2000} = \overset{15{,}35}{\cancel{16{,}923}}$ feet.—3. If a 500 lbs. sledge has entirely lost, through friction on its path, its velocity of 16 feet, after describing a space of 100 feet, then is the resistance of friction $P = \frac{h G}{s} = 0{,}0155 \times 16^2 \times \frac{500}{100} = 0{,}0155 \times 256 \times 5 = 19{,}84$ lbs.

§ 72. The formula found for the work in the foregoing paragraph:

$$P s = (h - h_1)\, G$$

is not only good for constant, but also for variable forces, if, instead of P, the mean value of the force (from § 70) be introduced; for if the whole space (s) of motion be considered as consisting of equal and uniformly accelerated parts described $\left(\frac{s}{n}\right)$, then we have the amount of work for these:

$$P_1\left(\frac{s}{n}\right) = \frac{v_1^2 - c^2}{2g} G,$$

$$P_2\left(\frac{s}{n}\right) = \frac{v_2^2 - v_1^2}{2g} G,$$

$$P_3\left(\frac{s}{n}\right) = \frac{v_3^2 - v_2^2}{2g} G,$$

&c., in so far as v_1, v_2, v_3, &c., stand for the velocities acquired at the end of these parts of space; and by the addition of all these works we have the whole work required for the transformation of the velocity c into v:

$P s = (P_1 + P_2 + P_3 + \ldots)\frac{s}{n} = \frac{v^2 - c^2}{2g}\, G$, because for an infinite number (n) of forces $(P_1+P_2+P_3+\ldots) \div n$, it transforms itself into a mean force, and because the members on the right hand of the equation $\frac{v_1^2}{2g}\,G$ and $-\frac{v_1^2}{2g}\,G$, as also $\frac{v_2^2}{2g}\,G$ and $-\frac{v_2^2}{2g}\,G$, &c. are opposed to each other, so that the members $\frac{v^2}{2g}\,G$ and $\frac{c^2}{2g}\,G$, determined by the terminal velocity v and the initial velocity c, only remain.

---

* i. e. Working power.

The formula $P\,s = \left(\frac{v^2-c^2}{2g}\right) G = (h-h_1)\,G$ is not used merely for the determination of the work, but not unfrequently, also, for the measurement of the terminal velocity. In the last case h is put $= h_1 + \frac{Ps}{G}$ or $v = \sqrt{c^2+2g\,\frac{Ps}{G}}$. If by the constant motion of a body, the terminal velocity $v =$ the initial velocity c, the work done $=$ zero, *i. e.* as much work is performed by the accelerated, as is expended by the retarded part of the motion.

Example.—A carriage of 2500 lbs. proceeding upon a railroad without friction, has acquired by an augmentation of its velocity, which at the commencement amounted to 10 ft., a mechanical work of 8000 lbs., its velocity after this work will be:

$$v = \sqrt{10^2+64{,}4\cdot\frac{8000}{2500}} = \sqrt{100+206} = 17{,}49 \text{ feet.}$$

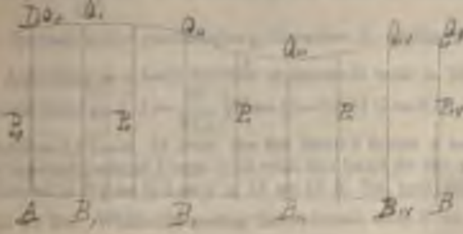
Remark. The product of the mass $M = \frac{G}{g}$ and the square of the velocity (v^2): Mv^2 is called, without attaching to it any definite idea, the living force (*vis viva*) of the moved mass: and hereafter, the mechanical work which a moved mass acquires, may be put equal to half of the *vis viva* of the same. If a mass enters from a velocity c into another v, the work performed is equal to half the difference of the *vis viva* at the commencement and end of the change of velocity. This law of the mechanical performance of bodies by means of their inertia, is called the *principle of living forces*, or the *vis viva*.

§ 73. *Composition of Forces.*—Two forces P_1 and P_2 act upon one and the same body, in the same or in an opposite direction, the effect is the same as if only one force acted upon the body, which is the sum or difference of these forces; for these forces impart to the mass M the acceleration, $p_1 = \frac{P_1}{M}$ and $p_2 = \frac{P_2}{M}$, consequently from § 28, the acceleration resulting from both, is

$p = p_1 \pm p_2 = \frac{P_1 \pm P_2}{M}$, and accordingly the force corresponding to this, is: $P = Mp = P_1 \pm P_2$.

The equivalent force P derived from these two is called the ***resultant***; its constituents P_1 and P_2 the *components*.

Example.—1. A body lying flat upon the hand presses so long only upon it with its absolute weight as the hand is at rest, or is moved up and down uniformly with the body; but if the hand be raised quickly, it suffers a greater pressure; on the other hand, if it be suddenly dropped, the pressure is then less than the weight; it becomes null if the hand be drawn back with the acceleration of gravity. If the pressure on the hand $= P$, the body falls with a force $G-P$, whilst its mass $M = \frac{G}{g}$; if we put the accelera-



Simpson's Rule: See Art 70

Let AB = s = whole space
P_0, P_1 &c the different ~~velocities~~ forces, then we have for the whole work done

$$Ps = (\tfrac{1}{2}P_0 + P_1 + P_{ii} + P_{iii} + \cdots\cdots + \tfrac{1}{2}P_n)\frac{s}{n}$$

$$\therefore P = (\frac{P_0}{2} + P_1 + P_{ii} + P_{iii} + \cdots\cdots + \frac{P_n}{2}) \div n$$

By dividing AB into a new set of spaces AB_{ii}, B_iB_{ii} &c each equalt to $\frac{2}{3}$ of $\frac{1}{n}$ of s then dropping perpendiculars we have the ~~velocities~~ forces $Q_0, Q_1, Q_{ii}, \ldots Q_n$ therefore

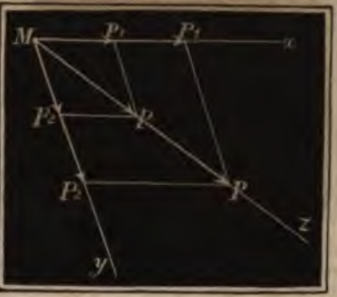
$$Ps = (\frac{Q_0}{2} + Q_1 + Q_{ii} + Q_{iii} \cdots\cdots + \frac{Q_n}{2})\frac{2s}{3n}$$ But $Q_0 = P_0$, $Q_1 + Q_{ii} = 2P_1$, $Q_{iii} = P_{ii}$ &c by substituting these values in the above equations we have

$$Ps = (\frac{P_0}{2} + 2P_1 + P_2 + 2P_3 + \cdots\cdots + \frac{P_n}{2})\frac{2s}{3n},$$
$$= (P_0 + 4P_1 + 2P_2 + 4P_3 + \cdots\cdots + P_n)\frac{s}{3n}$$

tion with which the hand falls $= p$, $G - P = \frac{G}{g} p$, and therefore the pressure $P = G - \frac{p}{g} G = \left(1 - \frac{p}{g}\right) G$. If the body on the hand be raised with the acceleration p, $-p$ is then opposed to the acceleration g, therefore the pressure upon the hand $P = \left(1 + \frac{p}{g}\right) G$. According as a body ascends or descends with a 20 feet acceleration, the pressure upon the hand $= \left(1 - \frac{20}{32,2}\right) G = (1 - 0,62)\, G = 0,38$, of the weight of the body, or $= 1 + 0,62 = 1,62$.—2. If with the flat hand I throw a body of 3 lbs. 14 feet perpendicularly upwards, whilst I urge it on with the hand for the first 2 feet, the mechanical work performed is $P s = G h = 3 \times 14 = 42$ ft. lbs., and the pressure upon the hand, $P = \frac{42}{2} = 21$ lbs. Whilst the resting body presses with 3 lbs., it reacts upon the hand during the projection with 21 lbs.

§ 74. *Parallelogram of Forces.*—When a material point M, Fig. 30, is acted upon by two forces, P_1, P_2, whose directions MX and MY make, with each other, the angle $XMY = \alpha$, these lines generate the accelerations in these directions, $p_1 = \frac{P_1}{M}$ and $p_2 = \frac{P_2}{M}$, and from their union, there arises a mean acceleration (§ 34) in the direction MZ, both of which are given by the diagonal of a parallelogram formed from p_1, p_2, and the angle α; this mean or resultant acceleration $p = \sqrt{p_1^2 + p_2^2 + 2 p_1 p_2 \cos. \alpha}$, and for the angle ϕ which its direction makes with MX of the one acceleration p_1:

Fig. 30.



$$\sin. \phi = \frac{p_2 \sin. \alpha}{p}.$$

If we substitute in these formulæ the above values of p_1 and p_2:

$$p = \sqrt{\left(\frac{P_1}{M}\right)^2 + \left(\frac{P_2}{M}\right)^2 + 2\left(\frac{P_1}{M}\right)\left(\frac{P_2}{M}\right) \cos. \alpha} \text{ and}$$

$$\sin. \phi = \left(\frac{P_2}{M}\right) \frac{\sin. \alpha}{p}.$$

If we multiply the first equation by M,

$$M p = \sqrt{P_1^2 + P_2^2 + 2 P_1 P_2 \cos. \alpha}, \text{ or,}$$

since $M p$ is the force corresponding to the acceleration:

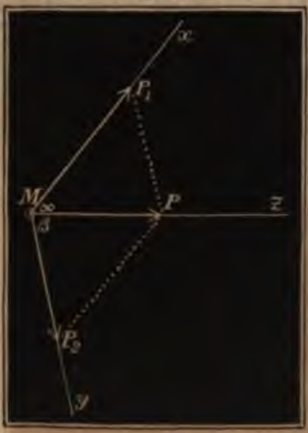
$$1.\ P = \sqrt{P_1^2 + P_2^2 + 2 P_1 P_2 \cos. \alpha}.$$

$$2.\ \sin. \phi = \frac{P_2 \sin. \alpha}{P}.$$

Thus, the resultant force is determined in magnitude and direction from the component forces exactly as the resultant acceleration from the component accelerations.

If we represent the forces by straight lines, and these lines be drawn, bearing the same proportions to each other as do weights, as pounds, &c., the mean force may be represented by the diagonal of the parallelogram whose sides are formed by the lateral forces, and one of whose angles is equal to that made by the directions of these lateral forces. The parallelogram which is constructed from the lateral forces, and whose diagonal is the mean force, is called the parallelogram of forces.

Fig. 31.



Example. When a body of 150 lbs. weight, resting upon a perfectly smooth table (Fig. 31) is acted upon by two forces $P_1 = 30$ lbs., and $P_2 = 24$ lbs., which make with each other an angle $P_1 M P_2 = \alpha + \beta = 105^\circ$: in what direction, and with what acceleration, will the motion take place? Since $\cos.(\alpha+\beta) = \cos. 105^\circ = -\cos. 75^\circ$, the mean force:

$$P = \sqrt{30^2 + 24^2 - 2 \times 30 \times 24 \cos. 75^\circ}$$
$$= \sqrt{900 + 576 - 1440 \cos. 75^\circ}$$
$$= \sqrt{1476 - 372{,}7} = 33{,}21 \text{ lbs.},$$

the acceleration corresponding with it is:

$$p = \frac{P}{M} = \frac{Pg}{G} = \frac{33{,}21 \times 32{,}2}{150} = 7{,}129 \text{ ft.}$$

The direction of motion makes with the direction of the first force an angle α, which is determined by:

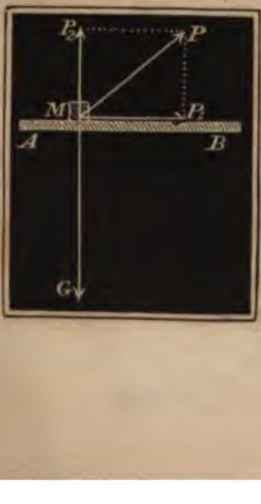
$$\sin. \alpha = \frac{24}{33{,}22} \sin. 105^\circ = 0{,}7224 \sin. 75^\circ = 0{,}6978, \text{ or } \alpha = 44^\circ, 15'.$$

Remark. The mean force P depends, from the formulæ found, only on the component forces, and not on the mass of the body upon which the forces act. For this reason, we find in many works on mechanics, the correctness of the parallelogram of forces proved without regard to the mass, but with the assumption of some fundamental law.

§ 75. ***Resolution of Forces.***—By help of the parallelogram of forces, not only two or more forces may be reduced to a single one, but also given forces under given relations may be resolved into two or more forces. If the angles α and β are given, which the components $MP_1 = P_1$, and $MP_2 = P_2$, make with the given force $MP = P$, the components may be found from the formulæ:

$$P_1 = \frac{P \sin. \beta}{\sin.(\alpha+\beta)}, \quad P_2 = \frac{P \sin. \alpha}{\sin.(\alpha+\beta)}.$$

Fig. 32.



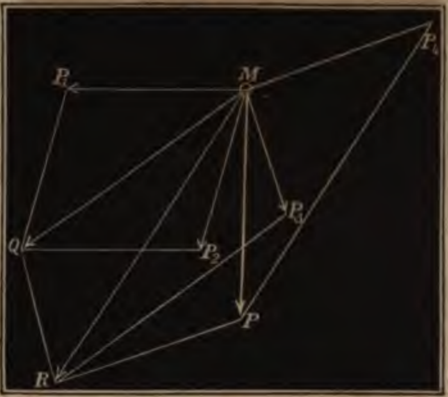
If the components are at right angles to each other, $\alpha+\beta = 90^\circ$, and $\sin.(\alpha+\beta) = 1$, and $P_1 = P \cos. \alpha$ and $P_2 = P \sin. \alpha$. If β and α be equal to one other, $P_2 = P_1$, viz:

$$P_2 = \frac{P \sin. \alpha}{\sin. 2\alpha} = \frac{P}{2 \cos. \alpha} = P_1.$$

Example 1. What is the pressure of a body M upon a table AB, Fig. 32, whose weight $G = 70$ lbs. and upon which a force $P = 50$ lbs. acts, and whose direction is inclined to the horizon at an angle $PMP_1 = \alpha = 40^\circ$? The horizontal component of P is $P_1 = P \cos. \alpha = 50 \cos. 40^\circ = 38{,}30$ lbs., and the vertical component $P_2 = P \sin. \alpha = 50 \sin. 40^\circ = 32{,}14$ lbs.; the latter strives to draw the body from the table, there remains then for the pressure: $G - P_2 = 70 - 32{,}14 = 37{,}86$ lbs.—2. If a body of 110 lbs. is so moved along an horizontal way, by two forces, that it describes in the first second a space of 6,5 feet, in a direction which deviates from the two directions of force by an angle $\alpha = 52°$ and $\beta = 77°$, the forces themselves are given as follows. The acceleration is twice the space in the first second, so that $p = 2 \times 6{,}5 = 13$ ft. Now the mean force is $P = \frac{pG}{g} = 0{,}031 \times 13 \times 110 =$ 44,33 lbs., therefore the one component $P_1 = \frac{P \sin. 77°}{\sin. (52° + 77°)} = \frac{44{,}33 \sin. 77°}{\sin. 51°} = 55{,}58$ lbs., and the other $P_2 = \frac{44{,}33 \sin. 52°}{\sin. 51°} = 45{,}59$ lbs.

§ 76. *Forces in a Plane.*—In order to find the mean force P for a system of forces P_1, P_2, P_3, &c., we may adopt exactly the same method (§ 33) as that followed in the composition of velocities, viz: by the repeated application of the parallelogram of forces, we may resolve them two and two and so on, till but a single force remains. The forces P_1 and P_2, for example, give from the parallelogram $M P_1 Q P_2$, the mean force $MQ = Q$, if this be joined to P_3, we have from the parallelogram $MQRP_3$, $MR = R$; and this last again forms a parallelogram with P_4 and gives the force $MP = P$ the last, and the resultant of the four forces P_1, P_2, P_3, P_4.

Fig. 33.



It is not necessary, in this way of composing forces, to complete the parallelogram, and draw its diagonal. We may form a polygon MP_1QRP, whose sides MP_1, P_1Q, QR, RP, are parallel and equal to the given components P_1, P_2, P_3, P_4, the last side MP completing the polygon will be the mean force sought, or rather its measure.

Remark. It is very useful to solve mechanical problems by construction also; though this method does not admit of such accuracy as that of calculation, it is free on the other hand from great errors, and may therefore serve as proof of the calculation. In Fig. 33 the forces meet each other under the given angles $P_1 M P_2 = 72°, 30'$; $P_2 M P_3 = 33°, 20'$, and $P_3 M P_4 = 92°, 40'$, and are so drawn that a pound is represented by a *line* or $\frac{1}{12}$ of a (Prussian*) inch. The forces $P_1 = 11{,}5$ lb., $P_2 = 10{,}8$ lbs., $P_3 = 8{,}5$ lbs., $P_4 = 12{,}2$ lbs., are therefore expressed by sides of 11,5 lines = 0,958 ... inches, 10,8 lines = 0,900 ... inches, 8,5 lines = 0,708 ... inches, 12,2 lines = 1,016 ... inches in length. A careful construction of the polygon of forces gives the magnitude of the mean force $P = 14{,}6$ lbs. and the variation of its direction MP from the direction MP_1 of the first force $= 86\frac{1}{2}°$.

§ 77. The resultant P is determined more simply and clearly if each of the given components P_1, P_2, P_3, &c., be resolved according to two axial directions $\overline{XX}$ and $\overline{YY}$, Fig. 34, at right angles to each other, into component forces as Q_1 and R_1, Q_2 and R_2, Q_3 and

---

* The Prussian inch (see § 15) is equal 1.031 English inches.—AM. ED.

R_3, &c., the forces lying in the same direction of axis, added together, and the resultants in magnitude and direction of these two rectangular forces be then sought for. If the angles $P_1 MX$, $P_2 MX$, $P_3 MX$, &c., which the directions of the forces P_1, P_2, P_3, make with the axis $\overline{XX} = \alpha_1$, α_2, α_3, &c., we have the components $Q_1 = P_1 \cos. \alpha_1$, $R_1 = P_1 \sin. \alpha_1$, $Q_2 = P_2 \cos. \alpha_2$, $R_2 = P_2 \sin. \alpha_2$, whence it follows from $Q = Q_1 + Q_2 + Q_3 + \ldots$,

1. $Q = P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + P_3 \cos. \alpha_3 + \ldots$, and

from $R = R_1 + R_2 + R_3 + \ldots$,

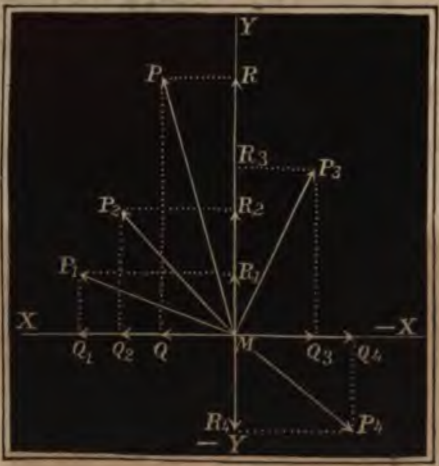
2. $R = P_1 \sin. \alpha_1 + P_2 \sin. \alpha_2 + P_3 \sin. \alpha_3 + \ldots$

From the two components Q and R so found, the magnitude of the resultant sought, is:

3. $P = \sqrt{Q^2 + R^2}$ and the angle $PMX = \phi$, whose direction with $\overline{XX}$ is given by

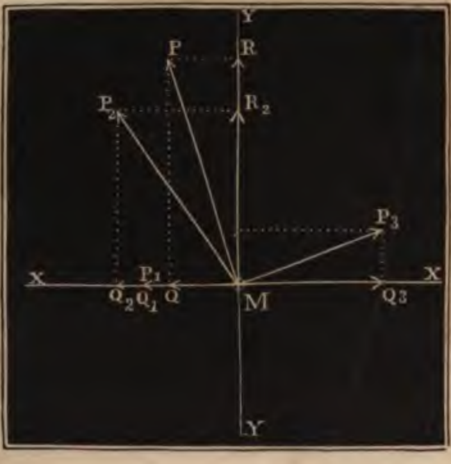
4. $tang. \phi = \dfrac{R}{Q}$.

Fig. 34.



In the algebraical addition of the forces, regard must be had to the sign, for if it be different in two forces, *i. e.* if the directions of these be upon opposite sides of the point of application M, this addition then becomes arithmetical subtraction (§ 73). The angle ϕ is acute, as long as Q and R are positive; it is between one and two right angles, when Q is negative and R positive; between two and three, when Q and R are both negative, and lastly, between three and four, when R only is negative.

Fig. 35.



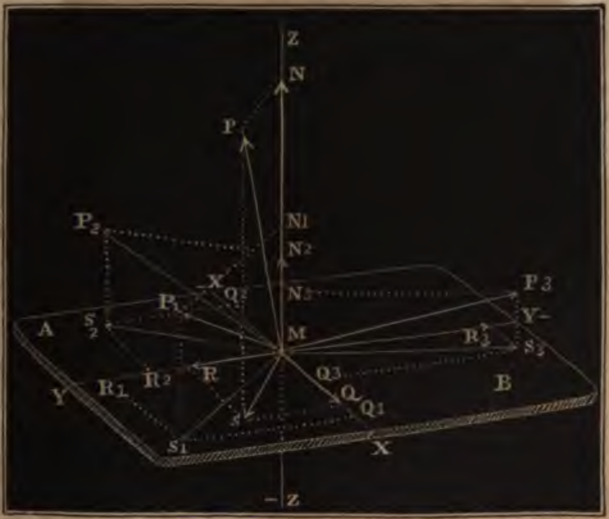
Example. What is the magnitude and direction of the resultant of the three components $P_1 = 30$ lbs., $P_2 = 70$ lbs., $P_3 = 50$ lbs., whose directions, lying in a plane, make between them the angles $P_1 M P_2 = 56°$ and $P_2\ M\ P_3 = 104$? If we draw the axis $\overline{XX}$

in the direction of the first force, we have $a_1 = 0$, $a_2 = 56^\circ$, and $a_3 = 56^\circ + 104^\circ = 160^\circ$; hence, 1. $Q = 30 \times \cos. 0^\circ + 70 \times \cos. 56^\circ + 50 \times \cos. 160^\circ = 30 + 39{,}14 - 46{,}98 = 22{,}16$ lbs.; and 2. $R = 30 \times \sin. 0^\circ + 70 \times \sin. 56^\circ + 50 \sin. 160^\circ = 0 + 58{,}03 + 17{,}10 = 75{,}13$ lbs. Hence, 3. $\text{tang.}\,\phi = \frac{75{,}13}{22{,}16} = 3{,}3903$; therefore, the angle which the resultant makes with the positive part of the axis MX or the force P_1 is $\phi = 73^\circ\ 34'$; lastly, the force itself $P = \sqrt{Q^2+R^2} = \frac{Q}{\cos.\,\phi} = \frac{R}{\sin.\,\phi} = \frac{75{,}13}{\sin.\ 73^\circ\ 34'} = \frac{75{,}13}{0{,}9591} = 78{,}33$ lbs.

§ 78. *Forces in Space.*—If the directions of the forces do not lie in one and the same plane, we must draw through the point of application a plane, and resolve each of the forces into two others, one lying in the plane, and the other at right angles to the plane; we must then find the resultant of the components so obtained in the plane, from the rule in the foregoing paragraph, and add together the components at right angles to the plane, and from the two rectangular components thus obtained, their resultant may be found according to the known rule (§ 74).

Fig. 36 puts the above mode of proceeding more clearly before us; let $MP_1 = P_1$, $MP_2 = P_2$, $MP_3 = P_3$ be the separate forces, AB the plane (of projection) and $Z\overline{Z}$ the axis at right angles to it. From the resolution of the forces P_1, P_2, &c., the forces S_1, S_2 are given in the plane, and those of N_1, N_2, &c., in the normal to it $Z\overline{Z}$. These are again resolved according to two axes $X\overline{X}$ and $Y\overline{Y}$ into the lateral forces Q_1, Q_2, &c., R_1, R_2, &c., and give the components Q and R, of which the resultant S consists, which, joined to the sum of all the normal forces, N_1, N_2, &c., gives P the resultant required.

Fig. 36.



If we put β_1, β_2, for the angles at which the directions of force are inclined to the plane AB or to the horizon, the forces in the plane are given, $S_1, = P_1 \cos. \beta_1$, $S_2 = P_2 \cos. \beta_2$, &c., and the normal forces, $N_1 = P_1 \sin. \beta_1$, $N_2 = P_2 \sin. \beta_2$, &c.; lastly, if we designate the angles which the projections of the directions of the forces lying in the plane AB, make with the axis $X\overline{X}$, by α_1, α_2, we obtain the three following forces, forming the sides of a rectangular parallelopiped.

$Q = S_1 \cos. \alpha_1 + S_2 \cos. \alpha_2 + S_3, \cos. \alpha_3$, or

1. $Q = P_1 \cos. \beta_1 \cos. \alpha_1 + P_2 \cos. \beta_2 \cos. \alpha_2 + \ldots,$
2. $R = P_1 \cos. \beta_1 \sin. \alpha_1 + P_2 \cos. \beta_2 \sin. \alpha_2 + \ldots$
3. $N = P_1 \sin. \beta_1 + P_2 \sin. \beta_2 + \ldots$

From these three follows the final resultant:

4. $P = \sqrt{Q^2 + R^2 + N^2}$, further

the angle of inclination to the plane of projection $PMS = \psi$, from

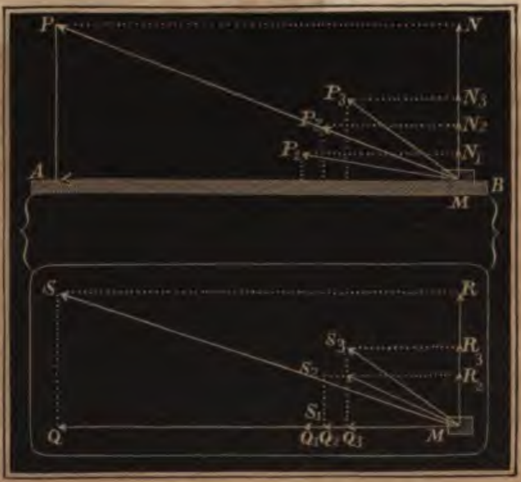
5. $tang. \psi = \dfrac{N}{S} = \dfrac{N}{\sqrt{Q^2 + R^2}}$, lastly

the angle $SMX = \phi$, which the projection of the resultant in the plane AB makes with the first axis $X\overline{X}$, by

6. $tang. \phi = \dfrac{R}{Q}$.

Example. Three workmen pull at the end of three ropes, which are attached to a load M lying upon a horizontal floor AB, Fig. 37, each with a force of 50 lbs.; the angles of inclination of these forces to the horizon are 10°, 20°, and 30°, and the horizontal angle between the first and second, and between the first and third, 20° and 35°; what is the magnitude and direction of the resultant, and how much is this less than the sum of all the forces which would result, if all three acted in the same direction? The vertical force pulling upward is:

Fig. 37.



$N = N_1 + N_2 + N_3 = 50 \times (\sin. 10° + \sin. 20° + \sin. 30°) = 50 \times 1{,}01567 = 50{,}78$ lbs.; by so much less than its own weight does the body press upon the floor.

The horizontal components are $S_1 = 50 \times \cos. 10° = 50 \times 0{,}9849 = 49{,}24$ lbs.; $S_2 = 50$

$\times \cos. 20° = 46{,}98$ lbs.; $S_3 = 50 \times \cos. 30° = 43{,}30$ lbs. If we draw the axis $X\overline{X}$ in the direction of the first force S_1, we obtain the lateral force in this axis $X\overline{X}$, $Q = Q_1 + Q_2 + Q_3 = S_1 \cos. a_1 + S_2 \cos. a_2 + S_3 \cos. a_3 = 49{,}24 \times \cos. 0° + 46{,}98 \times \cos. 20° + 43{,}30 \times \cos. 35° = 49{,}24 + 44{,}15 + 35{,}47 = 128{,}86$ lbs.; on the other hand, the lateral force in the second axis $Y\overline{Y}$: $R = R_1 + R_2 + R_3 = 49{,}24 \times \sin. 0° + 46{,}98 \times \sin. 20° + 43{,}30 \times \sin. 35° = 0 + 16{,}07 + 24{,}84 = 40{,}91$ lbs.

The horizontal mean force with which the body is drawn forward is from this:

$$S = \sqrt{Q^2 + R^2} = \sqrt{(128{,}86)^2 + (40{,}91)^2} = \sqrt{18278{,}7} = 135{,}2 \text{ lbs.}$$

The angle ϕ which this force makes with the axis $X\overline{X}$ is determined by the *tang.* $\phi = \frac{R}{Q} = \frac{40{,}91}{128{,}86} = 0{,}3175$: $\phi = 17°{,}37'$; the entire resultant is:

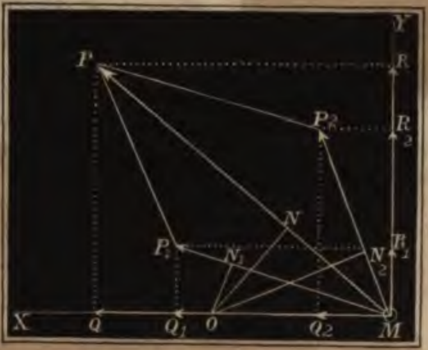
$$P = \sqrt{(135{,}2)^2 + (50{,}78)^2} = \sqrt{20856{,}6} = 144{,}42 \text{ lbs.}$$

If the forces act in the same direction, the resultant is $= 3 \times 50 = 150$ lbs., and the loss of force $= 150 - 144{,}42 = 5{,}58$ lbs.; further, because the horizontal force drawing the body forwards amounts only to 135,20 lbs., we have, with reference to the horizontal motion, the loss of force $150 - 135{,}20 = 14{,}80$ lbs.

The angle of inclination ψ of the mean force to the horizon is determined by the *tang.* $\psi = \frac{N}{S} = \frac{50{,}78}{135{,}20} = 0{,}3756$, wherefore ψ comes out $= 20°, 35'$.

§ 79.—From the rules found in the foregoing upon the composition of forces, two others of essential service for practical use may be deduced. In Fig. 37, let M be a material point, $MP_1 = P_1$ and $MP_2 = P_2$, the forces acting upon it; lastly, let $MP = P$, the resultant of P_1 and P_2. If we draw through M two axes, MX and MY, at right angles to each other, and resolve the forces P_1 and P_2, as well as their resultant P, into components in the direction of these axes, viz: P_1 into Q_1 and R_1, P_2 into Q_2 and R_2, and P into Q and R, we then obtain the forces in the one axis Q_1, Q_2 and Q, and those in the other R_1, R_2, R, and $Q = Q_1 + Q_2$, and $R = R_1 + R_2$.

Fig. 37.



If now we take in the axis MX any point O, and let fall from the same perpendiculars ON_1, ON_2 and ON on the directions of the forces P_1, P_2 and P we obtain rectangular triangles MON_1, MON_2, MON, which are similar to the triangles formed by the three forces, viz:

$$\triangle\, MON_1 \backsim \triangle\, MP_1Q_1$$
$$\triangle\, MON_2 \backsim \triangle\, MP_2Q_2$$
$$\triangle\, MON \backsim \triangle\, MPQ.$$

Principle of Virtual Velocities.—But from these similarities $\frac{MQ_1}{MP_1}$ *i. e.* $\frac{Q_1}{P_1} = \frac{MN_1}{MO}$, also $\frac{Q_2}{P_2} = \frac{MN_2}{MO}$ and $\frac{Q}{P} = \frac{MN}{MO}$; if we put the values hence derived of Q_1, Q_2, and Q into the equation $Q = Q_1 + Q_2$, we then obtain

$$P \,.\, MN = P_1 \,.\, MN_1 + P_2 \,.\, MN_2.$$

Likewise also $\frac{R_1}{P_1} = \frac{ON_1}{MO}$, $\frac{R_2}{P_2} = \frac{ON_2}{MO}$ and $\frac{R}{P} = \frac{ON}{MO}$, therefore

$$P \,.\, ON = P_1 \,.\, ON_1 + P_2 \,.\, ON_2.$$

These equations still hold good, if P the mean force be made up of three or more forces P_1, P_2, P_3, because generally

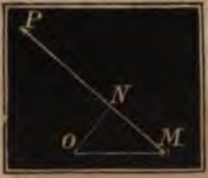
$$Q = Q_1 + Q_2 + Q_3 + \dots$$
$$R = R_1 + R_2 + R_3 + \dots$$

and, therefore, generally we may put:

1. $P \,.\, MN = P_1 \,.\, MN_1 + P_2 \,.\, MN_2 + P_3 \,.\, MN_3 + \dots,$
2. $P \,.\, ON = P_1 \,.\, ON_1 + P_2 \,.\, ON_2 + P_3 \,.\, ON_3 + \dots$

In both equations the mean force P must correspond to the forces P_1, P_2, P_3, and from these equations, not only the magnitude, but also the direction of this force may be determined.

Fig. 38.



§ 80. If the point of application M move in a straight line towards O, or if we imagine this point to have described the space $MO = s$, then the projection of this space $MN = s_1$ in the direction of the force MP is called the *space of the force* P, and the product Ps_1 of the force and its space, *the work or efficiency of* the force. If we substitute in the equation (1) of the last (§) these designations, we have

$$Ps = P_1 s_1 + P_2 s_2 + P_3 s_3 + \dots,$$

or the *work, or mechanical effect, of the resultant is equivalent to the sum of the works, or mechanical effects, of the components.*

In the summation of the mechanical effects, as in that of the forces, we must have regard to their signs. If a force (Q_3) of the forces Q_1, Q_2, &c., of the last § acts in an opposite direction to the rest, we must

Fig. 39.

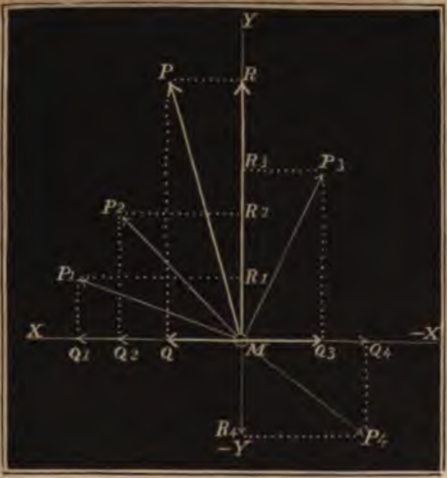


Fig. 40.

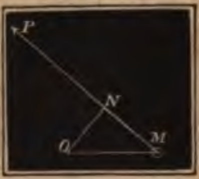
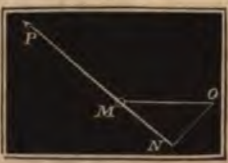


Fig. 41.



introduce it as negative, but this force Q_3, Fig. 39, is the component of a force P_3, which, acting in the circumstances set forth in the former §, opposed to their proper motion MN_3, we are, therefore, obliged to consider that force opposed to the motion MN, Fig. 40, as negative, and that one P, Fig. 41, acting in the direction of motion MN as positive.

If the forces are variable in magnitude or direction, the formula $Ps = P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots$ is only correct for infinitely small spaces s, s_1, s_2, &c.

The spaces of the forces σ_1, σ_2, σ_3, corresponding to an infinitely small displacement σ of a material point, are called their ***virtual velocities***; and the law corresponding to the formula $P\sigma = P_1\sigma_1 + P_2\sigma_2 + P_3\sigma_3$, ***the principle of virtual velocities.***

§ 81. *Transmission of Mechanical Effect.*—From the principle of vis viva, the mechanical effect (Ps) in rectilinear motion, which a force (P) generates in changing the velocity c of a mass M into another v is

$$Ps = \left(\frac{v^2 - c^2}{2}\right) M.$$

If P be now the mean force arising from other forces, P_1, P_2, &c., acting upon the mass M, and the spaces which these describe be s_1, s_2, whilst the mass itself M describes s, we then have from the foregoing:

$$Ps = P_1 s_1 + P_2 s_2 + \ldots$$

and, therefore, the following general formula:

$$P_1 s_1 + P_2 s_2 + \ldots = \left(\frac{v^2 - c^2}{2}\right) M,$$

which expresses that the sum of the mechanical effects of the single forces is equal to half the gain of vis viva of the mass taking up these forces.

If the velocity during the motion be invariable, that is $v = c$, and the motion itself be uniform, we have then $v^2 - c^2 = 0$, consequently neither loss nor gain of vis viva, and, therefore:

$$P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots . = 0,$$

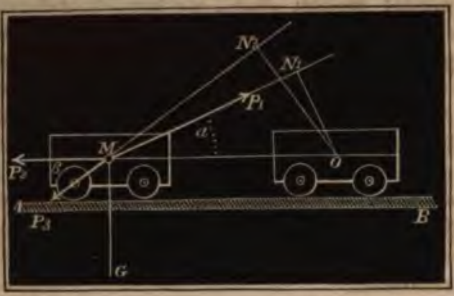
i. e. the sum of the mechanical effects of the single forces = 0.

If inversely the sum of the mechanical effects = 0, then the forces do not change the motion of the body in the given direction, nor impart to it in the given direction any motion which it had not before.

If the forces are variable, the variable velocity v after a certain time again passes into its initial velocity c, which takes place in all periodic motions as they present themselves in many machines. Now $v = c$ gives the effect $\left(\frac{v^2 - c^2}{2}\right) M = 0$; therefore within a period of the motion the loss or gain in mechanical effect is null.

Example. A carriage, of the weight $G = 5000$ lbs., Fig. 42, is moved forward upon a horizontal surface by means of a force $P_1 = 660$ lbs., ascending under an angle $\alpha = 24°$, and has during its motion two resistances to overcome; one, horizontal $P_2 = 350$ lbs., corresponding to the friction; and a resistance $P_3 = 230$ lbs., acting downwards, and inclined to the horizon at an angle $\beta = 35°$. What work will the force (P_1) perform, in order to convert the two feet initial velocity of the carriage into a velocity of 5 feet?

Fig. 42.



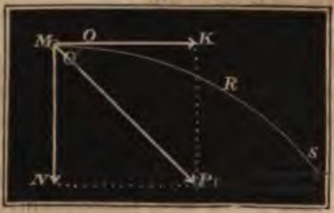
If we put the distance of the carriage $MO = s$, we then have for the work of the force $P_1 = P_1 . MN_1 = P_1\, s\, cos.\, \alpha = 660 \times s\, cos.\, 24° = 602{,}94 . s$; further, the work of the resisting force $= (-P_2) . s = -350 . s$; lastly, the work of $P_3 = (-P_3 . MN_3 = -P_3\, s\, cos.\, \beta = -230 \times s\, cos.\, 35° = -188{,}40 . s$. There then remains for the work of the effective force: $Ps = P_1\, s\, cos.\, \alpha - P_2\, s\, cos.\, 0 - P_3\, s\, cos.\, \beta = (602{,}94 - 350 - 188{,}40) . s = 64{,}54 . s$ ft. lbs.

The mass, however, requires for the change of its velocity, the mechanical effect: $\left(\frac{v^2 - c^2}{2g}\right) G = \left(\frac{5^2 - 2^2}{2g}\right) \times 5000 = 0{,}0155 \times (25 - 4) \times 5000 = 1627$ ft. lbs.

If now we equate both mechanical effects, we then obtain $64{,}54 . s = 1627$, consequently the distance of the carriage: $s = \frac{1627}{64{,}54} = 25{,}26$ feet; and lastly, the mechanical effect of the force $P : P_1\, s\, cos.\, \alpha = 602{,}94 \times 25{,}26 = 15230{,}2$ ft. lbs.

§ 82. *Curvilinear Motion.*—Provided that the spaces σ, σ_1, &c., be infinitely small, we may also apply the formula last found to curved paths. Let *MORS*, Fig. 43, be the path of a material point, and $MP_1 = P_1$ the resultant of all the forces acting upon it; if we resolve this force into two others, of which the one $MK = K$ is tangential, and the other $MN = N$ normal to the curve, we then term the one a *tangential*, and the other a *normal* force.

Fig. 43.



Whilst the material point describes the element $MO = \sigma$ of its curved path *MS*, and its velocity c is transformed into v_1, its mass M lays claim to the work $\left(\frac{v_1^2 - c^2}{2}\right) M$, but the tangential force K performs at the same time the work $K\sigma$, and the normal force the work $N . 0 = 0$; consequently $K\sigma = \left(\frac{v_1^2 - c^2}{2}\right) M$.

If the projection *MQ* of the elementary space *MO* in the direction of force be put $= \sigma_1$, then also $P_1\sigma_1 = K\sigma$; and, therefore,

$$P_1\sigma_1 = \left(\frac{v_1^2 - c^2}{2}\right) M.$$

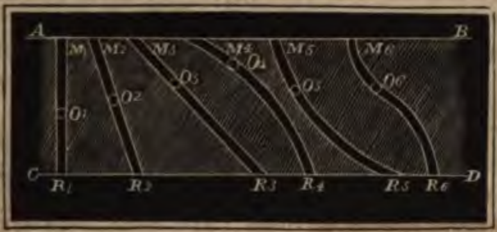
If the whole space described by the material point *MR* be decomposed into infinitely small parts, and each part be projected upon the direction of force at each moment, we then obtain the elementary space of the force at each moment, and the work at each moment by the multiplication of the space and force, and if we add together all these mechanical effects, we then have: $P_1\sigma_1 + P_2\sigma_2 + P_3\sigma_3 + \ldots = \left(\frac{v_1^2-c^2}{2}\right)_G M + \left(\frac{v_2^2-v_1^2}{2}\right) M + \left(\frac{v_3^2-v_2^2}{2}\right) M + \ldots = \left(\frac{v^2-c^2}{2}\right) M = (h-h_1) M$, if h_1 be the height due to the initial velocity c, and h that due to the terminal velocity v. Thus, *in curvilinear motion, the whole effect of the moving force is equal to half the gain of vis viva*, or equal to the product of the ~~mass~~ into the difference of the heights due to the velocities. weight

Remark and Example. The formula obtained which is derived from combining the principle of the vis viva with that of the virtual velocities, is especially applicable in cases where bodies are constrained by a fixed track or by suspension to describe a determinate path. If gravity alone act upon such a body, the work which it generates in a body of the weight G falling from a height corresponding to the vertical projection $M_1 R_1 = s$, is $= G s$, and therefore:

$$G s = (h-h_1)\, G, \text{ i. e. } s = h - h_1.$$

This is also the space which a body describes in falling from a horizontal plane AB, Fig. 44, to another CD; the difference of the heights due to the velocity is always equal to the perpendicular height of fall; bodies which begin to describe the paths $M_1\, O_1\, R_1$, $M_2\, O_2\, R_2$, $M_3\, O_3\, R_3$, &c., with equal velocity (c), acquire at the end of these paths, as well as at different times, equal velocities (v). If the initial velocity $c = 10$ feet, and the vertical height of fall $s = 20$ feet, then $h = s + h_1 = 20 + 0{,}0155 \,.\, 10^2 = 21{,}55$ feet, and the terminal velocity $v = \sqrt{2\,gh} = 8{,}02 \,.\, \sqrt{21{,}5} = 37{,}18$ feet, in whatever curved or right line the descent may take place.

Fig. 44.



SECTION III.

STATICS OF RIGID BODIES.

CHAPTER I.

GENERAL LAWS OF THE STATICS OF RIGID BODIES.

§ 83. *Transference of the Point of Application.*—Although every rigid body is changed in form by the action of forces upon it, *i. e.* becomes either compressed, extended, or bent, &c., it is nevertheless allowable for us to consider it for the most part as a rigid and invariable union of material points, partly because this change of form or displacement of parts is often very slight, and partly because it takes place in very short spaces of time. We shall, therefore, in the following, unless it be otherwise mentioned, regard every rigid body as a system of points, firmly connected, and we shall thereby essentially simplify the investigation.

A force P, Fig. 45, which acts upon a point A of a rigid body M, is transmitted in its proper direction $X\overline{X}$ uniformly throughout the body, and an equal and opposite force P_1 puts itself in equilibrium with it, then only when the point of application A_1 lies in the direction $X\overline{X}$ of the first force. The distance of A and A_1 is without influence on this condition of equilibrium. The two opposite forces hold themselves in equilibrium at every distance if the two points be rigidly connected. We may, therefore, assert *that the action of a force P*, Fig. 46, *remains the same at whatever point A_1, A_2, A_3, &c. of its direction it may be applied or may act directly upon the body.*

Fig. 45.

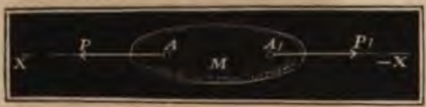
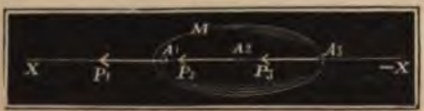
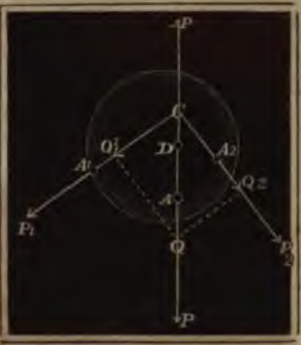


Fig. 46.



§ 84. When two forces P_1 and P_2 acting in the same plane are applied to a body at different points A_1 and A_2, their action upon the body is the same as if they had the point C, where the directions of the two forces intersect, for their common point of application, for from the proposition enunciated above, each of these points of application may be transferred to C without thereby producing any change in their effects. If, therefore, we make $CQ_1 = A_1P_1 = P_1$ and $CQ_2 = A_2P_2 = P_2$, and then complete the parallelogram $CQ_1\ QQ_2$, its diagonal will give us the resultant force $CQ = P$ of CQ_1 and CQ_2, and, therefore, also of the forces P_1 and P_2, and whose point of application may be any other point A in the direction of this diagonal.

Fig. 47.



If to the resultant force so found $AP = P$, there be put an opposite force $D\overline{P} = -P$ equally great at any point D of the direction of the diagonal C, the two forces P_1 and P_2 will be thereby held in equilibrium; P_1, P_2, and $-P$ are, therefore, three forces in equilibrium.

§ 85. If there be let fall from any point O, Fig. 48, in the plane of the forces perpendiculars ON_1, ON_2, and ON upon the directions of the component forces P_1 and P_2, and their resultant P, we have, according to § 79,

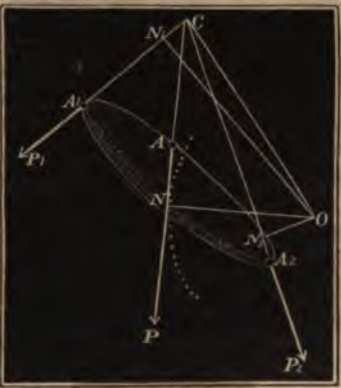
$$P \cdot ON = P_1 \cdot ON_1 + P_2 \cdot ON_2,$$

and the distance ON of the resultant force may be found from the perpendiculars or distances ON_1 and ON_2 of the component forces, if we put:

$$ON = \frac{P_1 \cdot ON_1 + P_2 \cdot ON_2}{P}.$$

Whilst we find the direction and magnitude of the resultant by the application of the parallelogram of forces, its position is given with the help of the last formula, by determining its distance ON.

Fig. 48.



If the prolonged direction of the forces includes between them an angle $P_1\ CP_2 = \alpha$, we then have:

1. The magnitude of the resultant $P = \sqrt{P_1^2 + P_2^2 + 2 P_1 P_2 \cos. \alpha}$.

Further, if the resultant makes with the direction of the component P_1 the angle $PCP_1 = \phi$, then:

$$2.\ \sin. \phi = \frac{P_2 \sin. \alpha}{P}.$$

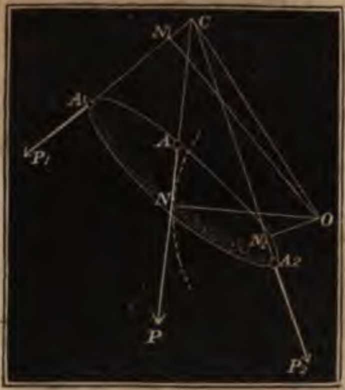
If the directions CP_1 and CP_2 of the given forces are distant $ON_1 = a_1$ and $ON_2 = a_2$ from an arbitrary point O, the distance $ON = a$ of the direction CP of the resultant from this point is:

3. $a = \frac{P_1 a_1 + P_2 a_2}{P}$.

With the help of this distance a, the position of the resultant is given without regard to the point C, if we describe a circle from O as a centre with radius a, and to this draw a tangent NP, whose direction is determined by the angle ϕ.

Example. There act upon a body the forces $P_1 = 20$ lbs. and $P_2 = 34$ lbs. whose directions meet under an angle $P_1 CP_2 = \alpha = 70°$, and are distant from a certain point $O = O\,N_1 = a_1 = 4$ feet, and $O\,N_2 = a_2 = 1$ foot; what is the magnitude, direction, and position of the resultant? The magnitude of the resultant is:

Fig. 49.



$$P = \sqrt{20^2 + 34^2 + 2 \times 20 \times 34\ cos.\ 70°}$$
$$= \sqrt{400 + 1156 + 1360 \times 0{,}34202}$$
$$= \sqrt{2021{,}15} = 44{,}96 \text{ lbs.};$$ further, for its direction, $sin.\ \phi = \frac{34 \times sin.\ 70°}{44{,}96}$,

Log. sin. $\phi = 9{,}8516384$, therefore, $\phi = 45°\ 17'$, the angle which this resultant makes with the direction of P_1. The position finally is determined by its distance $O\,N$ from O, which is:

$$a = \frac{20 \times 4 + 34 \times 1}{44{,}96} = \frac{114}{44{,}96} = 2{,}536 \text{ feet.}$$

§ 86. The normal distances $ON_1 = a_1$, $ON_2 = a_2$, &c., of the direction of the forces from an arbitrary point O, Fig. 50, are called the arms of the forces, because they form essential elements in the theory of the lever, to be treated of subsequently. The product Pa of the force and lever arm, is called the *statical moment* of the force. But since $Pa = P_1 a_1 + P_2 a_2$, the statical moment of the resultant is equivalent to the sum of the statical moments of the components.

In the addition of the moments, regard must be had to the signs plus and minus. If the forces P_1 and P_2, Fig. 50, act about the point O in like directions, and if the directions of force coincide with the direction of motion of the hands of a watch, these forces, as well as their statical moments, are said to have like signs; if the one be positive, the other must be positive likewise. If, on the other hand, Fig. 51, the directions of the forces about the point O be opposite to each other, then the same, as well as their statical moments, are of con-

Fig. 50. Cesar 1977

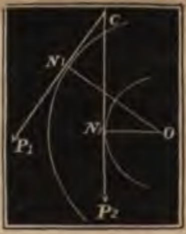
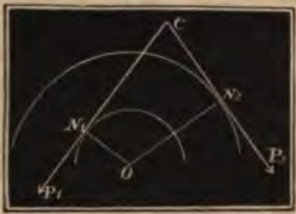
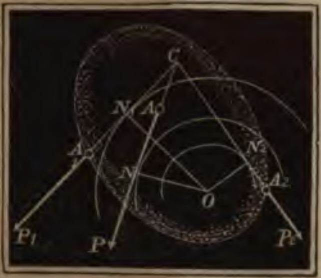


Fig. 51.



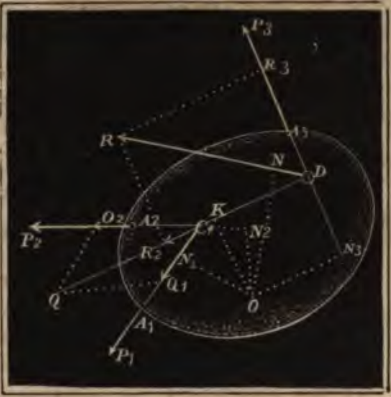
trary signs; if the one be negative, the other must be positive. In the composition of forces represented in Fig. 52, $Pa = P_1a_1 - P_2a_2$, because P_2 is opposed to the force P_1; its statical moment is, therefore, negative.

Fig. 52.



§ 87. *Composition of forces in a plane.*—If three forces, P_1, P_2, P_3, Fig. 53, act upon a body at the points A_1, A_2, A_3, two of these forces (P_1, P_2) by the last rule must be joined, and their resultant $CQ = Q$ found, this again joined to the third force (P_3), and the parallelogram DR_2RR_3 constructed from the forces $DR_2 = CQ$ and $DR_3 = A_3P_3$. The diagonal $DR = P$ is the required resultant of P_1, P_2, P_3. It is from this easy to see how the resultant might be found if a fourth force P_4 were to be introduced.

Fig. 53.



In this composition of the forces, the magnitude and direction of the resultant is as accurately found as if the forces acted in one single point (§ 77); the rules of calculation (§ 77) are, therefore, applicable for finding these two first elements of the resultant; but in order to find the third, viz., the position of the resultant or its line of action, we must make use of the equation between the statical moments. Here, also, $ON_1 = a_1$, $ON_2 = a_2$, $ON_3 = a_3$, and $ON = a$, are the arms of the three components P_1, P_2, P_3, and of their resultant P, with reference to an arbitrary point O. So that:

$Pa = Q \, . \, OK + P_3a_3$, and

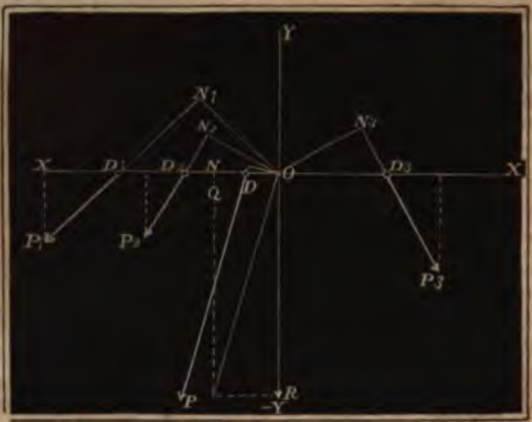
$Q \, . \, OK = P_1a_1 + P_2a_2$, provided Q is the resultant of P_1 and P_2, and OK the arm. If we combine these two equations we then obtain:

$Pa = P_1a_1 + P_2a_2 + P_3a_3$, and also for several forces:

$Pa = P_1a_1 + P_2a_2 + P_3a_3 + \ldots$, &c., *i. e.*, the (*statical*) *moment of the resultant is always equivalent to the algebraical sum of the* (statical) *moments of the components.*

§ 88. If P_1, P_2, P_3, Fig. 54, are the single forces of a system of forces; if, further, α_1, α_2, α_3, &c., are the angles P_1D_1X, P_2D_2X, P_3D_3X, &c., under which an arbitrarily chosen axis $X\overline{X}$ is intersected by the directions of force, and if a_1, a_2, a_3, designate the arms ON_1, ON_2, ON_3, of these forces with regard to the point of intersection O of both axes $X\overline{X}$ and $Y\overline{Y}$, we have from §§ 77 and 87:

Fig. 54.



1. The component parallel to the axis $X\overline{X}$:

$$Q = P_1 \, cos. \, \alpha_1 + P_2 \, cos. \, \alpha_2 + P_3 \, cos. \, \alpha_3 \ldots,$$

2. The component parallel to the axis $Y\overline{Y}$:

$$R = P_1 \, sin. \, \alpha_1 + P_2 \, sin. \, \alpha_2 + P_3 \, sin. \, \alpha_3 \ldots,$$

3. The resultant of the whole system:

$$P = \sqrt{Q^2 + R^2},$$

4. The angle which the resultant makes with the axis by

$$tang. \, \phi = \frac{R}{Q},$$

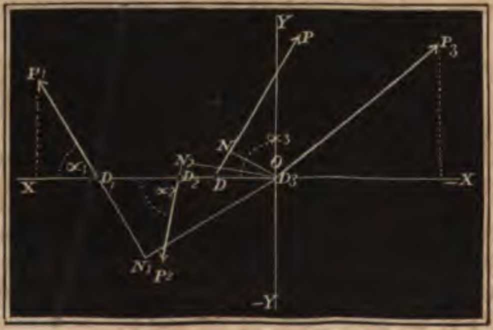
5. The arm of the resultant or the diameter of the circle to which the direction of the resultant is a tangent:

$$a = \frac{P_1 a_1 + P_2 a_2 + \ldots}{P}.$$

If this resultant be replaced by an equivalent opposite force $(-P)$, then the forces P_1, P_2, $P_3 \ldots . (-P)$ are in equilibrium.

Example. The forces $P_1 = 40$ lbs., $P_2 = 30$ lbs., $P_3 = 70$ lbs., Fig. 55, intersect the axis $X\overline{X}$ at angles $\alpha_1 = 60°$, $\alpha_2 = -80°$, $\alpha_3 = 142°$, and the distances of the points of intersection D_1, D_2, D_3, of the directions of the forces with the axis: $D_1 D_2 = 4$ ft.,

Fig. 55.



and $D_2 D_3 = 5$ ft. Required the elements of the resultant. The sum of the component forces parallel to $\overline{XX}$ is:

$$\begin{aligned} Q &= 40 \; cos. \; 60^\circ + 30 \; cos. \; (-80^\circ) + 70 \; cos. \; 142^\circ \\ &= 40 \; cos. \; 60^\circ + 30 \; cos. \; 80^\circ - 70 \; cos. \; 38^\circ \\ &= 20 + 5{,}209 - 55{,}161 = -29{,}952 \text{ lbs.} \end{aligned}$$

The sum of the components parallel to $\overline{YY}$:

$$\begin{aligned} R &= 40 \; sin. \; 60^\circ + 30 \; sin. \; (-80^\circ) + 70 \; sin. \; 142^\circ \\ &= 40 \; sin. \; 60^\circ - 30 \; sin. \; 80^\circ + 70 \; sin. \; 38 \\ &= 34{,}641 - 29{,}544 + 48{,}096 = 48{,}193 \text{ lbs.} \end{aligned}$$

The resultant sought is therefore:

$$P = \sqrt{Q^2 + R^2} = \sqrt{29{,}952^2 + 48{,}193^2} = \sqrt{3219{,}68} = 56{,}742 \text{ lbs.}$$

The angle ϕ, which it makes with the axis, is further determined by:

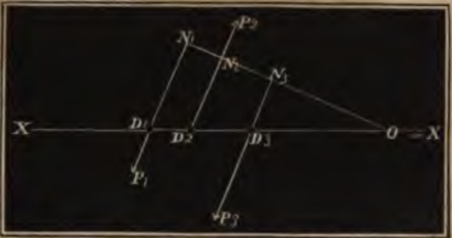
$$tang. \; \phi = \frac{R}{Q} = \frac{48{,}193}{-29{,}952} = -1{,}6090, \text{ it is therefore } \phi = 180^\circ - 58^\circ \; 8' = 121^\circ \; 52.'$$

The arm ON_1 of the force P_1 is $= OD_1 \; sin. \; \alpha_1 = (4+5) \; sin. \; 60^\circ = 9 \times 0.86603 =$ 7,794 feet; the arm ON_2 of $P_2 = OD_2 \; sin. \; \alpha_2 = 5 \; sin. \; 80^\circ = 4{,}924$ feet; lastly, the arm ON_3 of $P_3 = O$, when the point of application O is transferred to D_3. The arm of the resultant is finally given by:

$$a = \frac{40 \times 7{,}794 - 30 \times 4{,}924}{56{,}742} = \frac{311{,}76 - 147{,}72}{56{,}742} = \frac{164{,}04}{56{,}742} = 2{,}891 \text{ feet.}$$

§ 89. *Parallel Forces.*—If the forces P_1, P_2, P_3, &c., Fig. 56, of a rigid system are parallel, the arms ON_1, ON_2, ON_3, are in the same straight line; if now we draw through the point of application O an arbitrary line $\overline{XX}$, the directions of the forces cut off the parts OD_1, OD_2, OD_3, &c., which are proportional to the arms ON_1, ON_2, ON_3, &c., because $\triangle \; OD_1N_1 \backsim \triangle \; OD_2N_2 \backsim \triangle \; OD_3N_3$. If the angle $D_1ON_1 = D_2ON_2$ be designated by α, &c., the arms ON_1, ON_2, &c., by a_1, a_2, &c., the abscisses OD_1, OD_2, &c., by b_1, b_2, &c., we then have:

Fig. 56.



$$a_1 = b_1 \; cos. \; \alpha, \; a_2 = b_2 \; cos. \; \alpha, \; \&c.$$

If, lastly, these values be substituted in the formula:

$$Pa = Pa_1 + P_2a_2 + \ldots,$$

we then obtain:

$$Pb \; cos. \; \alpha = P_1b_1 \; cos. \; \alpha + P_2b_2 \; cos. \; \alpha + \ldots,$$

or if the common factor $cos. \; \alpha$ be left out:

$$Pb = P_1b_1 + P_2b_2 + \ldots$$

In every system of parallel forces it is allowable to replace the arms by the distances OD_1, OD_2, cut off from any line $\overline{XX}$. Because the magnitude and direction of the resultant is the same, the forces may act at one or at different points; hence the resultant of a system of parallel forces has the same direction with the single forces, and is equivalent to their algebraical sum.

Therefore

1. $P = P_1 + P_2 + P_3 + \ldots$ and
2. $a = \dfrac{P_1a_1 + P_2a_2 + \ldots}{P_1 + P_2 + \ldots}$, or also:

$$3.\ b = \frac{P_1 b_1 + P_2 b_2 + \ldots}{P_1 + P_2 + \ldots}.$$

Example. The forces $P_1 = 12$ lbs, $P_2 = -32$ lbs., $P_3 = 25$ lbs., and their directions intersect a straight line at the points D_1, D_2, and D_3, Fig. 56, whose distances from each other are $D_1D_2 = 21$ inches, $D_2D_3 = 30$ inches; required the resultant. The magnitude of this force is $P = 12 - 32 + 25 = 5$ lbs., its distance D_1O from D_1, is therefore:

$$b = \frac{12 \times 0 - 32 \times 21 + 25 \times (21 + 30)}{5} = \frac{0 - 672 + 1275}{5} = 120{,}6 \text{ inches.}$$

§ 90. *Couples.*—Two parallel, equal and opposite forces, P_1 and $-P_1$, Fig. 57, have the resultant

$$P = P_1 + (-P_1) = P_1 - P_1 = 0, \text{ with the arm}$$

$$a = \frac{P_1 a_1 + P_2 a_2}{0} = \infty\,.$$

Fig. 57.

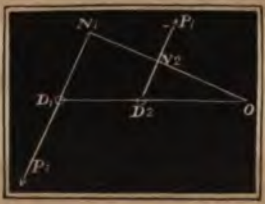
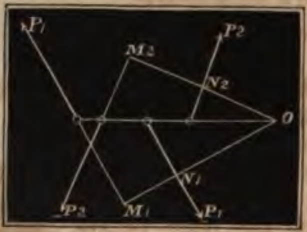


Fig. 58.



For restoring equilibrium to such a couple, according to this, a single finite force P acting at a finite distance, is not sufficient, but two such couples may easily hold each other in equilibrium. If P_1 and $-P_1$ and $-P_2$ and P_2, Fig. 58, are two such couples, and $OM_1 = a_1$, $ON_1 = OM_1 - M_1N_1 = a_1 - b_1$; if further, $OM_2 = a_2$ and $ON_2 = OM_2 - M_2N_2 = a_2 - b_2$ are the arms taken from a certain point O, we have for equilibrium:

$$P_1 a_1 - P_1(a_1 - b_1) - P_2 a_2 + P_2(a_2 - b_2) = 0, \textit{ i. e.}$$

$$P_1 b_1 = P_2 b_2.$$

Two such couples are, therefore, in equilibrium if the product of one force, and its distance from the opposite force, are as great in the one couple as in the other.

A pair of equal opposite forces is called simply a *couple*, and the product of one of the forces and its normal distance from the other force, the *moment of the couple*. From the above, two couples acting in opposite directions are in equilibrium, if they have equal moments.

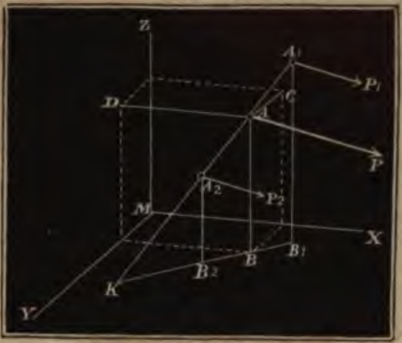
If we substitute in the formula (§ 87) for the arm a of the resultant:

$$a = \frac{P_1 a_1 + P_2 a_2 + \ldots}{P}$$

$P = 0$ without the sum of the statical moments becoming null; we obtain likewise $a = \infty$, a proof that in this case also there is no resultant, but only a couple, possible.

Example. If a couple consists of the forces $P_1 = 25$ lbs., and $-P_1 = -25$ lbs., and another of the forces $-P_2 = -18$ lbs., and $P_2 = 18$ lbs.; if, lastly, the normal distance of the first pair $= 3$ feet for the condition of equilibrium, the normal distance of the second must amount to $= \frac{25 \times 3}{18} = 4\frac{1}{6}$ feet.

Fig. 59.



§ 91. *Centre of Parallel Forces.*—If the parallel forces lie in different planes, their union may be effected in the following manner. If the straight line $A_1 A_2$, Fig. 59, which unites the points of application of two parallel forces P_1 and P_2, be prolonged to the plane XY between the rectangular axes MX, MY, and if the point of intersection K be taken for the initial point, we shall in this manner obtain for the point of application A of the resultant $(P_1 + P_2)$ of these forces,

$$(P_1 + P_2) \,.\, KA = P_1 \,.\, KA_1 + P_2 \,.\, KA_2.$$

As now B, B_1, and B_2 are the projections of the points of application A, A_1, A_2, on the plane XY, we have:

$AB : A_1B_1 : A_2B_2 = KA : KA_1 : KA_2$, and therefore also

$(P_1 + P_2)\, AB = P_1 \,.\, A_1\, B_1 + P_2 \,.\, A_2\, B_2$.

If we designate by z_1, z_2, z_3, &c., the normal distances A_1B_1, A_2B_2, A_3B_3, &c., of the points of application from the principal plane XY, and by w_1 that of the point A from the same plane, we have for the two forces:

$(P_1 + P_2)\, w_1 = P_1\, z_1 + P_2\, z_2$; for three or more, and generally

$(P_1 + P_2 + P_3 + \dots)\, w = P_1 z_1 + P_2 z_2 + P_3 z_3 \dots$ Consequently

$$1.\ w = \frac{P_1 z_1 + P_2 z_2 + \dots}{P_1 + P_2 + \dots}.$$

If we put likewise the distances AC and AD of the point of application of the resultant from the planes XZ and $YZ = v$ and u, we then obtain:

$$2.\ v = \frac{P_1 y_1 + P_2 y_2 + \dots}{P_1 + P_2 \dots} \text{ and}$$

$$3.\ u = \frac{P_1 x_1 + P_2 x_2 + \dots}{P_1 + P_2 + \dots}$$

The three distances u, v, w from the principal planes, as for example, from the floor and the two side walls of a room, fully determine the point A, for it is the eighth terminating point of the parallelopiped, constructed from u, v, w, consequently, in such a system there is but one single point of application of the resultant.

As the three formulæ for u, v, w, do not contain the angles which the forces make with the principal planes, the point of application is independent of these forces, and also of their directions; the whole system admits, therefore, of being turned about this point without its ceasing to be the point of application, provided only that in this turning the parallelism of the forces be preserved.

In such a system of parallel forces the product of a force, and the distance of its point of application from a plane or line, is called the moment of the force with reference to this plane or line, and generally, the point of application of the resultant is called the *centre of parallel forces*. The distance of the centre of a system of parallel forces from any plane or line whatever, (the latter when the forces lie in the same plane,) is obtained, when the sum of the moments is divided by the sum of the forces.

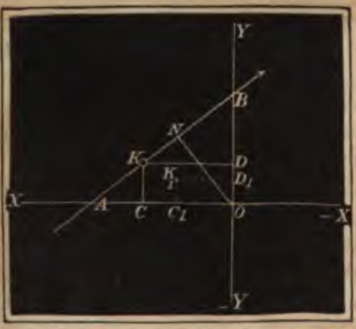
						SUM.
Example.	If the forces are	P_n	5	—7	10	3 lbs.
	the distances	x_n	1	2	0	9 ft.
	" "	y_n	2	4	5	3 "
	" "	z_n	8	3	7	10 "
	The moments are	$P_n\,x_n$	5	—14	0	36 ft. lbs.
	" "	$P_n\,y_n$	10	—28	50	12 "
	" "	$P_n\,z_n$	40	—21	70	40 "

Now, if the sum of the forces $= 19 - 7 = 12$ lbs., the distances of the central point of this system from the three principal planes are consequently:

$$u = \frac{5+36-14}{12} = \frac{27}{12} = \frac{9}{4} = 2{,}25 \text{ feet};$$
$$v = \frac{10+50+12-28}{12} = \frac{44}{12} = \frac{11}{3} = 3{,}66\ldots \text{feet};$$
$$w = \frac{40+70+40-21}{12} = \frac{129}{12} = \frac{43}{3} = 10{,}75 \text{ feet.}$$

§ 92. *Forces in Space.*—If it be required to unite a system constituted of differently directed forces, a plane must be carried through the system, the different points of application transferred to this plane, and each force resolved into two component forces, the one coinciding with the plane, the other at right angles to it. If $\beta_1, \beta_2 \ldots$ are the angles under which the plane is intersected by the directions of the forces, then the normal forces are $P_1 \sin. \beta$, $P_2 \sin. \beta \ldots$, and those in the plane $P_1 \cos. \beta_1$, $P_2 \cos. \beta_2$, &c. The latter from § 88, and the former from the last § 91 may be combined to a resultant. In general, the directions of both resultants will nowhere intersect each other, and accordingly a composition of these is impossible, but if the resultant of parallel forces passes through a point K, Fig. 60, in the direction AB of the resultant of the forces in the plane (the plane of the paper) a composition is then possible. If we put the distances $OC = DK = u$, and $OD = CK = v$ for the point of application of the first resultant, on the other hand the arm ON of the second $= a$, and the angle BAO, at which it intersects the axis XX $= \alpha$, the condition for the possibility of a composition is:

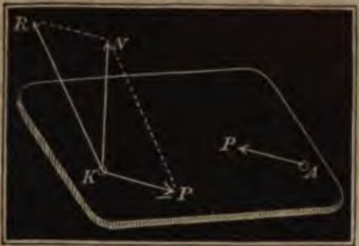
Fig. 60.



$$u \sin. \alpha + v \cos. \alpha = a.$$

If this equation is not satisfied, if, for example, the resultant of the normal forces passes through K_1, the reduction of the whole system of forces to a resultant is then impossible, but it readily admits of being reduced to a resultant R, Fig. 61, and a couple $P, -P$, if the resultant N of the parallel components is resolved into the forces $-P$ and R, of which the one is equal, and directed parallel and opposite to the resultant P of the forces in the plane.

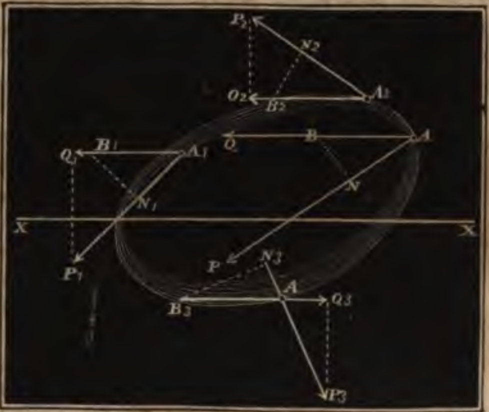
Fig. 61.



§ 93. *Principle of Virtual Velocities.*—If a system of forces P_1, P_2, P_3, acting in a plane, Fig. 62, is progressive, *i. e.* moves forward so that all the points of application $A_1, A_2, A_3, \ldots$ pass through equal parallel spaces $A_1 B_1, A_2 B_2, A_3 B_3$, the effect of the resultant (in the sense of § 80) is equivalent to those of the components, and in a state of equilibrium therefore $= 0$. If the projections $A_1 N_1$, $A_2 N_2$, &c., coinciding with the directions of the forces of the common spaces $A_1 B_1, = A_2 B_2$, &c., $= s_1, s_2$, then the mechanical effect of the resultant is:

$$Ps = P_1 s_1 + P_2 s_2 + \ldots$$

Fig. 62.



This law follows from one of the formulæ of § 88, according to which the component of the resultant running parallel with the axis XX is equal to the sum $Q_1 + Q_2 + \ldots$ &c., of the similarly running components of the forces P_1, P_2; now from the similarity of the triangles $A_1 B_1 N_1$ and $A_1 P_1 Q_1$, there follows the proportion

$$\frac{Q_1}{P_1} = \frac{A_1 N_1}{A_1 B_1} = \frac{s_1}{AB}, \text{ and from this:}$$

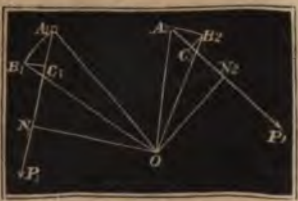
$$Q_1 = \frac{P_1 s_1}{AB},\ Q_2 = \frac{P_2 s_2}{AB}, \text{ \&c.,}$$

we may, therefore, in place of

$$Q = Q_1 + Q_2 + \ldots \text{ put}$$

$$Ps = P_1 s_1 + P_2 s_2 + \ldots$$

Fig. 63.



§ 94. If the system of forces P_1, P_2, &c., Fig. 63, be made to revolve a very little about the point O, the law of the principle of virtual velocities enunciated above in § 80 and § 93 holds equally good, as may be proved in the following manner. From § 86 the moment $P \,.\, ON$ of the resultant is equivalent to the sum of the moments of the components, so that:

$$Pa = P_1 a_1 + P_2 a_2 + \ldots$$

The space $A_1 B_1$ corresponding to a revolution through the small angle $A_1 O B_1 = \phi^0$ or the arc $\phi = \frac{\phi^0}{180} \cdot \pi$, is perpendicular to the diameter OA_1, therefore, the triangle $A_1 B_1 C_1$, which is formed if a perpendicular line $B_1 C_1$ be let fall on the direction of the force, is similar to the triangle $OA_1 N_1$ determined by the arm $ON_1 = a_1$, and accordingly

$$\frac{ON_1}{OA_1} = \frac{A_1 C_1}{A_1 B_1}$$

If the virtual velocity $A_1 C_1 = \sigma_1$ and the arc $A_1 B_1 = OA_1 \,.\, \phi$, we then obtain:

$$a_1 = \frac{OA_1 . \sigma_1}{OA_1 . \phi} = \frac{\sigma_1}{\phi}, \text{ also } a_2 = \frac{\sigma_2}{\phi}, \text{ \&c.}$$

If these values be substituted in the above equation for a_1, a_2, we then have

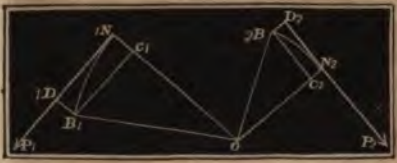
$$\frac{P\sigma}{\phi} = \frac{P_1 \sigma_1}{\phi} + \frac{P_2 \sigma_2}{\phi} + \ldots \text{ \&c.},$$

or, as ϕ is a common divisor,

$$P\sigma = P_1 \sigma_1 + P_2 \sigma_2 + \ldots \text{ \&c., the same as in § 80.}$$

So that, for small revolutions the mechanical effect $(P\sigma)$ of the resultant is equivalent to the sum of the mechanical effects of the components.

Fig. 64.



§ 95. The principle of virtual velocities holds likewise for arbitrarily great revolutions, if instead of the virtual velocities of the points of application, the projections $N_1 D_1$, $N_2 D_2$, &c., Fig. 64, of the spaces commencing at the points N_1, N_2, be introduced, and their values

$$B_1 C_1 = OB_1 \; sin. \; N_1 O B_1 = a_1 \; sin. \; \phi,$$
$$B_2 C_2 = OB_2 \; sin. \; N_2 O B_2 = a_2 \; sin. \; \phi, \text{ \&c.},$$

be substituted for σ_1, σ_2, we then obtain

$$Pa \; sin. \; \phi = P_1 a_1 \; sin. \; \phi + P_2 a_2 \; sin. \; \phi \ldots +,$$ or, dividing by $sin. \; \phi$,

$$Pa = P_1 a_1 + P_2 a_2 + \ldots,$$

the known equation for statical moments.

This principle is correct also for finite revolutions, if the directions of the forces revolve simultaneously with the system, or if, while the point of application incessantly changes, the arm $ON_1 = OB_1$ remains invariable, then from

$$Pa = P_1 a_1 + P_2 a_2,$$

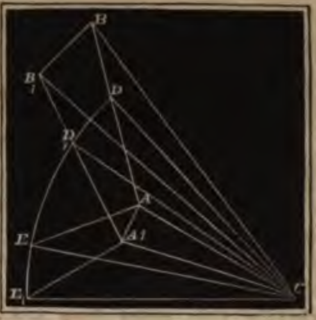
and multiplying by ϕ, we have

$$Pa\phi = P_1 a_1 \phi + P_2 a_2 \phi + \ldots, \; i.\, e.$$
$$P\sigma = P_1 \sigma_1 + P_2 \sigma_2 + \ldots,$$

if σ, σ_1, σ_2, &c., designate the circular arcs, $N_1 B_1$, $N_2 B_2$, &c., of the points N, N_1, &c.

§ 96. Every small motion or displacement of a body in a plane may be regarded as a small revolution about a movable centre, and may be proved in the following manner. Let two points A and B, Fig. 65, of this body (this surface or line) be advanced by a small motion to A_1 and B_1, let also $A_1B_1 = AB$. If at these points we draw perpendiculars to the small spaces described AA_1 and BB_1, they will intersect at a point C, from which as a centre AA_1 and BB_1 may be considered the circular arcs described. Now from the equalities $AB = A_1B_1$, $AC = A_1C$, and $BC = B_1C$, the triangles ABC and A_1B_1C are equal, therefore, also the $\angle B_1CA_1 = \angle BCA$ and the $\angle ACA_1 = \angle BCB_1$. If we make $A_1D_1 = AD$, we obtain from the equality of the $\angle$s D_1A_1C and DAC, and from that of the sides CA_1 and CA in CA_1D_1 and CAD, again two congruent triangles in which $CD_1 = CD$, and $\angle A_1CD_1 = \angle ACD$. Consequently any arbitrary point D in AB, by its small advancement, describes a circular arc DD_1. If lastly E be any point without the line AB, and rigidly connected with it, the small space EE_1 may be regarded as the arc of a circle from C as a centre, for if we make the $\angle E_1A_1B_1 = EAB$ and the distance $A_1E_1 = AE$, we again obtain two congruent triangles E_1A_1C and EAC with equal sides CE_1 and CE, and equal $\angle$s A_1CE_1 ACE, and the same may be shown for every other point rigidly connected with AB. We may consequently regard every small motion of a surface rigidly connected with AB, or of a rigid body, as a small revolution about a centre, which is given when the point of intersection C is determined, in which the perpendiculars to the paths AA_1 and BB_1 of the two points of the body intersect each other.

Fig. 65.



§ 97. From § 94, for a small revolution of a system of forces, the mechanical effect of the resultant is equivalent to the algebraical sum of its components; from § 95, every small displacement may be regarded as a small revolution: hence the law of the principle of virtual velocities above enunciated is, therefore, applicable to every small motion of a rigid body or system of forces.

If equilibrium obtain in a system of forces, *i. e.* if the resultant be null, the sum of the mechanical effects must be also null for a small arbitrary motion. If inversely for a small motion of the body, the sum of the effects be null, equilibrium does not from this necessarily follow; the sum for all possible small displacements must be $= 0$, if equilibrium is to take place. Since the formula expressing the law of virtual velocities only fulfils one condition of equilibrium, it is requisite for equilibrium that this law be satisfied, at least for as many motions as can be made from these conditions for example, in a system of forces in a plane, for the three motions independent of each other.

CHAPTER II.

CENTRE OF GRAVITY.

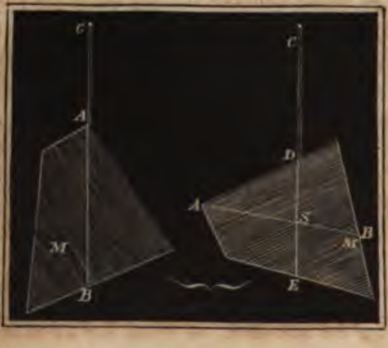
§ 98. *Centre of Gravity.*—The weights of the parts of a heavy body form a system of parallel forces, whose resultant is the weight of the whole, and whose centre may be determined from the three formulæ of § 91. This middle point of a body or system of bodies is called the centre of gravity, and also the centre of the mass of the body or system of bodies. If a body be turned about its centre of gravity, this point does not cease to be the central point of gravity, for if the three planes, to which the points of application of the separate weights are referred, revolve at the same time with the body, the position of the directions of force to these planes alone changes by this revolution, the distances of the points of application from these planes remain invariable. The centre of gravity is, therefore, that point of a body in which its weight acts vertically downwards, and which must be, therefore, supported, and fixed, in order that in every position the body may remain at rest.

§ 99. Every vertical straight line in which this point lies is called the *line of gravity;* and every plane passing through the centre of gravity, a *plane of gravity.* The centre of gravity is determined by the intersection of two lines of gravity, or that of a line of gravity and a plane of gravity, or by the intersection of the planes of gravity.

Since the point of application may be displaced at will in the direction of force, without changing the action of the force, so a body is in any position in equilibrium if a point in the vertical line passing through the centre of gravity is fixed.

If a body *M*, Fig. 66, be suspended by a thread *CA*, in its prolongation *AB* we have a line of gravity, and if it be similarly suspended by a second line, we get a second line of gravity *DE*. The intersection *S* of both lines is the centre of gravity of the body. If the body be suspended upon an axis, or be brought upon

Fig. 66.



a sharp edge (knife edge) into a state of equilibrium, we shall obtain in the vertical plane passing through the axis, or through the knife edge, a plane of gravity, &c. Experimental determinations of the centre of gravity, as just pointed out, are rarely applicable; we have generally to make use of geometrical rules, which will presently be given for the determination of this point with accuracy.

In many bodies, for example, in rings, the centre of gravity falls without the mass of the body. If such a body is to be fixed in its centre of gravity, it is necessary to connect a second body with the first, in such a manner that the centres of gravity of both may coincide.

§ 100. *Determination of the Centre of Gravity.*—If x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, &c., be the distances of the parts of a heavy body from the three planes xz, yz, xy, and the weights of these parts be P_1, P_2, P_3, &c., we then have the distances of the centre of gravity from these three planes,

$$x = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots},$$

$$y = \frac{P_1y_1 + P_2y_2 + P_3y_3 + \ldots}{P_1 + P_2 + P_3 + \ldots},$$

$$z = \frac{P_1z_1 + P_2z_2 + P_3z_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}.$$

If the volumes of these parts be V_1, V_2, V_3, &c., and their densities γ_1, γ_2, γ_3, &c., we may put therefore

$$x = \frac{V_1\gamma_1x_1 + V_2\gamma_2x_2 + \ldots}{V_1\gamma_1 + V_2\gamma_2 + \ldots}.$$

If the body be homogeneous, *i. e.* all parts of the same density γ, then:

$$x = \frac{(V_1x_1 + V_2x_2 + \ldots)\gamma}{(V_1 + V_2 + \ldots)\gamma},$$

or since the common factor γ above and below is cancelled:

$$1.\ x = \frac{V_1x_1 + V_2x_2 + \ldots}{V_1 + V_2 + \ldots},$$

$$2.\ y = \frac{V_1y_1 + V_2y_2 + \ldots}{V_1 + V_2 + \ldots},$$

$$3.\ z = \frac{V_1z_1 + V_2z_2 + \ldots}{V_1 + V_2 + \ldots}.$$

We may also, instead of the weights, substitute the volumes of the separate parts, and thereby make the determination of the centre of gravity a problem of pure geometry.

When bodies are a little extended in one or in two dimensions, as thin plates, fine wires, &c., they may be regarded as surfaces or lines, and their centres of gravity likewise determined with the help of the three last formulæ, if for the volumes V_1, V_2, the arms or lengths be substituted.

§ 101. In regular figures the centre of gravity coincides with the centre of figure, as in dice, cubes, spheres, equilateral triangles, circles, &c. Symmetric figures have their centre of gravity in the plane or axis of symmetry. The plane of symmetry *ABCD* divides a body *ADFE*, Fig. 67, into two congruent halves; the portions on both sides of this plane are equal; the moments also on the one side are equal to those on the other, and, consequently, the centre of gravity falls within this plane. Because the axis of symmetry *EF* cuts the plane surface *ABCD*, Fig. 68, into two congruent parts, here the portions on the one side are equal to those on the other; the moments also on both sides are equal, and the centre of gravity of the whole lies in this line. Lastly, the axis of symmetry *KL* of a body *ABGH*, Fig. 69, is its line of gravity, because it arises from the intersection of two planes of symmetry, *ABCD* and *EFGH*. For this reason, the centre of gravity of a cylinder, of a cone, and of a surface of revolution, or of a rotating body formed on the potter's wheel, lies in the axis of these bodies.

Fig. 67.

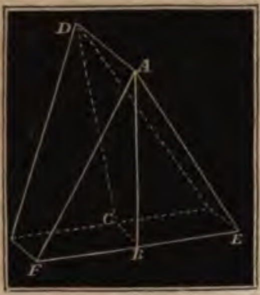


Fig. 68.

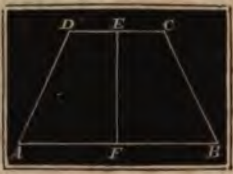


Fig. 69.

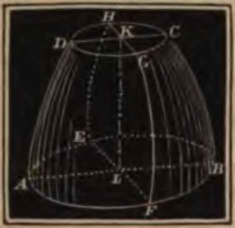
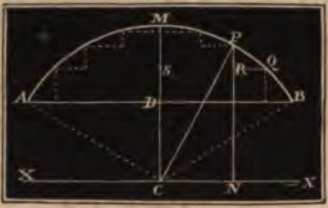


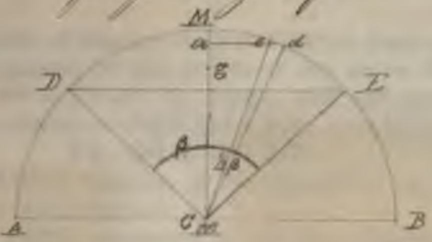
Fig. 70.



§ 102. *Centre of Gravity of Lines.*—The centre of gravity of a straight line lies in its middle.

The centre of gravity of a circular arc $AB = b$, Fig. 70, lies in the diameter *CM*, and passes through the middle *M* of the arc, for this diameter is the axis of symmetry of this arc. But in order to find the distance $CS = x$ of the centre of gravity *S* from the middle point, or *centre of the circle*, the arc must be divided into many elementary parts, and statical moments of these, with reference to an axis $\overline{XX}$

Centre of gravity of a circular Arc



Let EMD be a circular arc. It is known that the centre of gravity g will be upon the line CM which bisects the arc ED. The distance Cg is therefore the line required. Let the radius $AC = r$, the angle ECD which measures the arc $= \beta$, whence the angle $ECM = \frac{1}{2}\beta$. Take any angle $\varphi = MCe$ and the angle $d\varphi = eCd$, hence the moment of the line $ed = rd\varphi . r\cos\varphi$. For the line or arc $ed = rd\varphi$ and its arm with reference to AB is $Ca = r\cos\varphi$. The integral of this moment $(r^2 \cos\varphi\, d\varphi)$ taken between the limits $\frac{1}{2}\beta$ and $-\frac{1}{2}\beta$ will be the moment of the arc ED. The integral of $rd\varphi$ taken between the same limits gives the arc ED. The quotient of the first integral divided by the ~~first~~ second gives the arm of the whole arc ED with reference to the line AB.

passing through the centre C and parallel to the chord $AB = s$, be determined.

If PQ be a part of the arc, and PN be its distance from $X\overline{X}$, then the statical moment of this portion of the arc $= PQ \, . \, PN$. If now the radius $PC = MC = r$ be drawn, and QR parallel to AB, we obtain the two similar Δ^s PQR and CPN, for which:

$$PQ : QR = CP : PN,$$

from which the statical moment of the elementary arc $PQ \, . \, PN = QR \, . \, CP = QR \, . \, r$ is determined.

Now, for the statical moments of all the remaining arcs, the radius r is a common factor, and the sum of all the projections QR of the elementary arcs is equal to the chord corresponding to the projection of the whole arc; it follows, therefore, that the moment of the whole arc is also = the chord (s) times the radius r. If this moment be put equal to the arc (b) times the distance x, and therefore $bx = sr$, we then obtain:

$$\frac{x}{r} = \frac{s}{b}, \text{ and } x = \frac{sr}{b}.$$

So that the distance of the centre of gravity, from the middle point is to the radius, in the ratio of the arc to the chord.

If the angle at the centre ACB of the arc b be $= \beta^0$, the arc corresponding to the diameter 1 is $\beta = \frac{\beta^0}{180^0} \, . \, \pi$. We have then $b = \beta r$, and $s = 2 \, r \, sin. \frac{\beta}{2}$; whence it follows that, $x = \frac{2 \, sin. \frac{1}{2} \, \beta \, . \, r}{\beta}$.

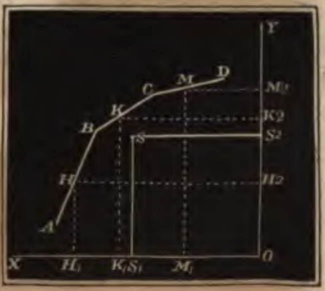
For the semicircle $\beta = \pi$ and $sin. \frac{\beta}{2} = 1$; therefore, $x = \frac{2}{\pi} r = 0{,}6366 \ldots r = \frac{7}{11} r$ nearly.

§ 103. To find the centre of gravity of a polygon or a connection of lines $ABCD$, Fig. 71, we must seek the distances of the middle points H, K, M, of the lines $AB = L_1$, $BC = L_2$, $CD = L_3$, &c., from two axes OX and OY, viz: $HH_1 = y_1$, $HH_2 = x_1$, $KK_1 = y_2$, $KK_2 = x_2$, &c.; the distances of the centre of gravity sought from these axes are then:

$$SS_2 = x = \frac{L_1 x_1 + L_2 x_2 + \ldots}{L_1 + L_2 + \ldots},$$

$$SS_1 = y = \frac{L_1 y_1 + L_2 y_2 + \ldots}{L_1 + L_2 + \ldots},$$

Fig. 71.

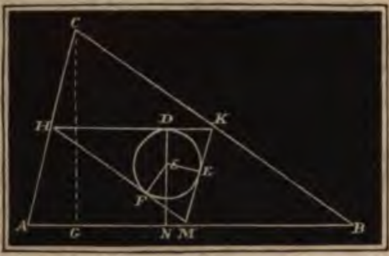


For example, the distance of the centre of gravity S of a wire bent into the form of a Δ ABC, Fig. 72, from the base is:

$$NS = x = \frac{\frac{1}{2} \, ah + \frac{1}{2} \, bh}{a + b + c} = \frac{a + b}{a + b + c} \, . \, \frac{h}{2},$$

if the sides opposite to the angles A, B, C be designated by a, b, c, and the height CG by h.

Fig. 72.

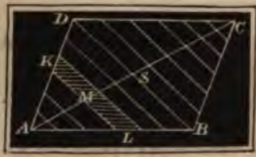


If the middle points H, K, M, of the sides of the triangle be connected with each other, and in the triangle so obtained a circle be described, its centre will coincide with the centre of gravity S, for the distance SD from one side HK is

$$= DN - SN = \frac{h}{2} - \frac{a+b}{a+b+c} \cdot \frac{h}{2} = \frac{ch}{2(a+b+c)} = \frac{\Delta ABC}{a+b+c} =$$

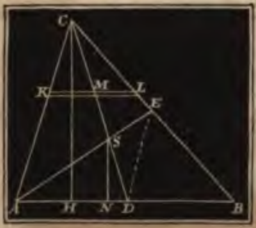
the distances SE and SF from the other sides.

Fig. 73.



§ 104. *Centre of Gravity of Plane Figures.*—The centre of gravity of a parallelogram $ABCD$, Fig. 73, lies in the point of intersection of its diagonals, for all strips, such as KL, which are formed by drawing lines parallel to one of its diagonals BD, are bisected by the other diagonals AC; each of the diagonals, therefore, is a line of gravity.

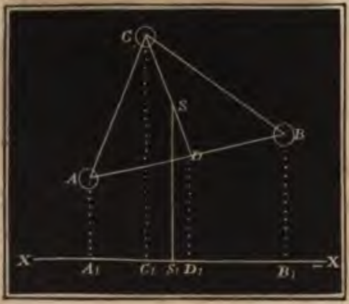
Fig. 74.



In a plane $\Delta\ ABC$, Fig. 74, every line CD from one angle to the middle D of the opposite side AB, is a line of gravity, for the same bisects all the elements KL of the Δ which are given when lines parallel to AB are drawn. If from a second angle A a second line of gravity be drawn to the middle E of the opposite side BC, the point of intersection of the two will give the centre of gravity of the whole Δ.

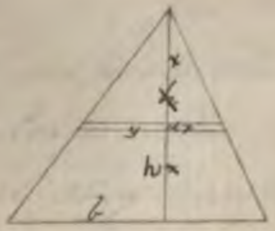
Because $BD = \frac{1}{2} BA$ and $BE = \frac{1}{2} BC$, DE is parallel to AC and $= \frac{1}{2} AC$, and $\Delta\ DES$ similar to the $\Delta\ CAS$, and lastly, $CS = 2\,SD$. If further we add SD, it follows that $CS + SD$, or $CD = 3\,DS$, and, therefore, inversely, $DS = \frac{1}{3}\,CD$. The centre of gravity S lies at $\frac{1}{3}$ of the line CD from the middle point D of the base, and at $\frac{2}{3}$ of the same from the angle C. If CH and SN be drawn perpendicular to the base, we have also $SN = \frac{1}{3}\,CH$; the centre of gravity S is at $\frac{1}{3}$ of the height from the base of the Δ.

Fig. 75.



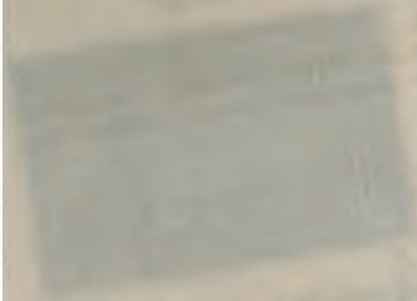
The distance SS_1 of the centre of gravity of a $\Delta\ ABC$, Fig. 75, from an axis $X\bar{X}$ is $= DD_1 + \frac{1}{3}$

Centre of gravity of a triangle



Let $y\,dx$ = Area of elementary section.
Then $y\,dx\,x$ = Moment thereof.
We have $\frac{x}{y} = \frac{h}{b}$. $y = \frac{b}{h}x$. $\int_{x=0}^{x=h} y\,dx\,x = \frac{\int_{x=0}^{x=h}\frac{b}{h}x^2dx}{\int_{x=0}^{x=h}\frac{b}{h}x\,dx} = \left.\frac{\frac{x^3}{3}}{\frac{x^2}{2}}\right\}_{x=0}^{x=h} = \frac{2}{3}x = \frac{2}{3}h$ = distance of the centre of gravity from the vertex of the triangle.

[illegible]



[illegible]

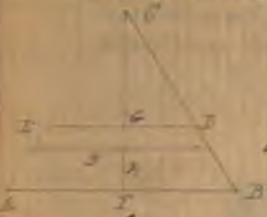
[illegible]

[illegible]

[illegible]

$(CC_1 - DD_1)$, but $DD_1 = \frac{1}{2}(AA_1 + BB_1)$, consequently, $x = SS_1$ $= \frac{1}{3} CC_1 + \frac{2}{3} \cdot \frac{1}{2}(AA_1 + BB_1)$:
$= \frac{AA_1 + BB_1 + CC_1}{3}$, *i. e.*, the arithmetical mean of the distances of the three angular points.

Since the distance of the centre of gravity is determined in the same manner by three equal weights at the angular points of a △, so the



To find the centre of gravity of a Trapezium. Let us desig the two parallel bases AB an ED by B and b respectively. Term GF the height of the trapezium h and prolon AE and DB until they meet making the triangle ABC. We have now the proportion $CF : CF - h :: B : b \therefore CF = \frac{Bh}{B-b}$. Let us now take an ele-ment ydx its arm with respect to the line AB is x hence its moment with respect to this line equals $yxdx$ which divided by ydx gives its arm with rega to AB or integrating within the proper limits the distance of ~~its~~ the centre of gravity of the trapezium from the line AB. But in ora

fore

$$= bh \cdot \frac{h}{2} + \frac{(B-b)h}{2} \cdot \frac{h}{3} = (B+2b)\frac{h^2}{6},$$

but the area of the trapezium is $= (B+b)\frac{h}{2}$; it follows, therefore, that the normal distance of the centre of gravity S from the base is

$$HS = \frac{\frac{1}{6}(B+2b)h^2}{\frac{1}{2}(B+b)h} = \frac{B+2b}{B+b} \cdot \frac{h}{3}.$$

to get an expression which we can in-tegrate we must substitute for one of the variables its value in terms of the other. We have from the fig. the proport

$$y : B :: \frac{Bh}{B-b} - x : \frac{Bh}{B-b} \qquad y = \frac{\frac{B(Bh-(B-b)x)}{B-b}}{\frac{Bh}{B-b}} = \frac{\cancel{B}(Bh-(B-b)x)}{\cancel{B}\ h}$$

$$\frac{\int_0^h y\,dx\,x}{\int y\,dx} = \int_0^h \frac{\frac{Bhx\,dx-(B-b)x^2dx}{h}}{\frac{Bh\,dx-(B-b)x\,dx}{h}} = \frac{\frac{1}{2}Bh^2-\frac{1}{3}(B-b)h^3}{Bh^2-\frac{1}{2}(B-b)h^2} = \frac{1}{6}\ \frac{3Bh-2Bh+2bh}{\frac{1}{2}(2B-B-b)}$$

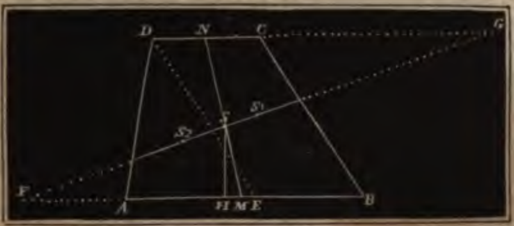
$= \frac{1}{3}h\frac{B+2b}{B+b}$ „ equals the height of the center of Gravity of a Trapezium from the major base.

$(CC_1 - DD_1)$, but $DD_1 = \frac{1}{2}(AA_1 + BB_1)$, consequently, $x = SS_1$ $= \frac{1}{3} CC_1 + \frac{2}{3} \cdot \frac{1}{2} (AA_1 + BB_1)$: $= \frac{AA_1 + BB_1 + CC_1}{3}$, *i. e.*, the arithmetical mean of the distances of the three angular points.

Since the distance of the centre of gravity is determined in the same manner by three equal weights at the angular points of a △, so the centre of gravity of a plane triangle coincides with the centre of gravity of these three equal weights.

§ 105. The determination of the centre of gravity S of a trapezium $ABCD$, Fig. 76, may be made in the following manner. The straight line MN, which connects the middle points of the two bases AB and CD with each other, is a line of gravity of the trapezium; for lines drawn parallel to the bases decompose the trapezium into elementary parts, whose middle points or centres of gravity lie in MN. Now to determine completely the centre of gravity S, we have only, therefore, to find its distance SH from a base AB.

Fig. 76.



Let B represent the one, and b the other of the parallel sides AB and CD of the trapezium, h the height or the normal distance of these sides. Let DE be now drawn parallel to the side BC, we shall then obtain a parallelogram $BCDE$ of the area bh, and whose centre of gravity is S_1, and distance from $AB = \frac{h}{2}$, and a △ ADE of the area $\frac{(B - b)h}{2}$ and centre of gravity S_2, and whose distance from $AB = \frac{h}{3}$.

The statical moment of the trapezium, about the line AB, is therefore

$$= bh \cdot \frac{h}{2} + \frac{(B - b)h}{2} \cdot \frac{h}{3} = (B + 2b)\frac{h^2}{6},$$

but the area of the trapezium is $= (B + b)\frac{h}{2}$; it follows, therefore, that the normal distance of the centre of gravity S from the base is

$$HS = \frac{\frac{1}{6}(B + 2b)h^2}{\frac{1}{2}(B + b)h} = \frac{B + 2b}{B + b} \cdot \frac{h}{3}.$$

To find the centre of gravity by construction, let the two bases be prolonged, the prolongations CG made $= B$ and $AF = b$, and the two extreme points obtained, F and G, connected by a straight line: the point of intersection S with the middle line MN will be the centre of gravity sought; for, from $HS=\frac{B+2b}{B+b}\cdot\frac{h}{3}$, it follows that

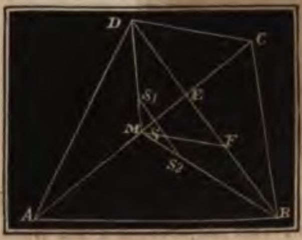
$$MS=\frac{B+2b}{B+b}\cdot\frac{MN}{3} \text{ and } NS=\frac{2B+b}{B+b}\cdot\frac{MN}{3}; \text{ and}$$

$$\frac{MS}{NS}=\frac{B+2b}{2B+b}=\frac{\frac{1}{2}B+b}{B+\frac{1}{2}b}=\frac{MA+AF}{CG+NC}=\frac{MF}{NG},$$

which actually arises from the similarity of the triangles MSF and NSG.

§ 106. To find the centre of gravity of any other four-sided figure $ABCD$, Fig. 77, we may decompose it by the diagonal AC into two triangles, and from the foregoing, determine their centres of gravity S_1 and S_2, and thereby a line of gravity S_1S_2. If now the four-sided figures be decomposed into two other triangles by the diagonal BD, and their centres of gravity determined, we obtain another line of gravity, whose intersection with the first will give the centre of gravity of the whole figure.

Fig. 77.



We may effect this more simply if we bisect the diagonal AC in M, apply the greater part BE of the second diagonal to the less, so that $DF=BE$, join FM and divide it into three equal parts; the centre of gravity lies in the first point S from M, as may be proved in the following manner. $MS_1=\frac{1}{3}MD$ and $MS_2=\frac{1}{3}MB$, consequently S_1S_2 are parallel to BD, but SS_1 times $\triangle ACD = SS_2$ times $\triangle ACB$, or $SS_1 \cdot DE = SS_2 \cdot BE$; therefore, $SS_1 : SS_2 = BE : DE$. Now, $BE=DF$ and $DE=BF$, consequently $SS_1 : SS_2 = DF : BF$. The straight line MF intersects, therefore, the line of gravity S_1S_2 in the centre of gravity of the figure.

§ 107. If it be required to find the centre of gravity S of a polygon $ABCDE$, Fig 78, we must decompose the polygon into triangles, and determine their statical moments with reference to two rectangular axes $X\overline{X}$ and $Y\overline{Y}$.

If the co-ordinates $OA_1=x_1$, $OA_2=y_1$, $OB_1=x_2$, $OB_2=y_2$, &c., of the extremities are given, the statical moments of the triangles ABO, BCO, COD, &c., may be determined simply in the following manner. The area of $\triangle ABO$, from the remark below, $= D_1 = \frac{1}{2}(x_1y_2-x_2y_1)$; of the following $\triangle BCO=D_2=\frac{1}{2}(x_2y_3-x_3y_2)$, &c., the distance of the centre of gravity of $\triangle ABO$ from $Y\overline{Y}$ according to § 104 $=u_1=\frac{x_1+x_2+0}{3}=\frac{x_1+x_2}{3}$, from $X\overline{X}=v_1=\frac{y_1+y_2}{3}$; of the centre of gravity of $\triangle\, BCO = u_2 = \frac{x_2 + x_3}{2}$ and $v_2 = \frac{y_2 + y_3}{3}$, &c. If these distances are multiplied by the areas of the triangles, the moments of these last are obtained; and if the values so obtained are substituted in the formulæ:

$$u = \frac{D_1 u_1 + D_2 u_2 + \ldots}{D_1 + D_2 + \ldots}$$

$$v = \frac{D_1 v_1 + D_2 v_2 + \ldots}{D_1 + D_2 + \ldots},$$

we have the distances u and v of the centre of gravity from the axes YY and XX.

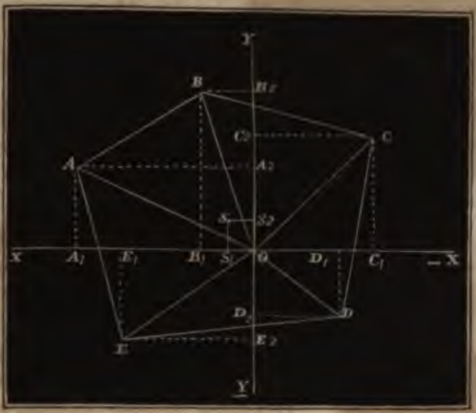


Fig. 78.

Example. A pentagon *ABCDE*, Fig. 78, is given by the following co-ordinates of its extremities *A, B, C,* &c.: to find the co-ordinates of its centre of gravity:

Co-ordinates given.		Twice the area of triangles.	Triple co-ordinates of centre of gravity.		Six times the statical moments.	
x	y		$3\,u_n$	$3\,v^n$	$6D_n u_n$	$6D_n v_n$
24	11	24 . 21 — 7 . 11 = 427	31	32	13237	13664
7	21	7 . 15 + 21 . 16 = 441	— 9	36	—3969	15876
—16	15	16 . 9 + 12 . 15 = 324	—28	6	—9072	1944
—12	— 9	12 . 12 + 18 . 9 = 306	+ 6	—21	1836	—6426
18	—12	18 . 11 + 24 . 12 = 486	+42	— 1	20412	— 486
		Sum: 1984			22444	24572

The distance of the centre of gravity from the axis $Y\overline{Y}$ is:

$$SS_2 = u = \frac{1}{3} \cdot \frac{22444}{1984} = 3{,}771,$$

and from the axis $X\overline{X}$:

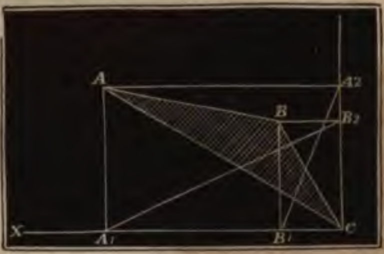
$$SS_1 = v = \frac{1}{3} \cdot \frac{24572}{1984} = 4{,}128.$$

Remark. If $CA_1 = x_1$, $CB_1 = x_2$, $CA_2 = y_1$, and $CB_2 = y_2$, the co-ordinates of the two angles of a triangle ABC, Fig. 79, whose third angle C coincides with the point of application of the system of co-ordinates, we have the area of the same:

$$D = \text{trapezium } ABB_1A_1 + \text{triangle } CBB_1 - \text{triangle } CAA_1$$
$$= \left(\frac{y_1 + y_2}{2}\right)(x_1 - x_2) + \frac{x_2 y_2}{2} - \frac{x_1 y_1}{2}$$
$$= \frac{x_1 y_2 - x_2 y_1}{2}.$$

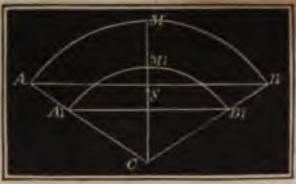
The area of this triangle is the difference of two other triangles, CB_2A_1 and CA_2B_1, and the one co-ordinate of a point is the base of the one, and the other co-ordinate the height of the other triangle, and inversely.

Fig. 79.



§ 108. The centre of gravity of the sector of a circle ACB, Fig. 80, coincides with that of a circular arc A_1B_1 which has the same angle with the sector, and whose radius CA_1 is two-thirds of the radius CA of the sector; for the sector may be divided by an infinity of radii into very small triangles, whose centres of gravity are distant two-thirds of the radius from the centre C, and these form by their continuity the arc $A_1M_1B_1$. The centre of gravity S of the sector lies in the radius CM, bisecting the surface, and at the distance $CS = x = \frac{\text{chord}}{\text{arc}} \cdot \frac{2}{3} CA = \frac{4}{3} \cdot \frac{\sin. \frac{1}{2}\beta}{\beta} \cdot r$; r representing the radius CA of the sector, and β the arc which measures the angle at the centre ACB.

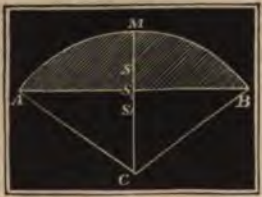
Fig. 80.

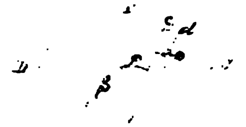


For the semi-circle $\beta = \pi$, $\sin. \frac{1}{2}\beta = \sin. 90^0 = 1$, therefore $x = \frac{4}{3\pi} r = 0{,}4244\, r$ or about $\frac{14}{33} r$. For a quadrant $x = \frac{4}{3} \cdot \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}\pi} r = \frac{4\sqrt{2}}{3\pi} r = 0{,}6002\, r$ and for a sixth part $x = \frac{4}{3} \cdot \frac{\frac{1}{2}}{\frac{1}{3}\pi} r = \frac{2}{\pi} r = 0{,}6366\, r$.

§ 109. The centre of gravity of a segment of a circle ABM, Fig. 81, is given, if we put the moment of the sector $ACBM$ equal to the sum of the moments of the segment and the moment of the triangle ACB. If r be the radius CA, s the chord AB, and A the area of the segment ABM, the moment of the sector = the sector × $CS_1 = \frac{r \,.\, \text{arc}}{2} \cdot \frac{\text{chord}}{\text{arc}} \cdot \frac{2}{3} r = \frac{1}{3} s r^2$, further the moment of the triangle = triangle × $CS_2 = \frac{s}{2} \sqrt{r^2 - \frac{s^2}{4}}$.

Fig. 81.





To find the centre of Gravity of a Sector of circle. Let the angle BEC $=\beta$ then FEC $=\frac{1}{2}\beta$. Lay off any line Ec making with EF the angle ϕ. Let us now consider the elementary sector cEd of which the side cd $= r\,d\phi$ may be considered a right line and therefore the elementary sector becomes a triangle whose area $= r\,d\phi\times\frac{1}{2}r$, ($r$ being the height of the triangle) $=\frac{1}{2}r^2d\phi$ and its moment with regard to the line AB equals its area into its arm $\frac{2}{3}r\cos\phi$ (from what we have previously proved with respect to the position of the centre of gravity of a triangle) hence the moment $=\frac{1}{3}r^3\cos\phi\,d\phi$, ∴ centre of gravity lies in line EF at distance from E $=\dfrac{\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\frac{1}{3}r^3\cos\phi\,d\phi}{\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\frac{1}{2}r^2d\phi}=\dfrac{\frac{1}{3}r^3 2\sin\frac{1}{2}\beta}{\frac{1}{2}r^2\,2\times\frac{1}{2}\beta}$

$=\frac{4}{3}r\dfrac{\sin\frac{1}{2}\beta}{\beta}$."

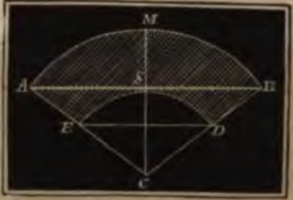
and on the straight CED. Hence [illegible]
[illegible] the difference [illegible]
and divide it [illegible]
[illegible]

$\frac{2}{3}\sqrt{r^2 - \frac{s^2}{4}} = \frac{sr^2}{3} - \frac{s^3}{12}$, and from this the moment of the segment: $A \,.\, CS = Ax = \frac{1}{3}sr^2 - \left(\frac{sr^2}{3} - \frac{s^3}{12}\right) = \frac{s^3}{12}$; consequently the distance sought is $x = \frac{s^3}{12\,A}$.

For the semi-circle $s = 2\,r$ and $A = \frac{1}{2}\pi\,r^2$, hence $x = \frac{8\,r^3}{12 . \frac{\pi\,r^2}{2}} = \frac{4\,r}{3\pi}$, as found above.

In like manner we may find the centre of gravity S of a portion of a ring $ABDE$, Fig. 82, which is the difference of two sectors ACB and DCE. If the radii be $CA = r$ and $CD = r_1$, and the chords $AB = s$ and $DE = s_1$, the statical moments of the sectors are: $\frac{s\,r^2}{3}$ and $\frac{s_1\,r_1^2}{3}$, therefore the statical moments of the portion of ring: $= \frac{sr^2 - s_1 r_1^2}{3}$, or (since $\frac{s_1}{s} = \frac{r_1}{r}$) is $= \frac{r^3 - r_1^3}{3} \cdot \frac{s}{r}$. But the area $= \frac{\beta\,r^2}{2} - \frac{\beta r_1^2}{2} = \beta\left(\frac{r^2 - r_1^2}{2}\right)$, provided that β represents the arc corresponding to the angle at the centre ACB; the centre of gravity, therefore, of the portion follows from the distance $CS = x = \frac{\text{moment}}{\text{area}} = \frac{r^3 - r_1^3}{r^2 - r_1^2} \cdot \frac{2}{3} \cdot \frac{s}{r\beta} = \frac{2}{3}\left(\frac{r^3 - r_1^3}{r^2 - r_1^2}\right) \cdot \frac{\text{chord}}{\text{arc}} = \frac{4}{3}\,\frac{\sin.\,\frac{1}{2}\,\beta}{\beta} \cdot \frac{r^3 - r_1^3}{r^2 - r_1^2}$.

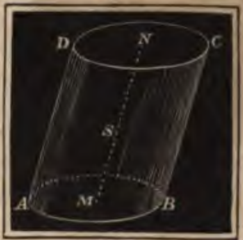
Fig. 82.



Example. The radii of the surfaces of a dome are: $r = 5$ ft., $r_1 = 3\frac{1}{2}$ ft., and the angle at the centre, $\beta^0 = 130^0$, then is the distance of the centre of gravity of these surfaces from their central point:

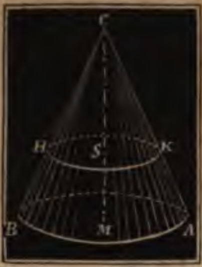
$$x = \frac{4}{3}\,\frac{\sin.\,65^0}{arc.\,130^0} \cdot \frac{5^3 - 3{,}5^3}{5^2 - 3{,}5^2} = \frac{4 \,.\, 0{,}9063}{3 \,.\, 2{,}2689} \cdot \frac{125 - 42{,}875}{25 - 12{,}25} = \frac{3{,}6252 \times 82{,}125}{6{,}8067 \times 12{,}75}$$
$$= 3{,}430 \text{ feet.}$$

Fig. 83.



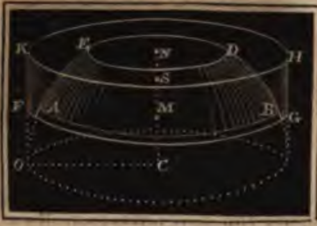
§ 110. *Centre of Gravity of Curved Surfaces.*—The centre of gravity of a curved surface (envelope) of a cylinder $ABCD$, Fig. 83, lies in the middle S of the axis MN of this body, for all the annular elements of the cylindrical envelope which are obtained by sections drawn through the body parallel to the base, are equal, and their centres of gravity lie in the axis; these centres of gravity form a uniform line of gravity. For the same reason the centre of gravity of the surfaces of a prism lies in the middle point of the straight lines connecting the centres of gravity of both the bases.

Fig. 84.



The centre of gravity of the envelope of a right cone ABC, Fig. 84, lies in the axis of the cone, and is one-third of this line from the base, or two-thirds from the vertex; for this curved surface may be decomposed into an infinite number of small triangles by straight lines, which are called the sides of the cone whose centres of gravity form a circle HK, which is distant two-thirds of the axis from the vertex, and whose centre of gravity or centre S lies in the axis CM.

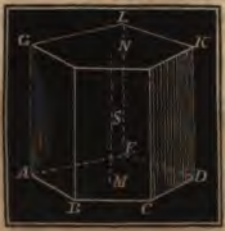
Fig. 85.



The centre of gravity of a spherical zone $ABDE$, Fig. 85, and likewise that of a spherical cup lies in the centre S of its height MN; for from the rules of geometry the zone has the same surface as a cylindrical envelope $FGHK$, whose height is equal to that of MN, and whose radius is equal to that of the radius CO of the spherical zone; and this equality also exists in the annular elements, which are obtained by carrying an infinite number of planes parallel to the circular bases through the same; according to this the centre of gravity of the zone coincides with that of the cylindrical envelope.

Remark. The centre of gravity of the surface of an oblique cone or oblique pyramid lies at about one-third of the height from the base, but not in the straight line passing from the vertex to the centre of gravity of the base, because slices parallel to the base decompose the surface into rings, which vary in breadth at different parts of their surface.

Fig. 86.



§ 111. *Centre of Gravity of Bodies.*—The centre of gravity of a prism AK, Fig. 86, is the centre S of the straight line which connects the centres of gravity M and N of both bases AD and GK, for the prism may be decomposed by sections parallel to the base into exactly congruent slices, whose centres of gravity lie in MN, and by their superposition make the line MN a uniform line of gravity.

For the same reason the centre of gravity of a cylinder lies in the middle of its axis.

The centre of gravity of a pyramid ADF, Fig. 87, lies in the straight line MF from the vertex F to the centre of gravity M of the base, for all slices as $NOPQR$, have from their similarity with the base, their centres of gravity in this line.

If the pyramid be triangular as $ABCD$, Fig. 88, each of the four

Centre of gravity of a portion of a Rin

Let BB_1C_1C be a portion of a ring. Let $AB = r$ and $AC = r_1$; let the angle B_1AB β ∴ the angle BAM $\frac{1}{2}\beta$. Let φ = any angle, $d\varphi$ = elementary angle; ρ = variable radius. $\rho\,d\rho\,d\varphi$ = elementary portion; $\rho\cos\varphi$ = arm of elementary portion. $\rho\,d\rho\,d\varphi \times \rho\cos\varphi$ = moment of elementary portion. The quotient of the integral of the moments taken between the limits of $\frac{1}{2}\beta$ and $-\frac{1}{2}\beta$, for φ, and r and r_1 for ρ divided by the integral of the arm of the elementary portion taken between the same limits is the distance AG of the centre of gravity G above the centre of A. Hence we now have

$$\frac{\int_{r_1}^{r}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\rho\,d\rho\,d\varphi\cos\varphi}{\int_{r_1}^{r}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\rho\,d\rho\,d\varphi} = \text{the distance of the}$$

centre of gravity above A. $= \frac{4}{3}\left(\frac{r^3-r_1^3}{r^2-r_1^2}\right)\frac{\sin\frac{1}{2}\beta}{\beta}$

Art 112 Remark The Pyramid should be considered as the aggregate of four right truncated prisms. These prisms are those in the figure which have for their bases 02_03_0, 02_01_0, 01_03_0, $2_01_03_0$, which are respectively truncated by the planes 02_03_0, 02_01_0, 01_03_0, $2_01_03_0$. The principle under which we are deducing this formula is, that the solidity of every solid is the product ~~into~~ of the area of the base into the height above the plane XY of the centre of gravity of the upper base. We follow this principle by taking the area of the bases of these prisms and finding the mean height of the centres of gravity. By taking ⅓ of the product of the respective bases by their respective heights we obtain the respective solidities of these prisms. By subtracting the solidity of truncated prism 02_03_0 from the sum of the solidities of the others we obtain the volume of the pyramid $1^{\circ}2^{\circ}3^{\circ}0$, which has its vertex 0 at the origin: the coördinates of the points 1, 2, 3 being respectively x_1, y_1, z_1, x_2, y_2, z_2, and $x_3\ y_3\ z_3$, angular points may be considered as vertices, and the opposite surfaces as bases; the centre of gravity S is determined by the intersection of two straight lines drawn from D and A to the centres of gravity M and N of the opposite surfaces ABC and BCD.

Fig. 87.

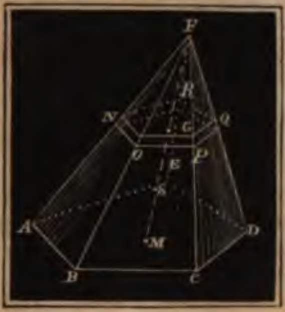
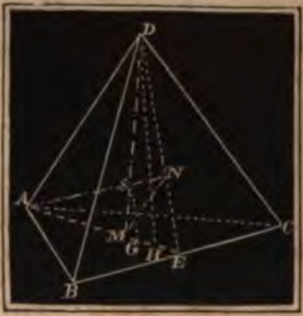


Fig. 88.

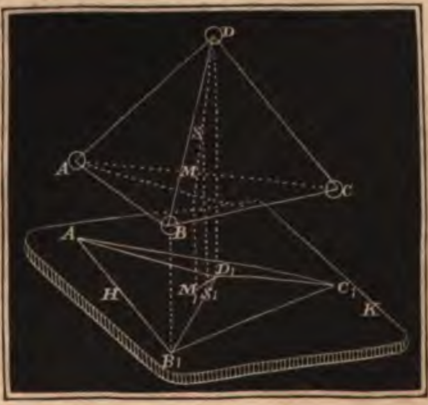


If the straight lines EA and ED be given, we then have from § 104 $EM = \frac{1}{3} EA$ and $EN = \frac{1}{3} ED$; therefore MN is parallel to AD and $= \frac{1}{3} AD$, and the $\triangle\ MNS$ similar to $\triangle\ DAS$. Again from this similarity we have $MS = \frac{1}{3} DS$, or $DS = 3\ MS$, also $MD = SD + MS = 4\ MS$, and inversely $MS = \frac{1}{4} MD$. Hence the centre of gravity is found to be one-fourth of the line joining the centre of gravity M of the base with the vertex D.

Further, if the heights DH and SG be given, and HM be drawn, we then obtain the two similar $\triangle^s$ DHM and SGM, in which from the foregoing $SG = \frac{1}{4} DH$. We may, therefore, say that the distance of the centre of gravity S of a triangular pyramid from the base is equal to one-fourth, and that from the vertex three-fourths of the height of the pyramid.

As every pyramid, and also every cone, is made up of an infinite number of three sided pyramids of the same height, the centre of gravity of every pyramid and cone is a fourth of the height from the base and three-fourths from the vertex. We may, therefore, find the centre of gravity of a pyramid or cone, if a plane be drawn parallel to the base at a distance one-fourth from the base, and the centre of gravity of the section or its intersection with the line joining the vertex and the centre of gravity of the base be determined.

Fig. 89.



§ 112. If the distances AA_1, BB_1, of the four angles of a triangular pyramid *ABCD*, Fig. 89, from a plane *HK* be known, the distance of the centre of gravity *S* from this plane is found from the mean value

$$SS_1 = \frac{AA_1+BB_1+CC_1+DD_1}{4},$$

The distance of the centre of gravity *M* of the base *ABC* from this plane is (§ 104):

$$MM_1 = \frac{AA_1+BB_1+CC_1}{3},$$

and that of the pyramid *S* is:

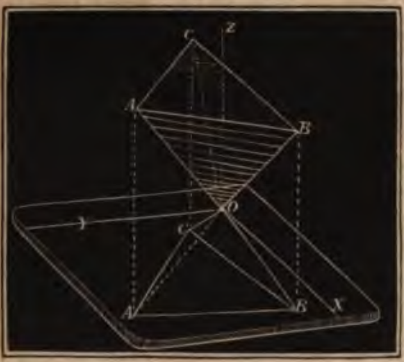
$$SS_1 = MM_1 + \tfrac{1}{4}(DD_1 - MM_1),$$

where DD_1 is the distance of the vertex: hence it follows by combining the two last equations, that:

$$SS_1 = \tfrac{3}{4}MM_1 + \tfrac{1}{4}DD_1 = \frac{AA_1+BB_1+CC_1+DD_1}{4}.$$

The distance of the centre of gravity of four equal weights applied to the angles of a triangular pyramid, is equivalent to the arithmetical mean $\frac{AA_1+BB_1+CC_1+DD_1}{4}$, consequently the centre of gravity of the pyramid corresponds with that of the system of weights.

Fig. 90.



Remark. The determination of the volume of a triangular pyramid, from the co-ordinates of its angles, is simple. If we draw planes *XY*, *XZ*, *YZ*, through the vextex *O* of such a pyramid *ABCO*, Fig. 90, and represent the distances of the angles *ABC* from these planes by z_1, z_2, z_3, y_1, y_2, y_3, and x_1, x_2, x_3, the volume of the pyramid will be

$$V = \tfrac{1}{6}(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1),$$

which will be given, if the pyramid be considered as an aggregate of four oblique prisms.

The distances of the centre of gravity of these pyramids from the three planes are:

$$x = \frac{x_1+x_2+x_3}{4},\ y = \frac{y_1+y_2+y_3}{4},\ \text{and}\ z = \frac{z_1+z_2+z_3}{4}.$$

§ 113. Since every polyhedron as *ABCDO*, Fig. 91, may be decomposed into triangular pyramids *ABCO*, *BCDO*, we may also find its centre of gravity *S* if we calculate the volumes, and the statical moments of the single pyramids.

If the distances of the angles *A*, *B*, *C*, &c., from the co-ordinate planes passing through the common vertex *O* of all the pyramids, are x_1, x_2, x_3, &c., y_1, y_2, y_3, &c., z_1, z_2, z_3, &c., the volumes of the single pyramids are:

$$V_1 = \pm\tfrac{1}{6}(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1),$$

$$V_2 = \pm\tfrac{1}{6}(x_2y_3z_4 + x_3y_4z_2 + x_4y_2z_3 - x_2y_4z_3 - x_3y_2z_4 - x_4y_3z_2),$$

and the distances of their centres of gravity:

To determine the volume of a triangular pyramid from the coördinates of its angle In Fig "90" page "100" designate the angles, C, A, B, by 1, 2, 3, and the projections of these angles by $1_0, 2_0, 3_0$. Then we have,

Area
$$O2_03_0 = \frac{x_3y_2 - x_2y_3}{2}$$
$$O1_03_0 = \frac{x_3y_1 - x_1y_3}{2}$$
$$O1_02_0 = \frac{x_1y_2 - x_2y_1}{2}$$
$$1_02_03_0 = \frac{x_3y_2 - x_2y_3 - x_3y_1 + x_1y_3 - x_1y_2 + x_2y_1}{2}$$

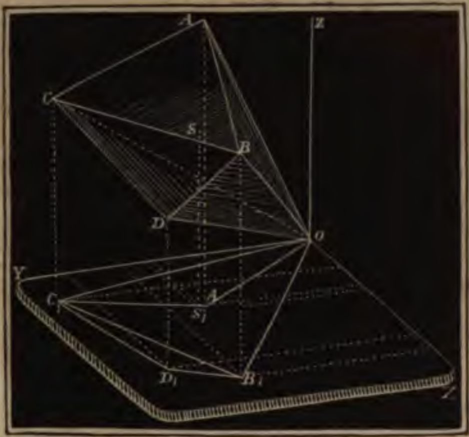
Height
$$1_02_03_0 = \frac{z_1 + z_2 + z_3}{3}.$$
$$O1_02_0 = \frac{z_1 + z_2}{3}$$
$$O1_03_0 = \frac{z_1 + z_3}{3}$$
$$O2_03_0 = \frac{z_2 + z_3}{3}$$

Solid $1_02_03_0$
$$\begin{cases} x_3y_2z_1 - x_2y_3z_1 - \underline{x_3y_1z_1} + \underline{x_1y_3z_1} - \underline{x_1y_2z_1} + \underline{x_2y_1z_1} \\ + \underline{x_3y_2z_2} - \underline{x_2y_3z_2} - x_3y_1z_2 + x_1y_3z_2 - \underline{x_1y_2z_2} + \underline{x_2y_1z_2} \\ \underline{x_3y_2z_3} - \underline{x_2y_3z_3} - \underline{x_3y_1z_3} + \underline{x_1y_3z_3} - x_1y_2z_3 + x_2y_1 \end{cases}$$

$$O1_03_0 = (\underline{x_3y_1z_1} - \underline{x_1y_3z_1} + \underline{x_3y_1z_3} - \underline{x_1y_3z_3})/6$$
$$O1_02_0 = (\underline{x_1y_2z_1} - \underline{x_2y_1z_1} + \underline{x_1y_2z_2} - \underline{x_2y_1z_2})/6$$
$$O2_03_0 = (\underline{x_3y_2z_2} - \underline{x_2y_3z_2} + \underline{x_3y_2z_3} - \underline{x_2y_3z_3})/6$$

Solid Entire $= \frac{1}{6}(x_3y_2z_1 - x_2y_3z_1 - x_3y_1z_2 + x_1y_3z_2$
$- x_1y_2z_3 + x_2y_1z_3)$

Fig. 91.



$$u_1=\frac{x_1+x_2+x_3}{4},\ v_1=\frac{y_1+y_2+y_3}{4},\ w_1=\frac{z_1+z_2+z_3}{4},$$

$$u_2=\frac{x_2+x_3+x_4}{4},\ v_2=\frac{y_2+y_3+y_4}{4},\ w_2=\frac{z_2+z_3+z_4}{4},\ \text{\&c.}$$

From these values the distances of the centre of gravity of the whole body may be finally calculated by the formula:

$$u=\frac{V_1u_1+V_2u_2+\dots}{V_1+V_2+\dots},\ v=\frac{V_1v_1+V_2v_2+\dots}{V_1+V_2+\dots},$$

$$w=\frac{V_1w_1+V_2w_2+\dots}{V_1+V_2+\dots}.$$

Example. A body bounded by six triangles *ADO*, Fig. 91, is determined by the following values for the co-ordinates of angles; whence the co-ordinates of its centre of gravity may be found.

Given co-ordinates			Six times the area of the triangular pyramids.	Four times the co-ordinates of centre of gravity.			Twenty-four times the statical moments.		
x	y	z		$4u_n$	$4v_n$	$4w_n$	$24V_nu_n$	$24V_nv_n$	$24V_nw_n$
20 45	23 29	41 30	$6\,V_1=\begin{Bmatrix}20.29.28\\41.45.40\\23.30.12\end{Bmatrix}-\begin{Bmatrix}20.40.30\\23.28.45\\41.12.29\end{Bmatrix}=31072$	77	92	99	2392544	2858624	3076128
12 38	40 35	28 20	$6\,V_2=\begin{Bmatrix}45.35.28\\29.20.12\\30.38.40\end{Bmatrix}-\begin{Bmatrix}45.40.20\\29.28.38\\30.12.35\end{Bmatrix}=17204$	95	104	78	1634380	1789216	1341912
			Sum: 48276				4026924	4647840	4418040

From the results of this calculation, the distances of the centres of gravity from the three planes *YZ*, *XZ*, and *XY* follow.

$$u=\frac{1}{4}\cdot\frac{4026924}{48276}=20,853,$$

$$v = \frac{1}{4} \cdot \frac{4647840}{48276} = 24{,}069,$$
$$w = \frac{1}{4} \cdot \frac{4418040}{48276} = 22{,}879.$$

§ 114. The centre of gravity of a truncated pyramid $ADQN$ (Fig. 88), lies in the line MG, which connects the centres of gravity of the two parallel bases; in order to determine the distance of this point from one of the bases, we must determine the volumes and moments of the entire pyramid ADF, and the supplementary pyramid NQF. If the areas of the bases AD and $NQ = G$ and g, and the normal distance of both $= h$, the height of the supplementary pyramid will be given from the formulæ:

$\frac{G}{g} = \frac{(h+x)^2}{x^2}$, or $\frac{h}{x} + 1 = \sqrt{\frac{G}{g}}$, and $x = \frac{h\sqrt{g}}{\sqrt{G}-\sqrt{g}}$, as also $h + x = \frac{h\sqrt{G}}{\sqrt{G}-\sqrt{g}}$.

The moment of the whole pyramid with reference to the base G is now

$\frac{G\,(h+x)}{3} \cdot \frac{h+x}{4} = \frac{1}{12} \cdot \frac{h^2 G^2}{(\sqrt{G}-\sqrt{g})^2}$, that of the supplementary pyramid $= \frac{gx}{3}\left(h+\frac{x}{4}\right) = \frac{1}{3} \frac{h^2\sqrt{g^3}}{\sqrt{G}-\sqrt{g}} + \frac{1}{12} \cdot \frac{h^2 g^2}{(\sqrt{G}-\sqrt{g})^2}$; hence it follows that the moment of the truncated pyramid:

$$= \frac{h^2}{12\,(\sqrt{G}-\sqrt{g})^2} \cdot \left(G^2 - 4(\sqrt{G g^3} - g^2) - g^2\right)$$
$$= \frac{h^2(G^2 - 4g\sqrt{Gg} + 3g^2)}{12\,(G - 2\sqrt{Gg} + g).} = \frac{h^2}{12} \cdot (G + 2\sqrt{Gg} + 3g).$$

Now the solid contents of the truncated pyramid are:

$$V = \frac{h}{3}\,(G + \sqrt{Gg} + g);$$

hence it follows finally that the distance of its centre of gravity S from the base is

$$MS = y = \frac{h}{4} \cdot \frac{G + 2\sqrt{Gg} + 3g}{G + \sqrt{Gg} + g}.$$

The radii of the bases of a truncated cone are R and r, and therefore $G = \pi R^2$ and $g = \pi r^2$, we have then for this

$$y = \frac{h}{4} \cdot \frac{R^2 + 2Rr + 3r^2}{R^2 + Rr + r^2}.$$

Example. The centre of gravity of a truncated cone of the height $h = 20$ inches, and radius $R = 12$ and $r = 8$ inches, always lies in the line joining the centres of the two circular bases, and is distant from the greater by:

$$y = \frac{20}{4} \cdot \frac{12^2 + 2.12.8 + 3.8^2}{12^2 + 12.8 + 8^2} = \frac{5.528}{304} = \frac{2640}{304} = 8{,}684 \text{ inches.}$$

§ 115. A pontoon is a body enclosed by two dissimilar rectangular

[illegible]

[illegible]

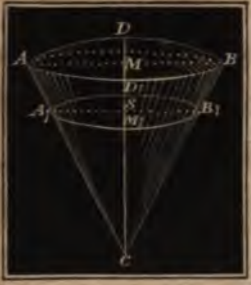
Hence it follows that the distance of the centre of gravity from the base is [illegible]

[illegible]

Fig. 93.

§ 116. If the sector of a circle AC
dius CD, th
tor ACB, w
determine.
containing
pyramids, wh
C, and who
face ADB.
these pyram
sphere from
a second sph
dius CA_1 =
vity S of thi
gravity of th
weights of the elementary pyramid
this surface, and therefore it is unifor

Fig. 94.



If we now put the radius $CA = C$
outer surface $= h$, we get for the inn
consequently (§ 110) $D_1S = \frac{1}{2} M_1D_1$
centre of gravity of the sector from

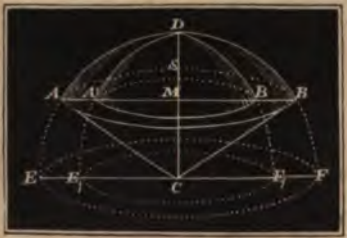
$$CS = CD_1 - D_1S = \frac{3}{4} r -$$

For the semicircle, for example, h
centre of gravity S from the centre C

$$CS = \frac{3}{4} \cdot \frac{r}{2}$$

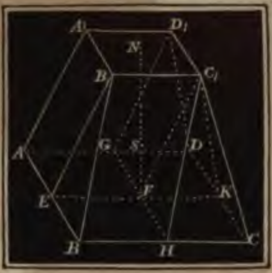
§ 117. The centre of gravity S of
Fig.
ment
of the
and th
if we
$= r$ a
ment
$r - h)$
the co
$(r - h)$
momen
is therefore $= \frac{1}{4} \pi h (2r - h) (r^2 - [r -$

Fig. 95.



bases and four trapeziums ACC_1A_1, Fig. 92, and may be decomposed into a parallelopiped AFC_1A_1, two triangular prisms EHC_1B_1, GKC_1D_1, and a quadrangular pyramid HKC_1; we may, therefore, with the help of these constituents, find the centre of gravity of the body.

Fig. 92.



It is easy to see that the line from the one bases to the other is the line of gravity of this body; there remains only to determine the distance of the centre of gravity from either base. If we represent the length BC and breadth AB of one base by l and b, and that of A_1B_1 and B_1C_1 of the other base by l_1 and b_1, and the height of the body by h, then the volume of the parallelopiped $= b_1l_1h$, and its moment $b_1l_1h\,.\,\frac{h}{2} = \frac{1}{2}\,b_1l_1h^2$, further the volumes of the two triangular prisms $= ([b-b_1]\,l_1 + [l-l_1]\,b_1)\,\frac{h}{2}$ and their moment $= ([b-b_1]\,l_1 + [l-l_1]\,b_1)\,\frac{h}{2}\,.\,\frac{h}{3}$, lastly the volume of the pyramid $= (b-b_1)\,.\,(l-l_1)\,\frac{h}{3}$ and its moment $= (b-b_1)\,.\,(l-l_1)\,\frac{h}{3}\,.\,\frac{h}{4}$. The volume of the whole body is, therefore:

$$V = (6b_1l_1 + 3bl_1 + 3lb_1 - 6b_1l_1 + 2bl + 2b_1l_1 - 2bl_1 - 2b_1l)\,.\,\frac{h}{6}$$

$$= (2\,bl + 2\,b_1l_1 + bl_1 + lb_1)\,.\,\frac{h}{6},$$ and its moment

$$Vy = (6b_1l_1 + 2bl_1 + 2lb_1 - 4b_1l_1 + bl + b_1l_1 - bl_1 - lb_1)\,.\,\frac{h^2}{12}$$

$$= (3b_1l_1 + bl + bl_1 + b_1l)\,\frac{h^2}{12}.$$

Hence it follows that the distance of the centre of gravity from the base bl is:

$$y = \frac{bl + 3\,b_1l_1 + bl_1 + b_1l}{2\,bl + 2\,b_1l_1 + bl_1 + b_1l}\,.\,\frac{h}{2}.$$

Remark. This formula is also applicable to bodies with elliptical bases. The axes of the one base are a and b, and of the other a_1 and b_1; the volume of such a body, therefore, is: $V = \frac{\pi h}{24}\,(2\,ab + 2\,a_1b_1 + ab_1 + a_1b)$, and the distance of the centre of gravity:

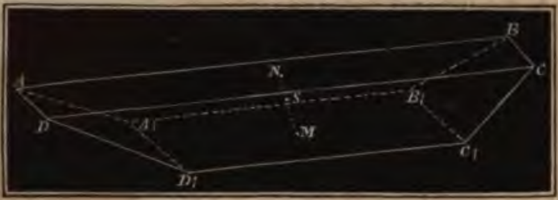
$$y = \frac{ab + 3\,a_1b_1 + ab_1 + a_1b}{2\,ab + 2\,a_1b_1 + ab_1 + a_1b}\,.\,\frac{h}{2}.$$

Example. A dam, ACC_1A_1, Fig. 93, is of the height 20 feet, 250 feet long at the bottom and 40 feet wide, at the top 400 feet long and 15 wide; to find the distance of its centre of gravity from the base. Here $b = 40$, $l = 250$, $b_1 = 15$, $l_1 = 400$, $h = 20$, therefore the vertical distance sought is:

$$MS = y = \frac{40\,.\,250 + 3\,.\,15\,.\,400 + 40\,.\,400 + 15\,.\,250}{2\,.\,40\,.\,250 + 2\,.\,15\,.\,400 + 40\,.\,400 + 15\,.\,250}\,.\,\frac{20}{2}$$

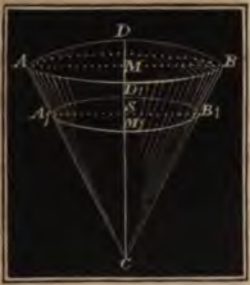
$$= \frac{4775}{5175}\,.\,10 = \frac{1910}{207} = 9{,}227 \text{ feet.}$$

Fig. 93.



§ 116. If the sector of a circle ACD, Fig. 94, revolves about its radius CD, there is generated the spherical sector ACB, whose centre of gravity we wish to determine. We may represent the body as containing infinitely many and infinitely thin pyramids, whose common vertex is the centre C, and whose base forms the spherical surface ADB. The centres of gravity of all these pyramids are at $\frac{3}{4}$ of the radius of the sphere from the centre C; they therefore form a second spherical surface $A_1D_1B_1$ of the radius $CA_1 = \frac{3}{4}\, CA$. But the centre of gravity S of this curved surface is the centre of gravity of the spherical sectors; because the weights of the elementary pyramids are uniformly distributed over this surface, and therefore it is uniformly heavy.

Fig. 94.



If we now put the radius $CA = CD = r$ and the height DM of the outer surface $= h$, we get for the inner $CD_1 = \frac{3}{4}\, r$, and $M_1D_1 = \frac{3}{4}\, h$; consequently (§ 110) $D_1S = \frac{1}{2}\, M_1D_1 = \frac{3}{8}\, h$, and the distance of the centre of gravity of the sector from the centre:

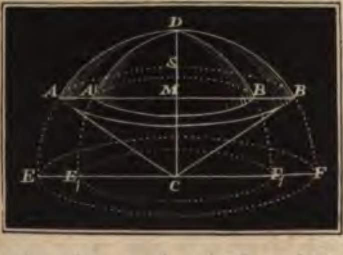
$$CS = CD_1 - D_1S = \tfrac{3}{4}\, r - \tfrac{3}{8}\, h = \tfrac{3}{4}\left(r - \frac{h}{2}\right).$$

For the semicircle, for example, $h = r$, therefore the distance of its centre of gravity S from the centre C is:

$$CS = \frac{3}{4} \cdot \frac{r}{2} = \frac{3}{8}\, r.$$

§ 117. The centre of gravity S of the segment of a sphere ABD, Fig. 95, is obtained when its moment is put equal to the difference of the moments of the sector $ADBC$ and that of the cone ABC. Again, if we put the radius of the cone $CD = r$ and the height $DM = h$, the moment of the sector $= \frac{2}{3}\pi r^2 h \cdot \frac{3}{8}\,(2r - h) = \frac{1}{4}\pi r^2 h\,(2r - h)$ and that of the cone $= \frac{1}{3}\pi h (2r - h) \cdot (r - h) \cdot \frac{3}{4}\,(r - h) = \frac{1}{4}\,\pi h\,(2r - h)\,(r - h)^2$; the moment of the segment of the sphere is therefore $= \frac{1}{4}\,\pi\, h\,(2r - h)\,(r^2 - [r - h]^2) = \frac{1}{4}\,\pi h^2\,(2r - h)^2$. The

Fig. 95.

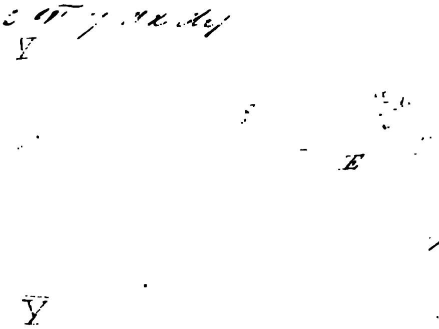


See next page

$m = \frac{3}{4}$ [illegible] $\frac{1 - \cos^2}{4}$ [illegible]

To [illegible] the centre of [illegible] [illegible] [illegible] segment of a sphere. [illegible] [illegible] lines of [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] dimensional. [illegible] [illegible] surface is dxdy and [illegible] [illegible] the [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] is

$2\pi y\,dx\,dy$ and its arm [illegible] [illegible] [illegible] [illegible] $V\bar{Y}$ [illegible] [illegible] [illegible] $2\pi y\,dx\,dy\,x$ is its moment and [illegible] [illegible] divided by its [illegible] [illegible] [illegible] Gives the [illegible]



$$\frac{2\pi \iint y\,dx\,dy\,x}{2\pi \iint y\,dx\,dy} = \frac{\int_{r-h}^{r} \frac{y^2}{2}x\,dx}{\int_{r-h}^{r} \frac{y^2}{2}dx}$$ [illegible] [illegible] arc ADB [illegible]

[illegible] a circular arc [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] $y^2 = r^2 - x^2$. Hence our expression [illegible] [illegible] [illegible] [illegible]

$$= \frac{\int_{r-h}^{r} \frac{r^2 - x^2}{2}x\,dx}{\int_{r-h}^{r} \frac{r^2 - x^2}{2}dx}$$ [illegible] [illegible] [illegible]

[illegible] the segment [illegible] [illegible] [illegible] [illegible] for x are CD and CE equal to r and r − h. [illegible] expression for the value of $\bar{x}$ [illegible] then $\bar{x} = \dfrac{\frac{r^2x^2}{2} - \frac{x^4}{4}}{r^2x - \frac{x^3}{3}} = \dfrac{3}{4}\left(\dfrac{2r^2x^2 - x^4}{3r^2x - x^3}\right)$, which taking the proper limits r and r − h, then

$$\bar{x} = \frac{3}{4}\,\frac{r^4 - 2r^2(r-h)^2 + (r-h)^4}{2r^3 - 3r^2(r-h) + (r-h)^3}.$$

Developing the expressions in the numerator it becomes

$$\left\{\begin{array}{l} \underline{r^4} \\ -\underline{2r^4} + \underline{4r^3h} - 2r^2h^2 \\ \underline{r^4} - \underline{4r^3h} + \underline{6r^2h^2} - 4rh^3 + h^4 \end{array}\right\} = 4r^2h^2 - 4rh^3 + h^4,$$

and [illegible] denominator [illegible]

$$\left\{\begin{array}{l} 2r^3 \\ -3r^3 + 3r^2h \\ r^3 - 3r^2h + 3rh^2 - h^3 \end{array}\right\} = 3rh^2 - h^3.$$

Hence our equation will become $\bar{x}$

$$= \frac{3}{4}\,\frac{4r^2h^2 - 4rh^3 + h^4}{3rh^2 - h^3} = \frac{3}{4}\,\frac{4r^2 - 4rh + h^2}{3r - h} = \frac{3}{4}\,\frac{(2r-h)^2}{3r-h}$$

which is the expression for the distance of the [illegible] [illegible] [illegible]

See next page

$arm = \frac{3}{4} r \frac{\frac{1-\cos 2\beta}{4}}{1-\cos\beta} = \frac{3}{4} r \frac{\sin^2\beta}{2(1-\cos\beta)} =$ (by mu-plying both numerator and denominator by r) $\frac{3}{4} r \frac{r^2\sin^2\beta}{2r(1-\cos\beta)} =$ (call $r - r\cos\beta = h$) $\frac{3}{4} \frac{r^2\sin^2\beta}{2h} = \frac{3}{4} \frac{r^2(1-\cos^2\beta)}{2h}$

$= \frac{3}{4} \cdot \frac{r[r(1-\cos\beta)(1+\cos\beta)]}{2h} = \frac{3}{8} r \frac{(r+r\cos\beta)(1-c}{h}$

$= \frac{3}{8} \frac{(r+r\cos\beta)(r-r\cos\beta)}{h} = \frac{3}{8} \frac{(r+r\cos\beta)h}{h}$.

$= \frac{3}{8}(r+r\cos\beta)$ But $r - r\cos\beta = h$ $\therefore$

$\cos\beta = \frac{r-h}{r}$ substituting which

$arm = \frac{3}{8} r + \frac{r(r-h)}{r} = \frac{3}{8}(2r-h) = \frac{3}{4}\left(r - \frac{h}{2}\right)$

$\int x^2 dx = \frac{1}{3}x^3 + C$ [illegible] for x are CD and CE equal to r and $r-h$. Integrating the expressions for the sum we have

$$Cm = \frac{\frac{r^2x^2}{2} - \frac{x^4}{4}}{r^2x - \frac{x^3}{3}} = \frac{3}{4}\left(\frac{2r^2x^2 - x^4}{3r^2x - x^3}\right),$$

and introducing our proper limits r and $r-h$

$$Cm = \frac{3}{4}\,\frac{r^4 - 2r^2(r-h)^2 + (r-h)^4}{2r^3 - 3r^2(r-h) + (r-h)^3}.$$

Developing the expressions in the numerator it becomes

$$\left\{\begin{array}{l} r^4 \\ -2r^4 + 4r^3h - 2r^2h^2 \\ r^4 + 4r^3h + 4r^2h^2 - 4rh^3 + h^4 \end{array}\right\} = 4r^2h^2 - 4rh^3 + h^4.$$

And now developing the denominator we have as follows

$$\left\{\begin{array}{l} 2r^3 \\ -3r^3 + 3r^2h \\ r^3 - 3r^2h + 3rh^2 - h^3 \end{array}\right\} = 3rh^2 - h^3.$$

Hence our equation has become

$$Cm = \frac{3}{4}\,\frac{4r^2h^2 - 4rh^3 + h^4}{3rh^2 - h^3} = \frac{3}{4}\,\frac{4r^2 - 4rh + h^2}{3r - h} = \frac{3}{4}\,\frac{(2r - h}{3r - h}$$

which is the expression for the distance of the centre of gravity in a spheroidal segment from the centre of the sphere.

See next page

arm $= \frac{3}{4} r \frac{\frac{1-\cos 2\beta}{4}}{1-\cos\beta} = \frac{3}{4} r \frac{\sin^2\beta}{2(1-\cos\beta)} =$ (by mu

-plying both numerator and

denominator by r) $\frac{3}{4} r \frac{r^2\sin^2\beta}{2r(1-\cos\beta)} =$ (call

$r - r\cos\beta = h$) $\frac{3}{4} \frac{r^2\sin^2\beta}{2h} = \frac{3}{4} \frac{r^2(1-\cos^2\beta)}{2h}$

$= \frac{3}{4} \frac{r[r(1-\cos\beta)(1+\cos\beta)]}{2h} = \frac{3}{8} r \frac{(r+r\cos\beta)(1 c}{h}$

$= \frac{3}{8} \frac{(r+r\cos\beta)(r-r\cos\beta)}{h} = \frac{3}{8} \frac{(r+r\cos\beta)h}{h}$.

$= \frac{3}{8}(r+r\cos\beta)$ But $r - r\cos\beta = h$ $\therefore$

$\cos\beta = \frac{r-h}{r}$ substituting which

arm $= \frac{3}{8} r + \frac{r(r-h)}{r} = \frac{3}{8}(2r-h) = \frac{3}{4}(r - \frac{h}{3})$

$\int$ [illegible]

$\int_{r-h}^{r} x^2 dx = \frac{1}{3} r^3 + c$. [illegible] $\frac{1}{3}(r-h)^3 - c =$

$\therefore \frac{1}{3} r^3 - \frac{1}{3}(r-h)^3 = \int_{r-h}^{r} x^2 dx$ [illegible]

Centre of Gravity of a Spherical Sector

We take the line of symmetry of the sector as an axis of rotation and a diameter of the sphere perpendicular to it for a line of reference cutting the variable radius ρ and the variable angle φ we have an elementary portion of the generating surface $\rho d\varphi d\rho$ This elementary surface being revolved generates a ring whose volume (the radius of the circumference passed over by the generatrix being $\rho \sin\varphi$) is $2\pi\rho^2 d\rho \sin\varphi d\varphi$ and the moment of this with respect to the line of reference is $2\pi\rho^3 d\rho \sin\varphi \cos\varphi d\varphi$ which divided by the volume gives the arm or distance of the center of gravity from the vertex $= \dfrac{2\pi\int_0^r\int_0^\beta \rho^3 d\rho \cos\varphi \sin\varphi d\varphi}{2\pi\int_0^r\int_0^\beta \rho^2 d\rho \sin\varphi d\varphi} = \dfrac{3}{4} r \dfrac{\int_0^\beta \cos\varphi \sin\varphi d\varphi}{\int_0^\beta \sin\varphi d\varphi}$

$= \dfrac{3}{4} r \dfrac{\int_0^\beta \frac{\sin 2\varphi}{2} d\varphi}{\int_0^\beta \sin\varphi d\varphi}$ ∴ The expression $-\dfrac{\cos 2\varphi}{4}$ being differentiated gives $\dfrac{\sin 2\varphi \, 2 d\varphi}{4} = \dfrac{\sin 2\varphi \, d\varphi}{2}$ and is therefore the integral of our numerator ∴ arm $= \dfrac{3}{4} r \dfrac{-\frac{\cos 2\varphi}{4}}{-\cos\varphi}$ $= \dfrac{3}{4} r \dfrac{-\left(\frac{\cos 2\beta - 1}{4}\right)}{-(\cos\beta - 1)}$ (since when we come to the limit 0 for φ its cosine is 1

back

volume of the segment $= \frac{1}{3}\pi h^2 (3r - h)$; hence, the distance in question is:

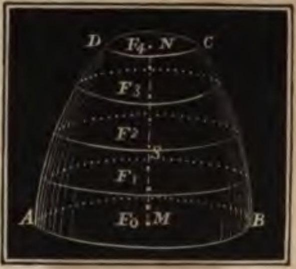
$$CS = \frac{\frac{1}{4}\pi h^2 (2r - h)^2}{\frac{1}{3}\pi h^2 (3r - h)} = \frac{3}{4} \cdot \frac{(2r-h)^2}{3r-h}.$$

If, again, we put $h = r$, the segment becomes a semicircle, and as above, $CS = \frac{3}{8} r$.

This formula holds good for the segment of a spheroid A_1DB_1, which is generated by the revolution of an elliptical arc DA_1 about its major semi-axis $CD = r$; for both segments may be divided into thin slices by planes parallel to the base AB, so that the ratio of any two is constant and $= \frac{\overline{MA_1}^2}{\overline{MA_1}^2} = \frac{CE_1^2}{CE^2} = \frac{b^2}{r^2}$, if b represent the semi-axis minor of the ellipse. The volume, as well as the moment of the segment of the sphere must be multiplied by $\frac{b^2}{r^2}$, in order to give the volume and moment of the segment of the spheroid, and thereby the quotient $CS = \frac{\text{moment}}{\text{volume}}$ will remain unchanged.

§ 118. To find the centre of gravity of an irregular body $ABCD$, Fig. 96, we must decompose it into thin slices, by planes equi-distant from each other, determine the solid contents of each slice, their moments with reference to the first parallel plane AB serving for the base, and finally connect them together by Simpson's rule.

Fig. 96.



The contents of these slices are F_0, F_1, F_2, F_3, F_4, and the whole height or distance of the outermost parallel plane is $= h$; the volume of the body, therefore, according to Simpson's rule (approximately) is:

$$V = (F_0 + 4F_1 + 2F_2 + 4F_3 + F_4)\frac{h}{12}.$$

If we multiply in this formula each of these volumes by their distance, we obtain the moment:

$$Vy = (0 . F_0 + 1 . 4F_1 + 2 . 2F_2 + 3 . 4F_3 + 4F_4)\frac{h}{4} \cdot \frac{h}{12};$$

lastly, by dividing one expression by the other, we get the distance required:

$$MS = y = \frac{(0 . F_0 + 1 . 4F_1 + 2 . 2F_2 + 3 . 4F_3 + 4F_4)}{F_0 + 4F_1 + 2F_2 + 4F_3 + F_4}\frac{h}{4}.$$

If the number of elementary slices $= 6$, we have:

$$y = \frac{0.F_0 + 1.4F_1 + 2.2F_2 + 3.4F_3 + 4.2F_4 + 5.4F_5 + 6.F_6}{F_0 + 4F_1 + 2F_2 + 4F_3 + 2F_4 + 4F_5 + F_6} \cdot \frac{h}{6}.$$

It is easy to understand how this formula may be altered when the number of slices is different from the above. This rule requires only that the number of the slices should be even, and, therefore, that of the surfaces uneven.

In most cases of application, the determination of one distance is enough, because, besides this, a line of gravity is known. The bodies commonly met with in practice are solids of rotation, generated in a lathe whose axis of rotation is the line of gravity.

This formula, lastly, is applicable to the determination of the centre of gravity of a surface, in which case the sections F_0, F_1, F_2, become lines.

Example.—1. For the parabolic conoid ABC, Fig. 97, which is generated by the revolution of a parabola ABM about its axis AM, we obtain by making the section DNE, the following:

The height $AM = h$, the radius $BM = r$, $AN = NM = \frac{h}{2}$, and hence the radius $DN = r\sqrt{\frac{1}{2}}$. The area of the section through A is $F_0 = 0$, of that through $N = F_1 = \pi \overline{DN^2} = \frac{\pi r^2}{2}$, and of that through $M = F_2 = \pi r^2$. Hence the volume of this body is:

$$V = \frac{h}{6}(0 + 4F_1 + F_2) = \frac{h}{6}(2\pi r^2 + \pi r^2) = \tfrac{1}{2}\pi r^2 h = \tfrac{1}{2} F_2 h;$$

on the other hand, the moment is $= \frac{h^2}{12}(1 . 2\pi r^2 + 2 . \pi r^2) = \frac{1}{3}\pi r^2 h^2 = \frac{1}{3} F_2 h^2$; lastly, the distance of the centre of gravity S from the vertex, is:

$$AS = \frac{\frac{1}{3} F_2 h^2}{\frac{1}{2} F_2 h} = \tfrac{2}{3} h.$$

Fig. 97.

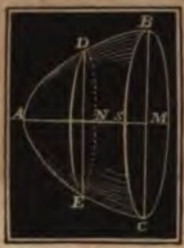
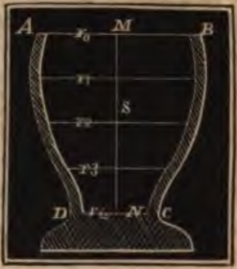


Fig. 98.



Example 2. A vessel $ABCD$, Fig. 98, has its mean half breadths, $r_0 = 1$ inch, $r_1 = 1{,}1$ inch, $r_2 = 0{,}9$ inch, $r_3 = 0{,}7$ inch, $r_4 = 0{,}4$ inch, with a height $MN = 2{,}5$ inch. The sections are $F_0 = 1 . \pi$, $F_1 = 1{,}21 . \pi$, $F_2 = 0{,}81 . \pi$, $F_3 = 0{,}49 . \pi$, $F_4 = 0{,}16 . \pi$; hence, the distance of the centre of gravity from the horizontal plane AB, is:

$$MS = \frac{0 . 1\pi + 1 . 4 . 1{,}21 . \pi + 2 . 2 . 0{,}81\pi + 3 . 4 . 0{,}49\pi + 4\ 0{,}16 . \pi}{1\pi + 4 . 1{,}21\pi + 2 . 0{,}81\pi + 4 . 0{,}49\pi + 0{,}16\pi} \frac{2{,}5}{4}$$

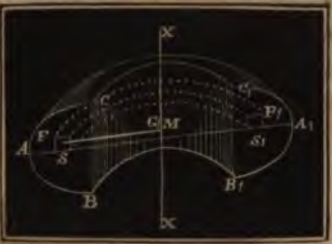
$$= \frac{14{,}60}{9{,}58} \cdot \frac{2{,}5}{4} = \frac{3650}{38{,}32} = 0{,}9502 \text{ inches.}$$

The capacity, therefore, is $= 9{,}58\pi . \frac{2{,}5}{12} = 6{,}270$ cubic inches.

§ 119. An interesting and sometimes very useful application of the laws of the centre of gravity is the *properties of Guldinus*, or the barocentric method. According to these, the volume of a body of revolution (or of a surface of revolution) is equal to the product of the generating surface (or generating line), and the space described by its centre of gravity during the generation of the body or surface of revolution. The correctness of this proposition may be made evident in the following manner.

Guldinus' Properties.—If the plane area ABC, Fig. 99, revolve about an axis $X\overline{X}$, each element F_1F_2, &c., of the same will describe an annulus; if the distances F_1G_1, F_2G_2, &c., of these elements from the axis of revolution $X\overline{X}_1$ be $= r_1, r_2$, &c., and the angle of revolution $AMA_1 = \alpha^0$, therefore the arc corresponding to the radius $1 = \alpha$, the circular paths of the elements will be $= r_1\alpha$, $r_2\alpha$, &c. The spaces described by the elements F_1, F_2, &c., may be considered as curved prisms having the bases F_1, F_2, &c., and the heights $r_1\alpha$, $r_2\alpha$, &c., and the volumes $F_1 r_1 \alpha$, $F_2 r_2 \alpha$, &c., and therefore the volume of the whole body $ABCB_1A_1C_1$: $V = F_1r_1\alpha + F_2r_2\alpha \ldots = (F_1r_1 + F_2r_2 + \ldots) . \alpha$. If $MS = x$ be the distance of the centre of gravity S of the generating surface from the axis of revolution, we have also $(F_1 + F_2 + \ldots)x = F_1r_1 + F_2r_2 + \ldots$, consequently the volume of the whole body $V = (F_1 + F_2 + \ldots)x\alpha$. But $F_1 + F_2 + \ldots$ are the contents of the whole surface F, and $x\alpha$ the circular arc w SS_1, described by the centre of gravity S; consequently, $V = Fw$, as above enunciated. This formula holds good also for the revolution of a line, because it may be considered as a surface made up of infinitely small breadths; F is namely $= Lw$: *i.e.* the surface of revolution is a product of the generating line (L) and the path (w) of its centre of gravity.

Fig. 99.

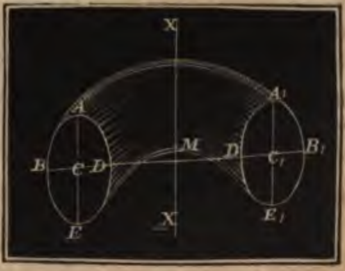


Example.—1. In a half ring of an elliptical section $ABED$, Fig. 100, let the semi-axis of the section be $CA = a$ and $CB = b$, and let the distance CM of the centre C from the axis $X\overline{X} = r$; then the elliptical generating surface $F = \pi ab$, and the path of the centre of gravity (C) $w = \pi r$; hence the volume of this half-ring $V = \pi^2 abr$, and that of the whole ring $= 2\pi^2 abr$. If the dimensions be, $a = 5$ inches, $b = 3$ inches, $r = 6$ inches, the volume of one-fourth of the ring $= \frac{1}{2} . \pi^2 . 5 . 3 . 6 = 9{,}8696 . 5 . 9 = 444{,}132$ cubic inches.

Example.—2. For a ring with a semicircular section ABD, Fig. 101, if $CA = CB = a$, represent the radius of this section, and $MC = r$ that of the hollow space or neck, the volume is

$$V = \frac{\pi a^2}{2} . 2\pi\left(r + \frac{4a}{3\pi}\right) = \pi a^2\left(\pi r + \frac{4}{3}a\right).$$

Fig. 100.



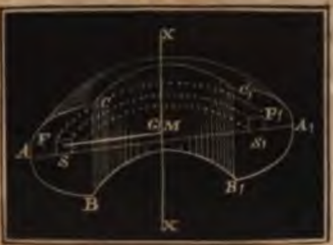
Example.—3. To find the surface and volume of a cupola ADB of the dome of a convent, Fig. 102, half the width $MA = MB = a$, and the height $MD = h$ are given. From both dimensions it follows that the radius $CA = CD$ of the generating circle $= r = \frac{a^2 + h^2}{2a}$, and the angle ACD subtended at the centre by $AD = \alpha^0$, if we put the $sin.\, \alpha =$

Guldinus' Properties

If we have any curves in space OP and MM′(Fig 1) whose equations are $f(x,y)$ and $f(X,Y)$ revolving around the axis of X, then the volume of the surface of revolution, (which the area intercepted between the two curves describes in Revolution) is equal to the product of that area into the path described by its centre of gravity. The equations of the two curves MM′$=f(X,Y)$ OP$=f(x,y)$ since the abscissas are the same for both curves and as the ordinates alone vary we have for the equation of the two curves, ... This is all gammon from K.., This is from Duhamel. The volume V generated by a plane comprised between the two curves whose ordinates are y_0 and Y, and parallel to the axis of y corresponding to the abscissas x_0 + X has for expression $V = \pi\int_{x_0}^{X}(Y^2-y_0^2)dx$. But the ordinate y_1 of the centre of gravity of the surface whose area we represent by A is given by the equation $Ay_1=\frac{1}{2}\int_{x_0}^{X}(Y^2-y_0^2)dx$; whence $V=2\pi y_1 A$ and consequently the volume generated by a plane area revolving around an axis in its plane is equal to the product of this area into the path described by its centre of gravity.

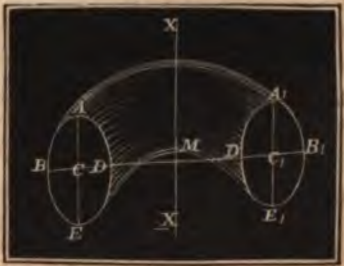
centre of gravity during the generation of the body or surface of revolution. The correctness of this proposition may be made evident in the following manner.

Fig. 99.



Guldinus' Properties.—If the plane area ABC, Fig. 99, revolve about an axis $X\overline{X}$, each element F_1F_2, &c., of the same will describe an annulus; if the distances F_1G_1, F_2G_2, &c., of these elements from the axis of revolution $X\overline{X}_1$ be $= r_1, r_2$, &c., and the angle of revolution $AMA_1 = \alpha^0$, therefore the arc corresponding to the radius $1 = \alpha$, the circular paths of the elements will be $= r_1\alpha$, $r_2\alpha$, &c. The spaces described by the elements F_1, F_2, &c., may be considered as curved prisms having the bases F_1, F_2, &c., and the heights $r_1\alpha$, $r_2\alpha$, &c., and the volumes $F_1 r_1 \alpha$, $F_2 r_2 \alpha$, &c., and therefore the volume of the whole body $ABCB_1A_1C_1$: $V = F_1r_1\alpha + F_2r_2\alpha \ldots = (F_1r_1 + F_2r_2 + \ldots) . \alpha$. If $MS = x$ be the distance of the centre of gravity S of the generating surface from the axis of revolution, we have also $(F_1 + F_2 + \ldots) x = F_1r_1 + F_2r_2 + \ldots$, consequently the volume of the whole body $V = (F_1 + F_2 + \ldots) x\,\alpha$. But $F_1 + F_2 + \ldots$ are the contents of the whole surface F, and $x\,\alpha$ the circular arc $w\,SS_1$, described by the centre of gravity S; consequently, $V = Fw$, as above enunciated. This formula holds good also for the revolution of a line, because it may be considered as a surface made up of infinitely small breadths; F is namely $= Lw$: *i.e.* the surface of revolution is a product of the generating line (L) and the path (w) of its centre of gravity.

Fig. 100.



Example.—1. In a half ring of an elliptical section $ABED$, Fig. 100, let the semi-axis of the section be $CA = a$ and $CB = b$, and let the distance CM of the centre C from the axis $X\overline{X} = r$; then the elliptical generating surface $F = \pi ab$, and the path of the centre of gravity (C) $w = \pi r$; hence the volume of this half-ring $V = \pi^2 abr$, and that of the whole ring $= 2\pi^2 abr$. If the dimensions be, $a = 5$ inches, $b = 3$ inches, $r = 6$ inches, the volume of one-fourth of the ring $= \frac{1}{2} . \pi^2 . 5 . 3 . 6 = 9{,}8696 . 5 . 9 = 444{,}132$ cubic inches.

Example.—2. For a ring with a semicircular section ABD, Fig. 101, if $CA = CB = a$, represent the radius of this section, and $MC = r$ that of the hollow space or neck, the volume is

$$V = \frac{\pi a^2}{2} . 2\pi \left(r + \frac{4a}{3\pi} \right) = \pi a^2 \left(\pi r + \frac{4}{3} a \right).$$

Example.—3. To find the surface and volume of a cupola ADB of the dome of a convent, Fig. 102, half the width $MA = MB = a$, and the height $MD = h$ are given. From both dimensions it follows that the radius $CA = CD$ of the generating circle $= r = \frac{a^2 + h^2}{2a}$, and the angle ACD subtended at the centre by $AD = \alpha^0$, if we put the *sin.* $\alpha =$

$\frac{h}{r}$. The centre of gravity S of an arc $DAD_1 = 2\,AD$ is determined by the distance $CS = r.\ \frac{\text{chord } MD}{\text{arc } AD} = \frac{r \sin.\alpha}{\alpha}$; further, $CM = r \cos.\alpha$, consequently the distance MS of the centre of gravity S from the axis $MD = \frac{r \sin.\alpha}{\alpha} - r \cos.\alpha = r\left(\frac{\sin.\alpha}{\alpha} - \cos.\alpha\right)$, and the path described by the centre of gravity in the generation of the surface $ADB = 2\pi r.\left(\frac{\sin.\alpha}{\alpha} - \cos.\alpha\right)$. The generating line $DAD_1 = 2\,r\,\alpha$, and since it only is required to determine the half ADB, this line $= r\,\alpha$, and consequently we must put the whole surface $O = r\,\alpha\,.\,2\,\pi\,r\left(\frac{\sin.\alpha}{\alpha} - \cos.\alpha\right) = 2\,\pi\,r^2\,(\sin.\alpha - \alpha \cos.\alpha)$.

Fig. 101.

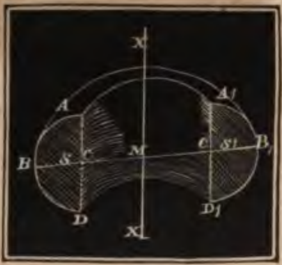
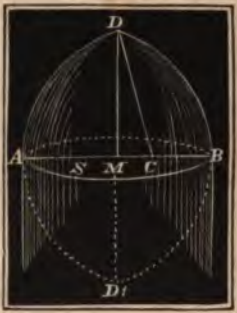


Fig. 102.



Very commonly $\alpha^0 = 60^0$; therefore, $\alpha = \frac{\pi}{3}$, $\sin.\alpha = \frac{1}{2}\sqrt{3}$, and the $\cos.\alpha = \frac{1}{2}$; hence it follows that $O = \pi\,r^2\left(\sqrt{3} - \frac{\pi}{3}\right) = 2{,}1515\,.\,r^2$. CESAR 1977

For the segment $DAD_1 = A = r^2\,(\alpha - \frac{1}{2}\sin.2\,\alpha)$ the distance of the centre of gravity from the centre C is $= \frac{(2\,.\,MD)^3}{12\,A} = \frac{2}{3}\,.\,\frac{r^3 \sin.\alpha^3}{A}$, hence the distance from the axis $MS = CS - CM = \frac{2}{3}\,.\,\frac{r^3 \sin.\alpha^3}{A} - r\cos.\alpha$; finally, the path of this centre of gravity described in one revolution is:

$$w = \frac{2\,\pi\,r}{A}\,(\tfrac{2}{3}\,r^3 \sin.\alpha^3 - A \cos.\alpha) = \frac{2\,\pi\,r^3}{A}\,(\tfrac{2}{3} \sin.\alpha^3 - [\alpha - \tfrac{1}{2}\sin.2\,\alpha]\cos.\alpha).$$

The volume of the whole body generated by the segment DAD_1, is given if this path be multiplied by A, and the volume of the dome found by taking the half of this: therefore, $V = \pi\,r^3\,(\frac{2}{3}\sin.\alpha^3 - [\alpha - \frac{1}{2}\sin.2\,\alpha]\cos.\alpha)$. For example, $\alpha^0 = 60^0 = \alpha\,\frac{\pi}{3}$ $\sin.\alpha = \frac{1}{2}\sqrt{3}$, and $\cos.\alpha = \frac{1}{2}$; hence:

$$V = \pi\,r^3\left(\frac{3}{8}\sqrt{3} - \frac{\pi}{6} = 0{,}3956\,.\,r^3.\right.$$

Remark. Guldinus' properties find their application in those bodies which arise when the generating surface so moves that in every position it remains perpendicular to the path of its centre of gravity, because we may assume every small part of a curvilinear motion to be circular. From this we may find the solid contents of the threads of screws, and sometimes also calculate the masses of earth, heaped up or removed, as in the case of canals, roads, railroads, &c.

§ 120. Another application of the doctrine of the centre of gravity, nearly allied to the last rule, is the following.

We may assume that every oblique prismatic body $ABCHKL$, Fig. 103, consists of an infinite number of thin prisms, similar to $F_1\,G_1$.

The surface A generated by the revolution of an arc of a plane curve revolving around an axis situated in its plane has for expression $A = 2\pi\int_{x_1}^{X} y\,ds$. But the ordinate y_1 of the centre of gravity of the line arc s is given by the expression $sy_1 = \int_{x_1}^{X} y\,ds$; whence $A = 2\pi y_1 s$, or that the area generated by an arc of a plane curve revolving around an axis situated in its plane is equal to this arc multiplied by the circumference described by its centre of gravity. These two propositions constitute what is called the Theory of Guldinus.

The volume of a truncated prism is equal to the volume of a right prism which stands upon the same base and has for its height the distance of the centre of gravity of the upper section from the base. $V = \iint dx\,dy\,z_1$, $F = \iint \frac{dx\,dy}{\cos\phi}$ ϕ being the angle which the upper section makes with the base. $\iint \frac{dx\,dy}{\cos\phi} z = Fz$, its statical moment. Z_1 = the distance of the centre of gravity from the base.

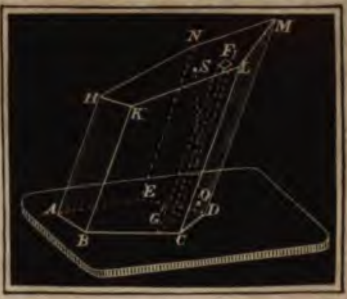
$\iint dx\,dy\,z = Fz, \cos\phi = V = Gz.$

If G_1, G_2 are the bases, and h_1, h_2 the heights of these elementary prisms, we have for their solid contents $G_1 h_1$, $G_2 h_2$, &c., and the volume of the whole oblique prism $V = G_1 h_1 + G_2 h_2 + \ldots$ Now an element F_1 of the oblique section HKL is to the element G_1 of the base ABC as the whole oblique surface F to the base G; therefore,

$$G_1 = \frac{G}{F} F_1,\ G_2 = \frac{G}{F} F_2, \text{ \&c. and}$$

$$V = \frac{G}{F}(F_1h_1 + F_2h_2 + \ldots).$$

Fig. 103.



And because $F_1h_1 + F_2h_2 + \ldots$ is the statical moment Fh of the whole oblique section, it follows that:

$$V = \frac{G}{F} \cdot Fh = Gh, \text{ i. e.,}$$

the volume of an oblique prism is equal to the volume of a perfect prism, which stands upon the same base, and whose height is equal to the distance SO of the centre of gravity S of the oblique surface from the base.

In a right or oblique triangular prism, if h_1, h_2, h_3, be the edges of the sides, the distance of the centre of gravity of the oblique surface from the base $h = \frac{h_1 + h_2 + h_3}{3}$, hence the volume

$$V = G\,\frac{(h_1 + h_2 + h_3)}{3}.$$

CHAPTER III.

EQUILIBRIUM OF BODIES RIGIDLY CONNECTED AND SUPPORTED.

§ 121. *Kinds of Support.*—The rules developed in the first chapter of this section, on the equilibrium of a rigid system of forces, are applicable to that of rigid bodies acted upon by forces, if we consider the weight of the body as a force applied to its centre of gravity, and acting vertically downwards.

Bodies balanced by forces, are either *freely movable, i. e.* yield to the action of forces, or they are *fixed by one or more points*, or *supported* by other bodies.

If a point of a rigid body is fixed, any other point may take up a motion whose path lies in the surface of a sphere, described from the fixed point as a centre by the distance of the other point as radius. If, on the other hand, two points of a body are fixed in every possible motion, the paths described by the remaining points are circles, which are the intersections of two spherical surfaces described from the fixed points. These circles are parallel to each other, and perpendicular to the straight line joining the two fixed points. The points of this line remain immovable; and the body revolves about this line, which is called the axis of revolution.

The radius of the circle in which each point moves, is found by letting fall from the point a perpendicular upon the axis of revolution. The greater this is, the greater also is the circle in which the point revolves.

If three points of a body, not falling within the same line, be fixed, the body can in no sense take up motion, because the three spherical surfaces, in which a fourth point must move, intersect each other in a point only.

§ 122. *Kinds of Equilibrium.*—If a body, fixed at one point, be balanced by one force or by the resultant of several forces, the direction of this force must pass through the fixed point; for a point is fixed when every force passing through it is counteracted. If this force consist merely of the weight of the body, it is then necessary that its centre of gravity should lie in the vertical line passing through the fixed point. If the centre of gravity coincide with the fixed, or the so called point of suspension, we then have indifferent equilibrium, because the body is balanced, in whatever direction it may revolve about the fixed point. If a body *AB*, Fig. 104, be fixed or sustained at a point *C* lying above its centre of gravity *S*, it then finds itself in a condition of stable equilibrium, because, if this body be brought into any other position, the component *N* of the weight *G* tends to bring it back into its first position, whilst the fixed point *C* counteracts the other component *P*. On the other hand, if a body *AB*, Fig. 105, be fixed at a point *C* below its centre of gravity *S*, the body is then in a state of unstable equilibrium; for if the centre of gravity be drawn out of the vertical passing through *C*, the component *N* of the weight of the body *G* not only does not bring it back into its former position, but draws it more and more out of that position, until the centre of gravity at last comes below the fixed point.

Fig. 104.

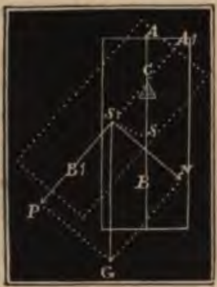
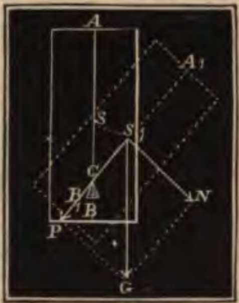
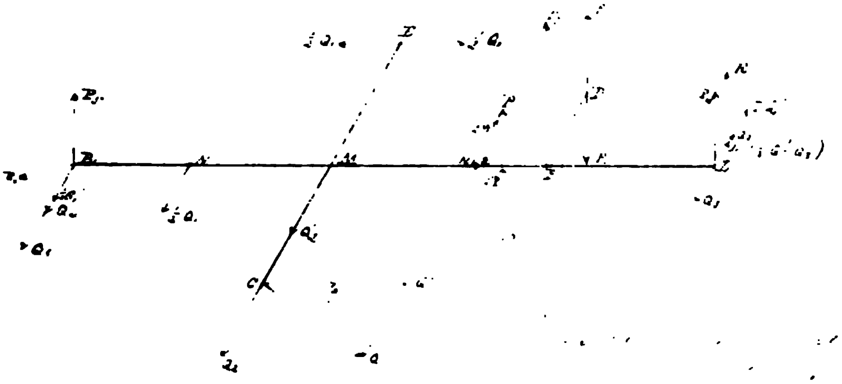


Fig. 105.




B
A
C
Y
Z

[illegible]

[illegible]

[illegible] $B_4 : P_6 :: BB_1 : BT_1$ and [illegible]

[illegible] and perpendicular [illegible]

[illegible] G_3 and G_4 [illegible]

[illegible] $G_3 : G_4 :: B.M : ML$ [illegible]

[illegible] applied [illegible]

[illegible] ME equal to MC and [illegible]

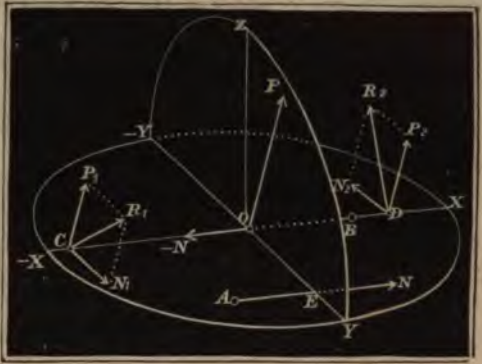
[illegible] equal and [illegible]

[illegible]

[illegible] and N, [illegible] at the gudgeon [illegible] and [illegible] at the gudgeon, at the extremity B, [illegible] and B [illegible] R, [illegible] and [illegible] producing pressure [illegible] gudgeons [illegible]

The same reasoning will also apply to the case of a body fixed by two points, or by an axis; it will be either in indifferent, stable, or unstable equilibrium, according as the centre of gravity lies vertically above or vertically below the axis.

Fig. 106.



§ 123. *Pressure on the Axis.*—If a body, acted upon by forces in space, be fixed by two points or by a line, relations then take place, which we will now investigate. We may reduce, according to § 92, every system of forces to two, viz., one running parallel to the fixed axis, and the other acting in the plane normal to this line. Let $AN = N$, Fig. 106, be the first, parallel to the axis $X\overline{X}$, passing through the fixed points C and D; and $OP = P$, the second force acting in the plane $YZ\overline{Y}$ at right angles to the axis $X\overline{X}$. If we introduce other forces, as $B\overline{N} = -N$, $CN_1 = N_1$, and $DN_2 = -N_1$, we change nothing in the condition of equilibrium or of motion, because these forces are entirely taken up by the axis. Now the forces N and $-N$ form together a first couple, and the forces N_1 and $-N_1$, acting in the plane XY and perpendicular to XX, a second couple; we may, therefore, so manage, that these shall perfectly replace each other. If EO is the normal distance between the force N and the axis $XX = y$, and CD that of the fixed point $= x$; from § 90, we have the moments of both couples $= Ny$ and N_1x, and these are equivalent to each other, if $Ny = N_1x$. We may also assume inversely that the force N is entirely taken up by the axis XX, whilst the axis has to sustain in its proper direction the pressure N, and the forces $N_1 = \frac{y}{x} N$, and $-N_1 = -\frac{y}{x} N$ applied perpendicularly to it at the points C and D.

That the body may be in a state of equilibrium, it is necessary that the direction also of the resultant acting in the normal plane YZ (at O) pass through the axis. This force P may be replaced by two parallel forces P_1 and P_2 applied at the points C and D, which may be determined, if we put $P_1 . CD = P . DO$ and $P_2 . CD = P . CO$; the axis $X\overline{X}$ will have, therefore, besides the forces $BN = -N$, $CN_1 = N_1$ and $DN_2 = -N_1$, also to react against the forces $P_1 = \frac{x_2}{x} . P$ and $P_2 = \frac{x_1}{x} . P$, which may be calculated from the distances $CD = x$, $OC = x_1$, and $OD = x_2$.

§ 124. From the results of the investigations of the foregoing paragraph we may easily calculate the forces sustained by the axis and the fixed points C and D. First, the axis has a pressure to sustain equivalent to the force $\mathcal{N}$ in its own direction, which may be entirely resisted by one or other of the two fixed points. Secondly, from the forces $\mathcal{N}_1 = \frac{y}{x}\mathcal{N}$, $P_1 = \frac{x_2}{x}P$ and $-\mathcal{N}_1 = -\frac{y}{x}\mathcal{N}$ and $P_2 = \frac{x_1}{x}P$, acting in planes normal to $X\overline{X}$, and applied at the points C and D, there arise the resultants R_1 and R_2, which must be also sustained by the fixed points C and D.

If we put the angle POY, which the direction of the force P makes with the plane XY containing the axis $X\overline{X}$ and the direction of the force $\mathcal{N} = \alpha$, the angle $\mathcal{N}_1 CP_1$ is also $= \alpha$; on the other hand, $\mathcal{N}_2 DP_2 = 180^\circ - \alpha$, and the resultant pressures are therefore given by:

$$R_1 = \sqrt{\mathcal{N}_1^2 + P_1^2 + 2\mathcal{N}_1 P_1 \cos. a}, \text{ and}$$
$$R_2 = \sqrt{\mathcal{N}_1^2 + P_2^2 - 2\mathcal{N}_1 P_2 \cos. a}.$$

Example. A set of forces of a body fixed by its axis $X\overline{X}$, is resolved into a normal force $P = 36$ lbs., and a parallel force $N = 20$ lbs.; the distance of the last from the axis is $y = 1\frac{1}{2}$ feet, and the distance $CD = x = 4$ feet. To find the forces sustained by the axis, or by the fixed points in it, with the condition that the direction of P deviate by an angle $\alpha = 65^0$ from the plane XY, and its point of application O be distant by $CO = x_1 = 1$ foot from the fixed point C? The force $N = 20$ lbs. imparts to the axis along its direction a thrust $N = 20$ lbs.; besides, it generates also the forces $N_1 = \frac{y}{x}N = \frac{1,5}{4}.20 = 7,5$ lb. and $-N_1 = -7,5$ lb., against which the fixed points C and D react. From the force P arise the forces $P_1 = \frac{x_2}{x}P = \frac{4-1}{4}.36 = 27$ lbs. and $P_2 = \frac{x_1}{x}P = \frac{1}{4}.36 = 9$ lbs. and by substitution of these values we have the resultant forces:

$$R_1 = \sqrt{7,5^2 + 27^2 + 2\,.\,7,5\,.\,27\,.\cos. 65^\circ} = \sqrt{56,25 + 729 + 171,160}$$
$$= \sqrt{956,410} = 30,926 \text{ lb., and}$$
$$R_2 = \sqrt{7,5^2 + 9^2 - 2\,.\,7,5\,.\,9\,.\cos. 65^\circ} = \sqrt{56,25 + 81 - 57,054}$$
$$= \sqrt{80,196} = 8,955 \text{ lbs.}$$

§ 125. *Equilibrium of Forces about an Axis.*—The force P is the resultant of all those component forces whose directions lie in one or more planes normal to the axis. But now in these cases, from § 86, the statical moment Pa of the resultant is equivalent to the sum $P_1a_1 + P_2a_2 + \ldots$ of the statical moments of the components, and for the condition of equilibrium of the fixed body the arm a of the resultant $= 0$, because this passes through the axis; hence the sum is also:

$$P_1a_1 + P_2a_2 + \ldots\ldots = 0;$$

i. e. a body fixed by its axis is in a state of equilibrium, and remains also without revolving, if the sum of the moments about this axis $= 0$, or if the sum of the moments of the forces acting in one direction of revolution, is equivalent to the sum of the moments of those acting in the opposite direction. By the help of this last formula we may find either a *force* or an *arm* for an element of a system of forces in equilibrium.

Example. The forces of rotation $P_1 = 50$ lbs., and $P_2 = -35$ lbs., act upon a body capable of turning about an axis at the arms $a_1 = 1\frac{1}{4}$ foot, and $a_2 = 2\frac{1}{2}$ feet; required, the force P_3 which shall act at the arm $a_3 = 4$ feet, in order to restore the balance, *i. e.* to prevent rotation about the axis? It is:

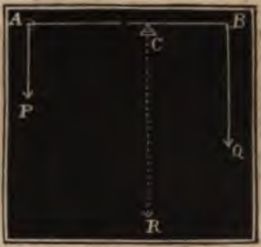
$$50 \,.\, 1{,}25 - 35 \,.\, 2{,}5 + 4\,P_3 = 0, \text{ hence}$$
$$P_3 = \frac{87{,}5 - 62{,}5}{4} = 6{,}25 \text{ lb.}$$

§ 126. *The Lever.*—A body capable of turning about a fixed axis, and acted upon by forces, is called a lever. If we imagine it to be devoid of weight, it is then called a *mathematical*, but otherwise, a *material* or *physical lever*.

It is generally assumed that the forces of a lever act in a plane at right angles to the axis, and that the axis is replaced by a fixed point, called the fulcrum. The perpendiculars let fall from this point on the direction of the forces, are called arms. If the directions of the forces of a lever are parallel, the arms form a single straight line, and the lever is called a straight lever. If the arms make an angle with each other, it is then called a bent lever. The straight lever acted upon by two forces, is either one-armed or two-armed, according as the points of application lie on the same or on opposite sides of the fulcrum. There is a distinction made of levers of the first, second and third order; the two-armed lever is termed a lever of the first order; the one-armed, of the second or third order, according as the weight acting vertically downwards, or the power acting vertically upwards, lies nearest to the fulcrum.

§ 127. The theory of the equilibrium of the lever has been already fully laid down; we have now, therefore, only to treat of each specially.

Fig. 107.



In the two-armed lever, ACB, Fig. 107, if the arm CA of the power P be designated by a, and the arm CB of the weight Q by b, from the general theory: $Pa = Qb$, *i. e.* the moment of the force is equal to the moment of the weight; or also, $P : Q = b : a$, *i. e.* the power is to the weight inversely as the arms. The pressure on the fulcrum is $R = P + Q$.

In the one-armed levers ABC, Fig 108, and BAC, Fig. 109, the same relation takes place between the power P and the weight Q, but here the direction

Fig. 108.

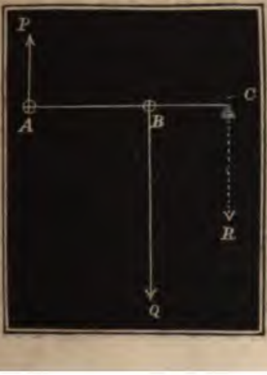
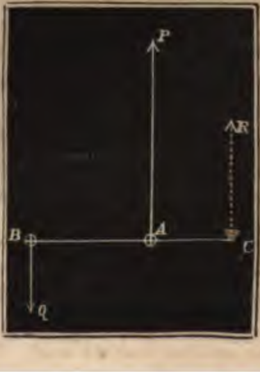


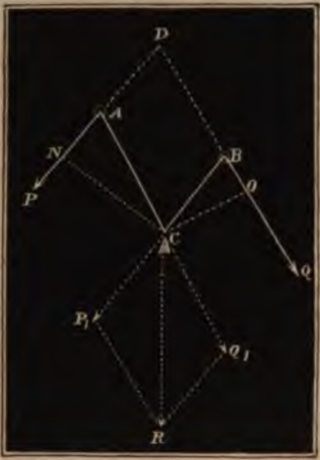
Fig. 109.



of the power is opposite to that of the weight, and therefore the pressure on the fulcrum is their difference, and in the first case $R = Q - P$, and in the second, $R = P - Q$.

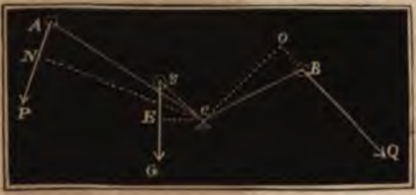
Also in the bent lever ACB, with the arms $CN=a$ and $CO=b$, Fig. 110, $P : Q=b : a$, here the pressure on the fulcrum is equivalent to the diagonal R of the parallelogram CP_1RQ_1, which may be constructed from the power P, the weight Q and the angle $P_1CQ_1=PDQ=\alpha$, which their directions make with each other.

Fig. 110.



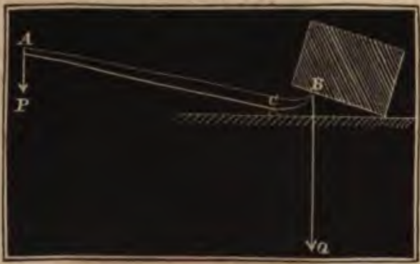
Let G be the weight of the lever, and $CE=e$, Fig. 111, the distance of the fulcrum C from the vertical line SG, passing through its centre of gravity; we shall then have to put $Pa \pm Ge = Qb$, and the plus or minus sign before G, according as the centre of gravity lies on the side of the power P, or on that of the weight Q.

Fig. 111.



Remark. The theory of the lever finds its application in many tools and machines, viz. in the different kinds of balances, crow-bars, the brakes of pumps, wheelbarrows, &c. The second part will treat fully of these.

Fig. 112.



Example.—1. If we press down the end A of a crow-bar ACB, Fig. 112, with a force $P=60$ lbs., and with the arm CA of the power equal to 12 times that of the arm CB of the weight, then will this, or rather the force exerted at B, be $=Q=12$ times that of $P = 12 \,.\, 60 = 720$ lbs.—2. If a load Q, hanging from a pole, Fig. 113, be carried by two men, which pole the one lays hold of at A and the other at B, we may readily find out what weight each has to sustain. Let the load $Q = 120$ lbs., the weight of the pole $G = 12$ lbs., the distance AB of both points of application $= 6$ ft., the distance BC of the load from one of these points $= 2\frac{1}{2}$ feet, the distance of the centre of gravity S of the pole from this same point $BS = 3\frac{1}{2}$ feet. If we take B for the fulcrum, the power P_1 has to balance at A the weights Q and G, therefore $P_1 \,.\, BA = Q \,.\, BC + G \,.\, BS$, i. e. $6\,P_1 = 2{,}5 \,.\, 120 + 3{,}5 \,.\, 12 = 300 + 42 = 342$; hence, $P_1 = \frac{342}{6} = 57$ lbs. On the other hand, if A be considered as the fulcrum, we must put $P_2 \,.\, AB = Q \,.\, AC + G \,.\, AS$, and in numbers, $6\,P_2 = 3{,}5 \,.\, 120 + 2{,}5 \,.\, 12 = 420 + 30 = 450$; hence, the power P_2 of the second man is $P_2 = \frac{450}{6} = 75$ lbs.; also, the sum of the forces $P_1 + P_2$ acting upwards, $= 57 + 75 = 132$ lbs., is exactly equal to the sum of the forces acting downwards,

[illegible]

[illegible]

[illegible]

Having a body A acted upon by a force P and suspended at the point B, we wish to find a force which will hold the body in position on the body A, which rests upon the plane XX. We will decompose the force P, into two forces, one normal to the plane of tangency at the point of contact and another force passing through the point of suspension, and parallel

If now we consider the body A, to be acted upon by a force P, this we will decompose into two one parallel to the plane X X and the other parallel to the direction of the normal force to the plane of tangency. If the angle $P_1OS = \alpha$, and $OP_1S = \beta$, we shall have $CP_1 = \frac{P_1 \sin\beta}{\sin(\alpha+\beta)}$ & $S = \frac{P_1 \sin\alpha}{\sin(\alpha+\beta)}$. If the angle $DCP = \alpha$ and $D_1CP = \beta$, we shall have $D = \frac{P \sin\beta}{\sin(\alpha+\beta)}$ and $S = \frac{P \sin\alpha}{\sin(\alpha+\beta)}$. (Which by the way proves Nothing whatsoever with respect to the original question. K

$Q+G=120+12=132$ lbs.—3. In a bent lever, ACB, Fig. 114, of 150 lbs. wt. the vertically pulling force $Q=650$ lbs., and the arm $CB=4$ ft., but the arm CA of the power $P=$ 6 ft., and that of the weight $CE=1$ foot, what is the amount of the power P, and the pressure on the pivot R required to restore the balance? $CA \, . \, P = CB \, . \, Q + CE \, . \, G$, i. e., $6 \, P = 4 \, . \, 650 + 1 \, . \, 150 = 2750$; consequently, the power $P = \frac{2750}{6} = 458\frac{1}{3}$ lbs.; the pressure on the pivot consists of the vertical force $Q + G = 650 + 150 = 800$ lbs., and the horizontal power $P = 458\frac{1}{3}$ lbs., and is therefore:

$$= R = \sqrt{(Q+G)^2 + P^2}$$
$$= \sqrt{(800)^2 + (458\tfrac{1}{3})^2}$$
$$= \sqrt{850070} = 922 \text{ lbs.}$$

Fig. 113.

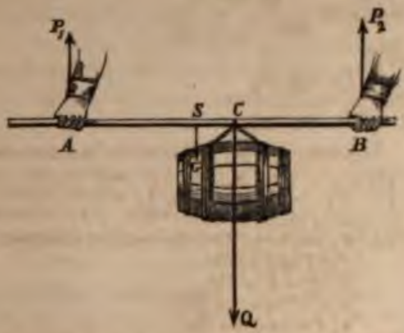
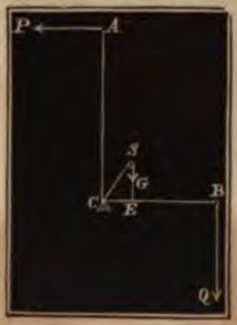
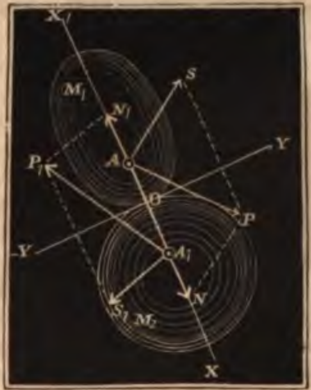


Fig. 114.



§ 128. *Pressure of Bodies on one another.*—The experimental law announced in § 62, that *action and reaction are equal to each other*, is the basis of the whole mechanics of machines. It is necessary in this place to make the meaning of this still clearer. When two bodies M_1 and M_2, Fig. 115, act upon each other with the forces P and P_1, whose directions deviate from the normal common $X\overline{X}$ to the two surfaces at their point of contact, a decomposition of the forces is always possible; the one component N or N_1, which is in the direction of the normal, passes over from the one body to the other, the other component S or S_1 remains in the body, and must be counteracted by another force or resistance, in order to maintain the bodies in equilibrium. From the principle set forth, perfect equilibrium is found to subsist between the normal components N and N_1.

Fig. 115.



If the direction of the force P deviates by the angle $NAP = \alpha$ from the normal AX and by the angle $SAP = \beta$ from the direction of the second component S, we have (§ 75)

$$N = \frac{P \sin. \beta}{\sin. (\alpha+\beta)},\ S = \frac{P \sin. \alpha}{\sin. (\alpha+\beta)}.$$

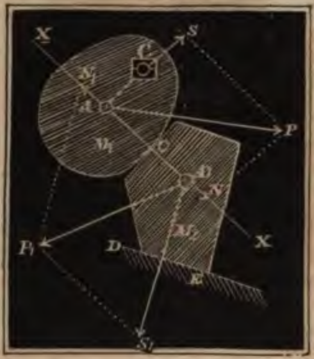
If we represent $N_1 A_1 P_1$ by α_1 and $S_1 A_1 P_1$ by β_1, we also have

$$N_1 = \frac{P_1 \sin. \beta_1}{\sin. (\alpha_1 + \beta_1)} \text{ and } S_1 = \frac{P_1 \sin. \alpha_1}{\sin. (\alpha_1 + \beta_1)};$$

lastly, from the equality $N = N_1$,

$$\frac{P \sin. \beta}{\sin. (\alpha+\beta)} = \frac{P_1 \sin. \beta_1}{\sin. (\alpha_1 + \beta_1)}.$$

Fig. 116.



Example.—What resolution of the forces takes place if a body M_2, Fig. 116, sustained by a support DE, be pressed upon by another, capable of revolving about an axis C with a force $P = 250$ lbs., the angles of direction being the following:

$PAN = \alpha = 35^0$,
$PAS = \beta = 48^0$,
$P_1 A_1 N_1 = \alpha_1 = 65^0$,
$P_1 A_1 S_1 = \beta_1 = 50^0$.

From the first formula the normal pressure between the two bodies is determined by

$$N = N_1 = \frac{P \sin. \beta}{\sin. (\alpha+\beta)} = \frac{250 \sin. 48^0}{\sin. 83^0} =$$

187,18 lbs.; from the second the pressure on the axis, or on the point C, is

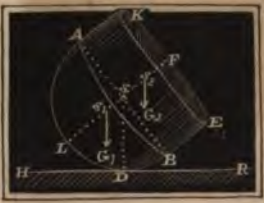
$$S = \frac{P \sin. \alpha}{\sin. (\alpha+\beta)} = \frac{250 \sin. 35^0}{\sin. 83^0} =$$

144,47 lbs.; and by combining the third and fourth equation, there follows finally for the component opposed to DE:

$$S_1 = \frac{N_1 \sin. \alpha_1}{\sin. \beta_1} = \frac{187,18 \sin. 65^0}{\sin. 50^0} = 221,46 \text{ lbs.}$$

§ 129. *Stability.*—When a body pressing against a horizontal plane is acted upon by no other force than gravity, it has no tendency to move forward, because the weight acting vertically downwards is exactly sustained by this plane; nevertheless, a revolution of the body is possible. If the body $ADBF$, Fig. 117, rests at a point D upon the horizontal plane HR, it will remain at rest, if its centre of gravity S be supported, *i. e.*, if it lie in the vertical line passing through D. If a body is supported at two points on the horizontal surface of another, it is requisite for its equilibrium that the vertical line of gravity should intersect the line connecting the two points. Lastly, if a body rests at three or more points on a horizontal plane, equilibrium subsists if the vertical line containing the centre of gravity passes through the triangle or polygon which is formed by the straight lines connecting the points of support.

Fig. 117.



In bodies which are supported, we must distinguish between stable and unstable equilibrium. The weight G of a body AB, Fig. 118, draws its centre of gravity downwards; if no resistance be opposed to this force it will cause the body to turn until its centre of gravity has attained its lowest position, and equilibrium will then be restored. We may mention that the equilibrium is stable when the centre of gravity is in its lowest possible position, Fig. 119, and unstable when in its highest, Fig. 120, and indifferent, when the centre of gravity in every position of the body remains at the same height, Fig. 121.

Fig. 118.

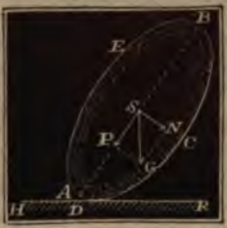


Fig. 119.

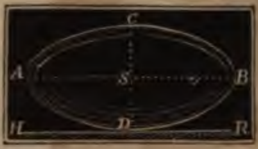


Fig. 120.

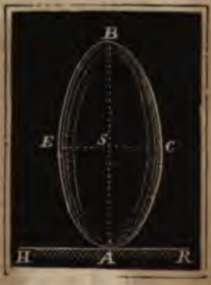
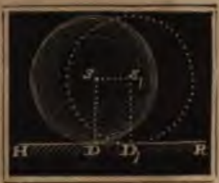


Fig. 121.

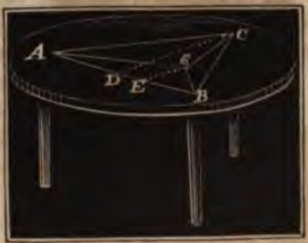


Example.—1. The homogeneous body $ADBF$, consisting of a hemisphere and a cylinder, Fig. 117, rests upon a horizontal plane HR. What height $SF = h$ must its cylindrical part have, that the body may be in equilibrium? The radius of a sphere is perpendicular to the corresponding plane of contact: now the horizontal plane is such a one; consequently the radius SD must be perpendicular to the horizontal plane, and the centre of gravity of the body lie in it. The axis FSL of the body passing through the centre of the sphere is its second line of gravity; the point S, the intersection of the two lines, is therefore the centre of gravity of the body.

Let us now put the radius of the sphere and cylinder $SA = SB = r$, and the height of the cylinder $SF = BE = h$, we then have for the volume of the hemisphere: $V_1 = \frac{2}{3} \pi r^3$, for the volume of the cylinder $V_2 = \pi r^2 h$; for the distance of the centre of gravity of the sphere $S_1 : SS_1 = \frac{3}{8} r$, and for that of the cylinder $S_2 : SS_2 = \frac{1}{2} h$. That the centre of gravity of the whole body may fall in S, the moment of the sphere $\frac{2}{3} \pi r^3 . \frac{3}{8} r$ must be put equal to the moment of the cylinder, $\pi r^2 h . \frac{1}{2} h$; from which we have:

$$h^2 = \frac{1}{2} r^2, \text{ i. e., } h = r \sqrt{\tfrac{1}{2}} = 0{,}7071 . r.$$

Fig. 122.



2. The pressure which each of the three legs, A, B, C, Fig. 122, of any loaded table has to sustain, is determined in the following manner. Let S be the centre of gravity of the table with its load, and SE, CD, perpendiculars upon AB. If G be the weight of the whole table, and R the pressure on C, we may, considering AB as the axis, put the moment of $R =$ to the moment of G, i. e., $R . CD = G . SE$, and we then obtain $R = \frac{SE}{CD} . G = \frac{\Delta ABS}{\Delta ABC} . G$; likewise also the pressure on $B = Q = \frac{\Delta ACS}{\Delta ABC} . G$, and that on $A = P = \frac{\Delta BCS}{\Delta ABC} . G$.

§ 130. Let us now take the case of a body having a plane base resting on a horizontal plane. Such a body possesses stability, or is in stable equilibrium when its centre of gravity is supported, *i. e.* when the vertical line containing the centre of gravity of the body passes through its base, because, in this case, the tendency of the weight of the body to cause it to turn is prevented by its own rigidity. When the line of gravity passes through the edge of the base, the body is then in unstable equilibrium, and when the line passes outside the base, no equilibrium subsists. The body falls to one side and overturns. The triangular prism $ABCDE$, Fig. 123, is, according to the above, stable, because the vertical SG passes through a point N of the base. The parallelopiped $ABCG$, Fig. 124, is in unstable equilibrium, because SG intersects a side CD of the base. The cylinder $ABCD$, Fig. 125, is without stability because SG no where intersects the base CD.

Fig. 123.

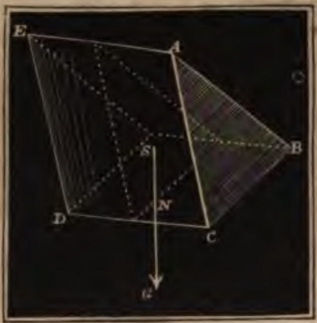


Fig. 124.

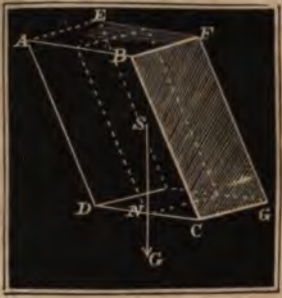
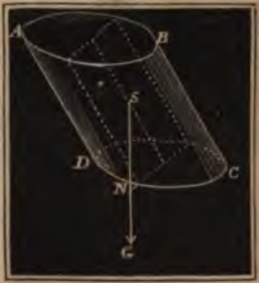


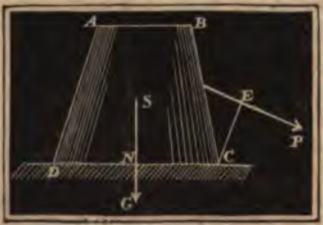
Fig. 125.



Stability is the power of a body to preserve its position by its weight alone, and to oppose resistance to any cause tending to overturn it. If we have to choose a measure of the stability of a body, we must distinguish whether this has reference to a displacement or to an actual overturning of the body. Let us now take into consideration the first only of these circumstances.

§ 131. *Formulæ of Stability.*—A force, P, not directed vertically, tends not only to overturn a body $ABCD$, Fig. 126, but also to push it forward; let us assume in the mean time that a resistance is opposed to the pushing or pulling forwards, as it may happen, and have regard only to its revolving about one of its edges C. If we let fall from this edge C a perpendicular $CE = a$ upon the direction of the force, and $CN = x$ upon the vertical line SG passing through the centre of gravity, we have

Fig. 126.



only to consider a bent lever ECN, for which $Pa = Gx$, so that $P = \frac{x}{a}\,G$; if the external force P be greater than $\frac{x\,G}{a}$ the body revolves about the point C, and, therefore, loses its stability. Hence the stability depends upon the product (Gx) of the weight of the body, and the shortest distance between a side of the perimeter of the base and the vertical line passing through the centre of gravity; Gx may therefore be regarded as a measure of the stability, and for this reason is properly called the stability itself.

Hence we see that the stability increases simultaneously with the weight G and the distance x, and may conclude that under otherwise similar circumstances a body twice or thrice as heavy does not possess more stability than one of the single weight with twice or thrice the distance or arm x, &c.

§ 132.—1. In a parallelopiped $ABCF$, Fig. 127, of the length $AE = l$, breadth $AB = CD = b$, and height $AD = BC = h$, the weight $G = V\gamma = bhl\gamma$, and the stability $S = G\,.\,KN = G.\,\frac{1}{2}\,CD = \frac{Gb}{2} = \frac{1}{2}\,b^2hl\gamma$, provided γ represent the density of the mass of the parallelopiped.

2. In a body $ACFH$ consisting of two parallelopipeds, Fig. 128, the stabilities about the two edges of the base C and E are different from one another. Let us take the heights BC and $EF = h$ and h_1, and the breadths CD and $DE = b$ and b_1, the weights of the parts G and $G_1 = bhl\gamma$ and $b_1h_1l\gamma$; then the arms about C will be $KN_1 = \frac{1}{2}\,b$ and $KN_2 = b_1 + \frac{1}{2}\,b_1$, and those about $E = b_1 + \frac{1}{2}\,b$ and $\frac{1}{2}\,b_1$. The stabilities accordingly are: first for the revolution about C,

$$S = \tfrac{1}{2}\,Gb + G_1\,(b + \tfrac{1}{2}\,b_1), = (\tfrac{1}{2}\,b^2h + bb_1h_1 + \tfrac{1}{2}b_1{}^2h_1)l\gamma,$$

secondly for that about E,

$$S_1 = G\,(b_1 + \tfrac{1}{2}b) + \tfrac{1}{2}\,G_1b_1 = (\tfrac{1}{2}b_1{}^2h_1 + bb_1h + \tfrac{1}{2}b^2h)\,l\gamma.$$

The latter stability is about $S_1 - S = (h - h_1)\,bb_1l\gamma$ greater than the former; if we wish to increase the stability of a wall AC by offsets, these must be placed on that side of the wall towards which the force of revolution (wind, water, pressure of earth, &c.) acts.

Fig. 127.

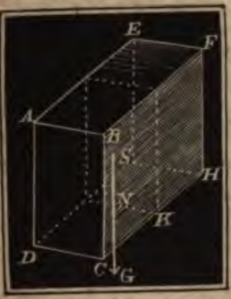
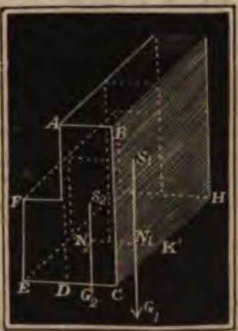
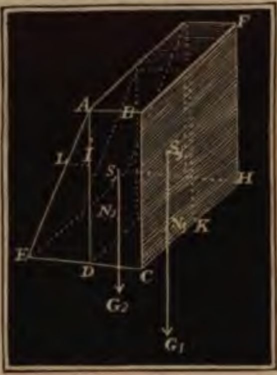


Fig. 128.



3. The following is the stability of a wall $ABCEF$, Fig. 129, battering on one side. The upper breadth $AB = b$, the height $BC = h$ and the length $CH = l$, and the batter $= n$, *i. e.* upon AI = a height of 1 foot; $IL = n$ feet or inches of batter, therefore upon h feet $ED = nh$. The weight of the parallelopiped ACF is $G_1 = bhl\gamma$, that of the three sided prism $ADE = G_2 = \frac{1}{2} nh \,.\, hl\gamma$, the arms for a revolution about E are $= DE + \frac{1}{2} b = nh + \frac{1}{2} b$ and $\frac{2}{3} DE = \frac{2}{3} nh$, consequently for the stability we have

Fig. 129.



$$S = G_1 \left(nh + \tfrac{1}{2} b\right) + \tfrac{2}{3} G_2 nh$$
$$= \left(\tfrac{1}{2} b^2 + nhb + \tfrac{1}{3} n^2h^2\right) hl\gamma.$$

A parallelopipedical wall of equal volume has the breadth $b + \frac{1}{2} nh$, hence the stability is:

$$S_1 = \tfrac{1}{2} \left(b + \tfrac{1}{2} nh\right)^2 hl\gamma = \left(\tfrac{1}{2} b^2 + \tfrac{1}{2} nhb + \tfrac{1}{8} n^2h^2\right) hl\gamma;$$

its stability is, therefore, about $S - S_1 = \left(b + \frac{5}{12} nh\right) . \frac{1}{2} nh^2 l\gamma$, less than that of the battered wall.

For a wall sloped upon the opposite side, the stability is $S_2 = \left(b^2 + nhb + \frac{1}{3} n^2h^2\right) . \frac{1}{2} hl\gamma$, less also than S, and indeed about $S - S_2 = \left(b + \frac{1}{3} nh\right) . \frac{1}{2} nh^2 l\gamma$, as well as about $S_2 - S_1 = \frac{1}{24} n^2h^3 l\gamma$ less than the stability of the parallelopipedical wall.

Example. What is the stability for each foot in length of a stone wall of 10 feet in height, and 1¼ feet of upper breadth with batter of 1 in 5 on the back? The specific gravity of this wall (§ 58) is taken at 2,4, its density γ is, therefore, $= 62{,}5 \,.\, 2{,}4 =$ 130 lbs.; now $l = 1$, $h = 10$, $b = 1{,}25$, and $n = \frac{1}{5} = 0{,}2$; hence it follows, that the stability sought is:

$$S = \left(\tfrac{1}{2} . [1{,}25]^2 + 0{,}2 \,.\, 1{,}25 \,.\, 10 + \tfrac{1}{3} . [0{,}2]^2 . 10^2\right) 10 \,.\, 1 \,.\, 130$$
$$= (0{,}78125 + 2{,}5 + 1{,}3333)\, 130 = 4{,}6146 \,.\, 130 = 603{,}4 \text{ ft. lbs.}$$

With the same quantity of material, and under otherwise similar circumstances, the stability of a parallelopipedical wall would be:

$$S_1 = \left(\tfrac{1}{2} . [1{,}25]^2 + \tfrac{1}{2} . 0{,}2 \,.\, 1{,}25 \,.\, 10 + \tfrac{1}{8} . 0{,}2^2 . 10^2\right) . 130$$
$$= (0{,}78125 + 1{,}25 + 0{,}5) \,.\, 130 = 2{,}531 \,.\, 130 = 329 \text{ ft. lbs.}$$

The same wall, with a sloping front, would have the stability:

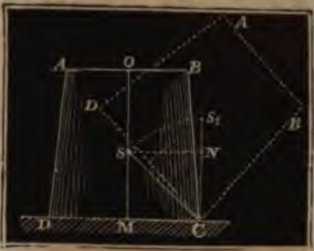
$$S_2 = \left(\tfrac{1}{2} . [1{,}25]^2 + \tfrac{1}{2} . 0{,}2 \,.\, 1{,}25 \,.\, 10 + \tfrac{1}{6} . [0{,}2]^2 . 10^2\right) . 130$$
$$= (0{,}78125 + 1{,}25 + 0{,}666 \ldots) \,.\, 130 = 2{,}6979 \,.\, 130 = 350{,}7 \text{ ft. lbs.}$$

Remark.—It is evident from the foregoing that it allows of a saving of material to batter walls, to construct them with counterforts, to give them offsets, or to place them upon plinths, &c. The second part will give a further extension of this subject, when we come to treat of the pressure of earth, and of vaults, chain bridges, &c.

§ 133. *Dynamical stability.*—We may distinguish from the measure of stability treated of in the last paragraph, still another to a certain degree dynamical measure of stability, when we consider the effect which is to be expended in order to overturn a body. Now the mechanical effect of a force is equal to the product of the force and the space, but the force of a heavy body is its weight G, and the space equal to the vertical projection of that described by its centre of gravity, we may consequently take for the dynamical measure of the stability of a body the product Gs, if s be the height to which the centre of gravity of the body must ascend in order to bring the body from its stable condition into an unstable one.

Let C be the axis of revolution and S the centre of gravity of a body $ABCD$, Fig. 130, whose dynamical stability we wish to find. If we cause the body to revolve so that its centre of gravity comes to S_1, *i. e.* vertically over C, the body will be in unstable equilibrium, for if it only revolve a little further it will fall over. If we draw the horizontal line SN, this will cut off the height $NS_1 = s$ to which the centre of gravity has ascended, from which the stability Gs is given. If now $CS = CS_1 = z$, $CM = SN = x$, and the height $CN = MS = y$, it follows that the space $S_1N = s = z - y = \sqrt{x^2 + y^2} - y$, and the stability in the last form of expression is

Fig. 130.



$$S = G\,(\sqrt{x^2 + y^2} - y).$$

If the body is a prism with a symmetrical trapezoidal transverse section, as Fig. 130 represents, and if the dimensions are the following: [redacted] height $MO = h$, lower breadth $CD = b_1$, upper breadth $AB = b_2$, we then have $MS = y = \frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}$ (§ 105) and $CM = x = \frac{1}{2} b_1$, hence

$$CS = \sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2},$$

and the dynamical stability, or the mechanical effect, required to overturn it:

$$S = G\left[\sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2} - \frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right].$$

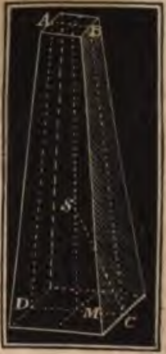
Example.—What is the dynamical stability or the mechanical effect necessary for the overturning of an obelisk $ABCD$, Fig. 131, of granite, if its height $h = 30$ ft., its upper length and breadth $l_1 = 1\frac{1}{2}$, and $b_1 = 1$ ft., and lower length and breadth $l_2 = 4$ ft., $b_2 = 3\frac{1}{2}$ ft.? The volume of this body is (§ 115):

$$V = (2\,b_1 l_1 + 2\,b_2 l_2 + b_1 l_2 + b_2 l_1)\,\frac{h}{6}$$

$$= (2 \,.\, \tfrac{3}{2} \,.\, 1 + 2 \,.\, 4 \,.\, \tfrac{7}{2} + 1 \,.\, 4 + \tfrac{3}{2} \,.\, \tfrac{7}{2})\,\tfrac{30}{6}$$

$= 40{,}25 \,.\, 5 = 201{,}25$ cubic feet. Now a cubic foot of granite weighs $= 3 \,.\, 62{,}5 = 187{,}5$ lbs.; the whole weight of this body is: $G = 201{,}25 \,.\, 187{,}5 = 37734{,}3$ lbs. The height of the centre of gravity above the base is:

Fig. 131.



$$y = \frac{b_2 l_2 + 3\,b_1 l_1 + b_2 l_1 + b_1 l_2}{2\,b_2 l_2 + 2\,b_1 l_1 + b_2 l_1 + b_1 l_2} \cdot \frac{h}{2}$$

$$= \frac{4 \,.\, \tfrac{7}{2} + 3 \,.\, \tfrac{3}{2} \,.\, 1 + 1 \,.\, 4 + \tfrac{3}{2} \,.\, \tfrac{7}{2}}{40{,}25} \cdot \frac{30}{2} = \frac{27{,}75 \,.\, 15}{40{,}25} = 10{,}342 \text{ ft.}$$

Provided it be a revolution about the longer edge of the base, the horizontal distance of the centre of gravity from this edge will be: $x = \frac{1}{2}\,b_2 = \frac{1}{2} \,.\, \frac{7}{2} = \frac{7}{4}$ ft.; hence, the distance of the centre of gravity from the axis will be:

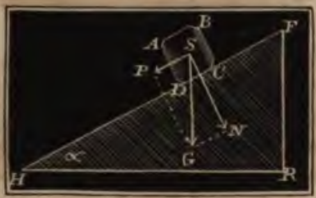
$CS = z = \sqrt{x^2 + y^2} = \sqrt{(1{,}75)^2 + (10{,}342)^2} = \sqrt{110{,}002} = 10{,}489$: and the height to which the centre of gravity must be raised to bring about an overthrow will be:

$s = z - y = 10{,}489 - 10{,}342 = 0{,}147$ ft; lastly, the corresponding mechanical effect or stability will be: $Gs = 37734.3 \,.\, 0{,}147 = 5547$ ft. lbs.

Remark. The factor $s = \sqrt{x^2 + y^2} - y$ gives for $y = 0$, $s = x$, for $y = x$, $s = x\,(\sqrt{2} - 1) = 0{,}414\,x$, for $y = nx$, $s = (\sqrt{n^2 + 1} - n)\,x$, approximately $= (n + \frac{1}{2n} - n)\,x = \frac{x}{2n}$, also for $y = 10\,x$, $s = \frac{x}{20}$, and for $y = \infty$, $s = \frac{x}{\infty} = 0$; the dynamical stability is therefore so much the greater, the lower the centre of gravity lies, and it approximates more and more to null, the higher the centre of gravity lies above the base. Sledges, carriages, ships, floating docks, &c., must on this account be so constructed and loaded, that the centre of gravity may lie as low as possible, and besides, be situated over the middle of the base.

§ 134. *Theory of the Inclined Plane.*—A body AC, Fig. 132, resting on an inclined plane, that is, on one inclined to the horizon, may take up two motions; it may slide down the inclined plane, and it may also revolve about one of the edges of its base and overturn. If the body is left to itself, its weight G is resolved into a force N normal, and to a force P parallel to the base, the first is resisted by the reaction of the plane, and the last urges the body down the plane. Let the angle of inclination FHR of the inclined plane to the horizon $= \alpha$, we have therefore the angle $GSN = \alpha$, and hence the normal pressure:

Fig. 132.



$$N = G\, cos.\, \alpha,$$

and the force parallel to the plane:

$$P = G\, sin.\, \alpha.$$

If the vertical line SG passes through the base CD as in Fig. 132, a sliding motion only can take place, but if this line passes outside the base, as in Fig. 133, an overturn ensues, and the body, therefore, is without stability. Besides, a body AC resting on the inclined plane FH, Fig. 134, has a stability different from that of one on a horizontal plane. If $DM = x$ and $MS = y$ are the rectangular co-ordinates of the centre of gravity S, we have the arm of the stability $DE = DO - MN = x\, cos.\, \alpha - y\, sin.\, \alpha$, while, if the body is on a horizontal plane, it is $= x$. Since $x > x\, cos.\, \alpha - y\, sin.\, \alpha$, the stability with reference to the lower edge D comes out less for the inclined than for the hori-

Fig. 133.

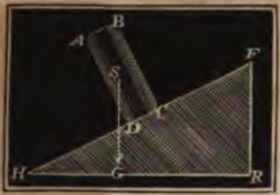
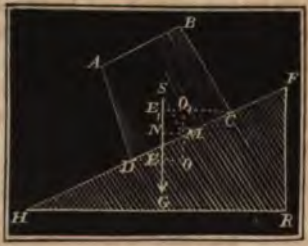


Fig. 134.



zontal plane; it is null for $x \cos. \alpha = y \sin. \alpha$, *i. e.* for $tang. \alpha = \frac{x}{y}$. When a body that is stable $G x$ on a horizontal plane is transferred to an inclined one, whose angle of inclination corresponds to the expression $tang. \alpha = \frac{x}{y}$, it will lose its stability. On the other hand, a body may acquire on an inclined plane the stability which is wanting to it on a horizontal one. For a turning about the upper edge C, the arm $CE_1 = CO_1 + MN = x_1 \cos. \alpha + y \sin. \alpha$, whilst in its position on the horizontal plane it is $= x_1$. If now x_1 is negative, the body has no stability so long as it remains on a horizontal plane, but if it rests on an inclined one, for whose angle of inclination $tang. \alpha$ is $> \frac{x_1}{y}$, the body is stable.

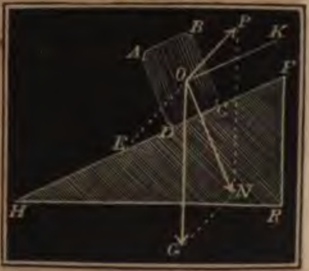
If another force besides gravity acts upon the body $ABCD$, Fig. 135, its stability continues if the direction of the resultant N of the weight G and the force P intersects the base CD of the body.

Example. The obelisk in the example of the preceding paragraphs has $x = \frac{7}{4}$ ft. and $y = 10{,}342$ ft., and will lose its stability, consequently, if transferred to an inclined plane, for whose angle of inclination:

$tang. \alpha = \frac{7}{4 \,.\, 10{,}342} = \frac{7000}{41368} = 0{,}16922$, and inclination $\alpha = 9^\circ\ 36'$.

§ 135. As the inclined plane only counteracts that pressure which is directed perpendicularly against it, the force P which is necessary to prevent a body supported upon an inclined plane from overturning, is determined by the condition that the resultant N of P and G, Fig. 135, must be at right angles to the inclined plane. From the theory of the parallelogram of forces we have $\frac{P}{G} = \frac{\sin. ONP}{\sin. PON}$, now the $\angle\, PNO = \angle\, GON = FHR = \alpha$, and $\angle\, PON = POK + KON = \beta + 90^\circ$, in so far as we represent by β the $\angle\, PEF = POK$, by which the direction of the force deviates from the inclined plane; hence we have

Fig. 135.



$$\frac{P}{G} = \frac{\sin. \alpha}{\sin. (90 + \beta)},\ i.\ e.\ \frac{P}{G} = \frac{\sin. \alpha}{\cos. \beta},$$

therefore the force which maintains the body on the plane is:

$$P = \frac{G \sin. \alpha}{\cos. \beta}.$$

For the normal pressure N

$\frac{N}{G} = \frac{\sin. OGN}{\sin. ONG}$, but the $\angle\, OGN = 90^\circ - (\alpha + \beta)$ and

$ONG = PON = 90 + \beta$, hence it follows

$$\frac{N}{G} = \frac{\sin. [90^\circ - (\alpha + \beta)]}{\sin. (90^\circ - \beta)} = \frac{\cos. (\alpha + \beta)}{\cos. \beta}$$

and for the normal pressure against the plane

$$\mathcal{N} = \frac{G \cos. (\alpha + \beta)}{\cos. \beta}.$$

If the force P is parallel to the plane, $\beta = 0$ and $\cos. \beta = 1$, since

$$P = G \sin. \alpha \text{ and } \mathcal{N} = G \cos. \alpha.$$

If P acts vertically $\alpha + \beta = 90°$, hence

$$\cos. \beta = \sin. \alpha,\ \cos. (\alpha + \beta) = 0 \text{ and}$$

$P = G$ and $\mathcal{N} = 0$, the inclined plane has then no control over the body.

Lastly, if the force acts horizontally, $\beta = -\alpha$, and $\cos. \beta = \cos. \alpha$, hence

$$P = \frac{G \sin. \alpha}{\cos. \alpha} = G \text{ tang. } \alpha; \text{ and } \mathcal{N} = \frac{G \cos. 0}{\cos. \alpha} = \frac{G}{\cos. \alpha}.$$

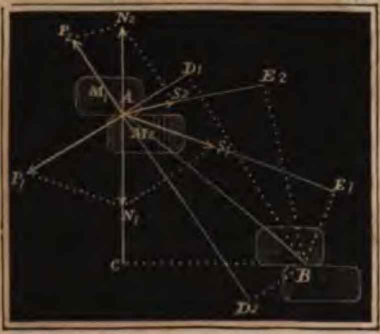
Example. To maintain a body of 500 lbs. upon an inclined plane of 50° inclination to the horizon, a force is applied whose direction makes an angle of 75° with the horizon, what is the magnitude of this force, and the pressure of the body against the plane? The force is:

$P = \frac{500 \sin. 50°}{\cos. (75-50)} = \frac{500 . \sin. 50°}{\cos. 25°} = 422{,}6$ lbs.; and the pressure on the plane:

$N = \frac{500 . \cos. 75°}{\cos. 25°} = 142{,}8$ lbs.

§ 136. *Principle of Virtual Velocities.*—If we combine the principle of the equality of action and reaction set forth in § 128, with that of virtual velocities (§ 80 and § 93), the following law transpires. If two bodies M_1 and M_2, Fig. 136, hold each other in equilibrium, *then for a finite rectilinear or infinitely small curvilinear motion of the point of contact or pressure $\mathcal{A}$, the sum of the mechanical effects of the forces of the one body is equivalent to the sum of the mechanical effects of those of the other.* If P_1 and S_1 be the forces of the one body, and P_2 and S_2 those of the other, then, for a displacement of the point of contact from $\mathcal{A}$ to B, the respective distances described are $\mathcal{A}D_1, \mathcal{A}E_1, \mathcal{A}D_2$ and $\mathcal{A}E_2$, and according to the above law:

Fig. 136.

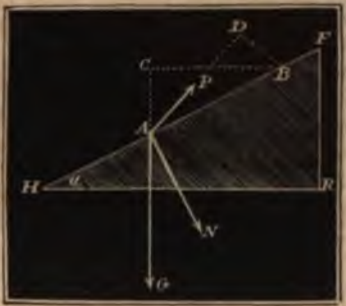


$$P_1 . \mathcal{A}D_1 + S . \mathcal{A}E_1 = P_2 . \mathcal{A}D_2 + S_2 . \mathcal{A}E_2.$$

The correctness of this proposition may be proved in the following manner. As the normal pressures $\mathcal{N}_1$ and $\mathcal{N}_2$ are equal, there is also equilibrium between their mechanical effects, $\mathcal{N}_1 . . \mathcal{A}C$ and $\mathcal{N}_2 . \mathcal{A}C$, with this difference, that the mechanical effect of the one force is positive, and that of the other negative. Now from what has preceded we have the mechanical effect $\mathcal{N}_1 . \mathcal{A}C$ of the resultant $\mathcal{N}_1$ equivalent to the sum $P_1 \mathcal{A}D_1 + S_1 . \mathcal{A}E_1$ of the mechanical effects of its components P_1 and S_1, and likewise $\mathcal{N}_2 . . \mathcal{A}C = P_2 . . \mathcal{A}D_2 + S_2 . \mathcal{A}E_2$; hence also $P_1 . \mathcal{A}D_1 + S_1 . \mathcal{A}E_1 = P_2 . \mathcal{A}D_2 + S_2 . \mathcal{A}E_2$.

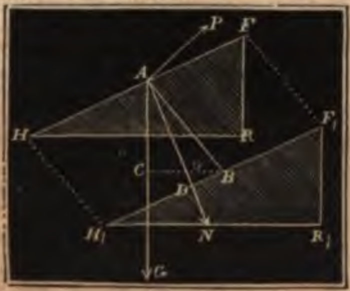
The application of the principle of virtual velocities thus made more general possesses great advantage in statical investigations, as by it the evolution of algebraical expressions becomes much simplified. If, for example, we move a body A up an inclined plane FH, Fig. 137, a distance AB, the corresponding path of the weight G, $= AC = AB\ sin.\ ABC = AB\ .\ sin.\ FHR = AB\ sin.\ \alpha$. On the other hand, the path of the force P is $AD = AB\ .\ cos.\ BAD = AB\ .\ cos.\ \beta$, and lastly, that of the normal force $N = 0$; now the mechanical effect of N is equivalent to that of G + that of P, hence we have to put $N\ .\ 0 = -G\ .\ AC + P\ .\ AD$, and so we find $P = \frac{AC}{AD}\ .\ G = \frac{G\ sin.\ \alpha}{cos.\ \beta}$, quite in accordance with the former paragraph.

Fig. 137.



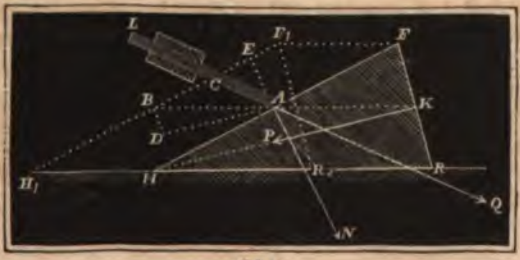
In order to find the normal pressure N, we must move forward the inclined plane HF, Fig. 138, through a space AB at right angles to the direction of the force AP, to determine the corresponding paths of the forces, and again put the mechanical effect of N equivalent to that of G + the mechanical effect of P. The path of N is $AD = AB\ cos.\ BAD = AB\ cos.\ \beta$, that of G is $AC = AB\ cos.\ BAC = AB\ cos.\ (\alpha + \beta)$ and that of $P = 0$, hence the mechanical effect $N\ .\ AD = G\ .\ AC + P\ .\ 0$, and $N = \frac{G\ .\ AC}{AD} = G\ .\ \frac{cos.\ (\alpha + \beta)}{cos.\ \beta}$, just as was found in the former paragraph.

Fig. 138.



§ 137. *Theory of the Wedge.*—After this the theory of the wedge comes out very simply. The wedge is a movable inclined plane,

Fig. 139.



formed by a triangular prism *FHR*, Fig. 139, generally the force *KP* is $= P$, and at right angles to the back *FR* of the wedge, and holds in equilibrium another force or load $AQ = Q$, which presses against its lateral surface *FH*. If $FHR = \alpha$ be the angle measuring the sharpness of its edge, and further, the angle by which the direction of the force *KP* or *AD* deviates from the surface *FH*~~, therefore~~ *FHK* $= HAD, = \delta$, and lastly the angle *LAH*, the deviation of the direction of *Q* from this same surface, $= \beta$, then the paths will be given which are described by the advance of the wedge from the position *FHR* into that of $F_1H_1R_1$, in the following manner. The path of the wedge is $AB = FF_1 = HH_1$, and that of the force is $= AD = AB \cos. BAD = AB \cos. (BAH - DAH) = AB \cos. (\alpha - \delta)$; further, the path of the bar *AL* or load is $AC = \frac{AB \sin. ABC}{\sin. ACB} = \frac{AB \sin. \alpha}{\sin. HAC} = \frac{AB \sin. \alpha}{\sin. \beta}$, and the simultaneous path of the normal pressure *N* between the wedge and the foot of the bar $= AE = AB \sin. \alpha$.

By the advance of the wedge a distance *AB*, the normal pressure *N* produces the mechanical effect $N . AE = N . AB \sin. \alpha$, the force, however, develops the mechanical effect $P . AD = P . AB \cos. (\alpha - \delta)$ and the resistance the mechanical effect, $Q . AC = Q . AB \frac{\sin. \alpha}{\sin. \beta}$, hence $N . AB \sin. \alpha = P . AB \cos. (\alpha - \delta)$ *i. e.* $N \sin. \alpha = P \cos. (\alpha - \delta)$, as also $N . AB \sin. \alpha = Q . AB \frac{\sin. \alpha}{\sin. \beta}$, *i. e.* $N \sin. \alpha = Q \frac{\sin. \alpha}{\sin. \beta}$, and from these equations the equation between the power and resistance sought is given:

$$P \cos. (\alpha - \delta) = \frac{Q \sin. \alpha}{\sin. \beta}, \text{ or}$$

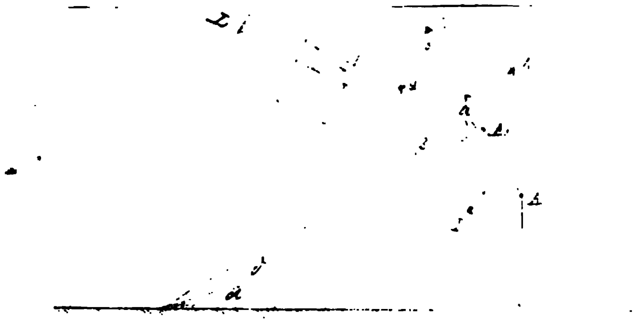
$$P = \frac{Q \sin. \alpha}{\sin. \beta \cos. (\alpha - \delta)},$$

which may likewise be obtained by the decomposition of the forces.

If the direction of the force is parallel to the base or lateral surface *HR*, $\delta = \alpha$, hence $P = \frac{Q \sin. \alpha}{\sin. \beta}$, and if, further, the direction of the load is perpendicular to the side *FH*, $\beta = 90°$, and *P* follows $= Q \sin. \alpha$.

Example. The ~~edge~~ angle *FHR* of a wedge $= \alpha = 25°$, the force is directed parallel to the base *HR*, therefore, $\delta = \alpha$, and the weight *Q* acts at right angles to the side *FH*, therefore $\beta = 90°$, in what proportions are the power and weight to each other? *P* is $= Q \sin. \alpha$, therefore $\frac{P}{Q} = \sin. 25° = 0{,}4226$. For a weight *Q* of 130 lbs. the power *P* comes out $= 130 . 0{,}4226 = 54{,}938$ lbs. In order to drive forward the weight or bar 1 foot, the wedge must pass over the space $AB = \frac{AC}{\sin \alpha} = \frac{1}{0{,}4226} 2{,}3662$ feet.

Remark. The theories of the inclined plane and the wedge will be more fully deveoped in the fifth chapter, where the effect of friction is taken into account.



[illegible] P = [illegible] AP. [illegible] perpendicular [illegible] N, and [illegible] infinite. [illegible] N [illegible] and [illegible] perpendicular and parallel [illegible] perpendicular [illegible] N, represents [illegible] against [illegible] parallel [illegible] $\frac{N}{P} = \frac{\sin\gamma}{\sin [illegible]}$ [illegible] $\gamma = 90^\circ - (\alpha - \delta)$ [illegible] $\frac{N}{P} =$ [illegible] $\frac{Q}{\sin\beta} = \frac{\cos(\alpha - \delta)}{\sin\alpha}$, $P = \frac{Q \sin\alpha}{\sin\beta \cos(\alpha - \delta)}$ 5558

CHAPTER IV.

EQUILIBRIUM IN FUNICULAR MACHINES.

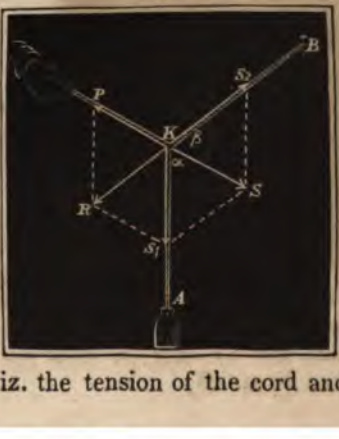
§ 138. *Funicular Machines.*—We have hitherto assumed that bodies, on which forces act, do not change their form in consequence of this action; we will now take up the equilibrium of such bodies as suffer a change in their form by the smallest forces. The former are called solid or rigid, the latter flexible bodies. In truth there is no body perfectly flexible; many of them, however, such as strings, ropes, cords, &c., and in some respects chains also, require so small a force to bend them that they may in many cases be regarded as perfectly flexible. Such bodies, which are moreover extensible, will be the subject of the following investigations.

We understand by a funicular machine, a cord or a connection of cords (the word cord taken in its general sense) which becomes stretched by forces, and in this chapter we will consider the theory of the equilibrium of these machines.

That point of a funicular machine to which the force is applied, and where the cord forms an angle with the direction of the force, is called a knot or node. This may be either fixed or movable. Tension is the force which a stretched cord transmits in the direction of its axis. The tensions at the ends of a straight cord or portion of a cord are equal and opposite, § 83; also a straight cord cannot transmit other forces than the tension acting in the direction of its axis, because it must otherwise bend, and, therefore, cannot remain straight.

§ 139. *Knots or Nodes.*—Equilibrium obtains in a funicular machine, when there is equilibrium at each of its nodes. Hence we must next find what are the relations of equilibrium at any one node.

Fig. 140.



Equilibrium takes place at a node K, which a portion of a cord AKB, Fig. 140, forms, when the resultant KS of the tensions of the cord $KS_1 = S_1$ and $KS_2 = S_2$ are equal and opposite to the force P applied at the node K, for the tensions S_1 and S_2 produce the same effects as equal and opposite forces, and three forces hold each other in equilibrium, if one of them is equal to and acts opposite to the resultant of the other two (§ 75). The resultant R of the force P and the first tension S_1 is equal and opposite to the second tension S_2, &c. In every case this equation may be used to find out two of the quantities to be determined, viz. the tension of the cord and its direction. Let, for example, the force be P, the tension S_1 and the ∠ between the two $AKP = 180^\circ - AKS = 180^\circ - \alpha$, we have for the other tension.

$$S_2 = \sqrt{P^2 + S_1^2 - 2\, PS_1 \cos.\, \alpha.}$$

and for its direction or deviation from KS, $BKS = \beta$, and

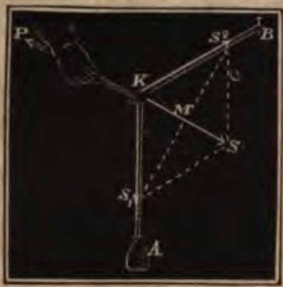
$$\sin.\, \beta = \frac{S_1 \sin.\, \alpha}{S_2}.$$

Example. If the cord AKB, Fig. 140, is fixed at the extremity B, and at the extremity A stretched by a weight $G = 135$ lbs., and the middle K by a force $P = 109$ lbs., which pulls upwards under an angle of 25°; required the direction and tension of the portion of cord KB. The magnitude of the tension is:

$$S_2 = \sqrt{109^2 + 135^2 - 2\,.\,109\,.\,135 \cos.\,(90^\circ - 25^\circ)}$$
$$= \sqrt{11881 + 18225 - 29430\,.\cos.\,65^\circ} = \sqrt{17668{,}3} = 132{,}92 \text{ lbs.}$$

For the angle β, $\sin.\,\beta = \frac{S_1 \sin.\,\alpha}{S_2} = \frac{135\,.\,\sin.\,65^\circ}{132{,}92}$, $Log.\,\sin.\,\beta = 0{,}964017 - 1$, hence $\beta = 67^\circ\, 0'$, and the inclination of the portion of the cord to the horizon $= \alpha + \beta - 90^\circ = 65^\circ + 67^\circ - 90^\circ = 42^\circ$.

Fig. 141.

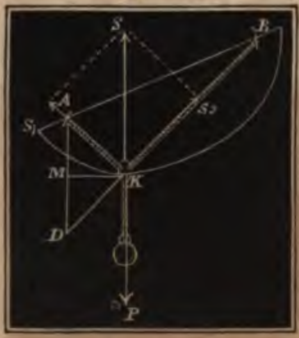


§ 140. If the node K is a running or movable one, or the force P acts by means of a ring running along the cord AKB, Fig. 141, the resultant S of the tensions S_1 and S_2 is equal and opposite to the force P at the ring; besides this, the tensions are equal, for if the cord be drawn a certain space s through the ring, each of the tensions S_1 and S_2 will pass over the space s, and the force P over a space $= 0$; consequently, provided there is perfect flexibility, the mechanical effect $P\,.\,0 = S_1\,.\,s - S_1\,.\,s$, *i. e.* $S_1 s = S_2 s$ and $S_1 = S_2$. From this equality of the tensions there follows the equality of the angles AKS and BKS, by which the resultant S deviates from the directions of the cords. If we put these angles $= \alpha$, the resolution of the rhomb $KS_1 SS_2$, gives

$$S = P = 2\, S_1 \cos.\, \alpha \text{ and inversely}$$

$$S_1 = S_2 = \frac{P}{2 \cos.\, \alpha}.$$

Fig. 142.



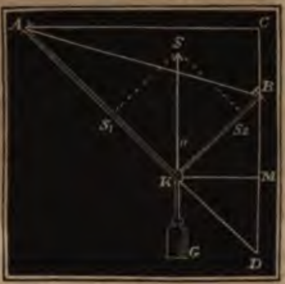
A and B are the fixed points of a cord AKB of given length $(2\,a)$ with a movable node K, the place of this node may be found by constructing an ellipse, whose foci are A and B, and whose major axis is equal to the length of the cord $2\,a$, and if a tangent is drawn to this curve at right angles to the given direction of the force, the resulting point of contact is the place of the node, because the normal to the ellipse KS makes equal angles with the radii vectores KA and KB, as does the resultant S with the tensions of the cord S_1 and S_2.

If AD be drawn parallel to the given direction of the force, and BD be made equal to the given length of the cord, AD bisected at M and the perpendicular MK be raised, the place of the node K may likewise be obtained without the construction of an ellipse, for since the $\angle AKM = \angle DKM$ and $AK = DK$, it follows that $\angle AKS$ also $= \angle BKS$ and $AK + KB = DK + KB = DB$.

Example. Between the points A and B, Fig. 143, a rope of 9 feet in length is stretched by a weight G of 170 lbs. suspended to it by a ring; the horizontal distance AC of the two points is $6\frac{1}{2}$ ft., and the vertical distance $BC = 2$ ft.; to find the position of the node, the tensions and directions of the rope. From the length $AD = 9$ ft. as hypothenuse and the horizontal line $AC = 6\frac{1}{2}$ ft.; it follows that the vertical $CD = \sqrt{9^2 - 6{,}5^2} = \sqrt{81 - 42{,}25} = \sqrt{38{,}75} = 6{,}225$ feet; and from this the base BD of the equilateral triangle $BDK, = CD - CB = 6{,}225 - 2 = 4{,}225$ ft. The similarity of the triangles DKM and DAC gives $DK = BK = \frac{DM}{DC} \cdot DA = \frac{4{,}225.9}{2.6{,}225} = 3{,}054$ ft.; hence it follows, that $AK = 9 - 3{,}054 = 5{,}946$ feet; and for the angles a, by which the sides of the rope are inclined to the vertical:

$$\cos. a = \frac{BM}{BK} = \frac{2{,}1125}{3{,}054} = 0{,}6917;$$

Fig. 143.



hence, $a = 46^\circ\, 14'$; and lastly, the tension of the rope $S_1 = S_2 = \frac{G}{2 \cos. a} = \frac{170}{2 \,.\, 0{,}6917} = 122{,}9$ lbs.*

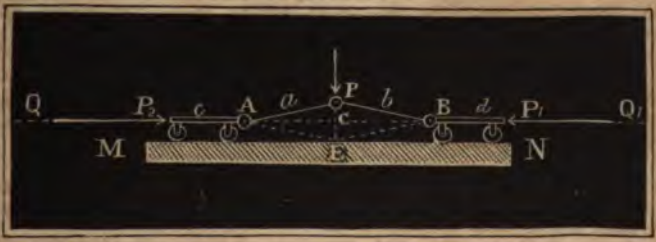
---

* If the demonstrations applied in the text to the simple funicular machine, where a single weight is represented as sustained by means of two parts of a flexible cord, attached to two fixed supports, be applied to the case of two rigid planes hinged together at a middle point, and also joined by hinges to two other planes capable of sliding to and from each other, but in opposite directions, then will the principles of the formulæ above given, be found to afford the relation between the force applied and the resistance which it is capable of overcoming, in the well-known machine called the *tricardo*, vulgarly the "toggle joint," which has been much applied of late years in the construction of printing, coining, and other presses.

When two ropes hang parallel to each other, the whole gravitating power of the *weight* is divided between them, and equally so between the points of support which sustain their upper extremities. The limit of the weight is the absolute strength of the ropes, and, in case of the tricardo, the force which could be applied to the planes would, *in that position*, be limited by the crushing force of the materials of the planes.

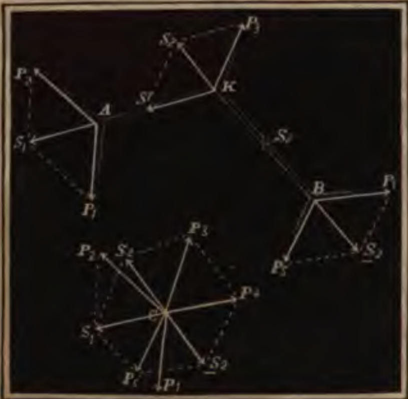
In the funicular machine, the question generally relates to the tension on the cords, not to the force tending to bring together the points of support, while, in the tricardo, the effort to separate the opposite extremities of the movable planes is the thing to be calculated. The following figure (143*) may render this more evident.

Fig. 143*.



Let a and b be the two planes of the *tricardo*, hinged at A and B to two other planes

Fig. 144.



§ 141. *Funicular Polygon.*—The relations of equilibrium in the funicular polygon, *i. e.* in a stretched cord which is acted upon by forces applied to different points, are in accordance with those of the equilibrium of forces, which are applied to one point. Let AKB, Fig. 144, be a cord stretched by the forces P_1, P_2, P_3, P_4, P_5: let P_1 and P_2 act at A, P_3 at K, and P_4 and P_5 at B. Let us put the tension of the portion $AK = S_1$ and that of $BK = S_2$, we shall then obtain S_1 for the resultant of P_1 and P_2 applied to A, and if we carry the point of application A of this tension from A to K, we shall again get S_2 for the resultant of S_1 and P_3, or of P_1, P_2, P_3; lastly, if we transport the point of application of S_2 from K to B, we shall then obtain in S_2, P_4 and P_5, or since S_2 is the resultant of P_1, P_2, P_3, also in P_1, P_2, P_3, P_4, P_5 a set of forces balancing each other. We may accordingly assert that, *when certain forces P_1, P_2, P_3, &c., hold a funicular polygon in equilibrium, they will hold each other in equilibrium also, if applied at a single point C, their direction and magnitude remaining invariable.*

If the cord $AK_1 K_2 \ldots B$, Fig. 145, be stretched at the points or nodes, K_1, K_2 by weights G_1, $G_2 \ldots$ and the extremities A and B by the vertical forces V_1 and V^n, and the horizontal forces H_1 and H^n, the sum of the vertical forces will be: $V_1 + V_n - (G_1 + G_2 + G_3 + \ldots)$ and of the horizontal forces: $H_1 - H_n$. The condition of equilibrium requires that both sums $= 0$; therefore

1. $V_1 + V_n = G_1 + G_2 + G_3 + \ldots$ and
2. $H_1 = H_n$; *i. e.*

In a funicular polygon stretched by weights, the sum of the vertical forces or vertical tensions at the extremities or points of suspension is

---

c and *d*, supposed to be capable of moving freely to and from each other along the plane *MN*. The hinge of *a* and *b* at *P* being supposed to be acted on by the small constant force *P*, the practical question is the relation of the resistances P_1, P_2 to this constant force *P*, in the different positions of the two planes *a* and *b*. If the angle PAC or $PBC = \alpha$ represent the angle of divergence of the planes *a* and *b* from the straight line QQ_1, it is evident that the force *P* will be represented by $CE = 2\ CP = 2\ sin.\ \alpha$, and the forces P_1 and P_2 each by $AB = 2\ cos.\ \alpha$. Hence $P : P_1 = sin.\ \alpha : cos.\ \alpha$; or as $tang.\ \alpha : rad.$ It is thus shown that the force applied at the central hinge of the "toggle joint" has to the force which resists the thrust of the planes, the relation of the sine to the cosine of the angle of inclination, or, what is the same, that of tangent to radius; or, in the case of the movable planes forming one and the same plane, that of 0 to 1, or 1 to ∞.

This subject will be found more fully treated of, and illustrated with figures of the tricardo, in a paper by the writer of this note in the *Journal of the Franklin Institute*, vol. iii. p. 354, for May, 1829.—AM. ED.

[illegible]

[illegible]

+ $R = 2PC = 2AP\sin a$

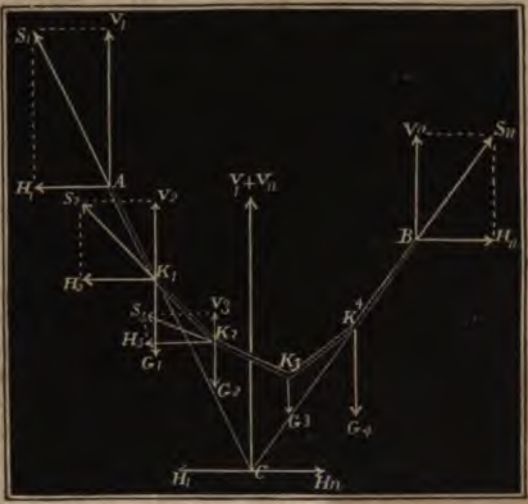
[illegible]

Equilibrium of an inelastic perfectly flexible cord ~~any point~~ of which all the points are under the action of several forces.

Let X, Y, Z, be the components of a force which act upon any point of a curve cord or a unit of length, so that an arc ds shall be acted upon by a force whose components shall be Xds, Yds, Zds. The extremities of the cord may be attached or acted upon by forces given in direction and intensity. It is required to find the conditions of equilibrium in the system which is but the Funicular Polygon of an infinite number of sides. In that case any infinitely small element will be held in equilibrium by means of the forces which are applied to it and inversely consequently if this takes place the entire thread will be in equilibrium. Let then ds be any element, it will be in equilibrium if its form is unchanged.(The forces which act upon it are the tensions applied to its extremities, which are directed respectively following the tangents at those points and in opposite directions, besides which forces, are Xds, Yds, Zds.

equivalent to the sum of the suspended weights, and the horizontal tension at the one extremity is equal and oppositely directed to the horizontal tension at the other extremity.

Fig. 145.



If the directions of the tensions S_1 and S_n at the cords *A* and *B* be prolonged to their intersection *C*, and the points of application of these tensions be transferred to this point, we shall then have the single force $P = V_1 + V_n$, because the horizontal forces H_1 and H_n counteract each other. Since this force holds in equilibrium the sum $G_1 + G_2 + G_3 + \ldots$ of the suspended weights, the point of application, or centre of gravity of these weights must, therefore, lie in the direction of the same, *i. e.*, in the vertical line passing through the point *C*.

§ 142. From the tension S_1 of the first portion AK_1 whose angle of inclination $S_1AH_1 = \alpha_1$, the vertical tension follows; $V_1 = S_1 \sin. \alpha_1$, and the horizontal $H_1 = S_1 \cos. \alpha_1$. If, now, we transfer the point of application of these forces from *A* to the first node K_1, the weight G_1 acting vertically downwards meets these tensions, and now for the following portion $K_1 K_2$, the vertical tension $V_2 = V_1 - G_1 = S_1 \sin. \alpha_1 - G_1$, for which the horizontal tension $H_2 = H_1 = H$ remains unchanged. Both forces united give the tension of the axis of the second portion $S_2 = \sqrt{V_2^2 + H^2}$ and its inclination α_2 by the formula

$$tang. \alpha_2 = \frac{V_2}{H} = \frac{S_1 \sin. \alpha_1 - G_1}{S_1 \cos. \alpha_1}, \text{ i. e.}$$

$$tang. \alpha_2 = tang. \alpha_1 - \frac{G_1}{H}.$$

If the point of application of the forces V_2 and H_2 is transferred from K_1 to K_2, we obtain in the weight G_2 meeting them another new vertical force, and therefore the vertical force of the third portion of the cord

$$V_3 = V_2 - G_2 = V_1 - (G_1 + G_2) = S_1 \, sin. \, \alpha_1 - (G_1 + G_2),$$
whilst the horizontal force H_3 remains $= H$. The whole tension of the third portion is

$S_3 = \sqrt{V_3^2 + H^2}$, and for its angle of inclination α_3, we have

$$tang. \, \alpha_3 = \frac{V_3}{H} = \frac{S_1 \, sin. \, \alpha_1 - (G_1 + G_2)}{S_1 \, cos. \, \alpha_1}, \text{ i. e.}$$

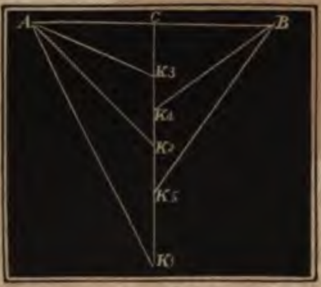
$$tang \, \alpha_3 = tang. \, \alpha_1 - \frac{G_1 + G_2}{H}.$$

For the angle of inclination of the fourth portion of the cord,

$$tang. \, \alpha_4 = tang. \, \alpha_1 - \frac{G_1 + G_2 + G_3}{H}, \text{ \&c.}$$

Besides, the tensions S_1, S_2, S_3, &c., as well as the angles of inclination α_1, α_2, α_3, &c., of the separate portions of the cord may easily be represented geometrically. If we make the horizontal line $CA = CB$, Fig. 146, $=$ the horizontal tension H and the vertical $CK_1 =$ vertical tension V_1 at the point of suspension A, the hypothenuse AK_1 gives the whole tension S_1 and the $\angle \, CAK_1$, also its inclination to the horizon; if now further we apply the weights G_1, G_2, G_3, &c., as parts K_1K_2, K_2K_3, &c., of CK, and draw the transversal lines AK_2, AK_3, &c., we shall have in them the tensions of the successive portions of the cord, and in the angles K_2AC, K_3AC, &c., the angles of inclination α_2, α_3, &c. of these portions.

Fig. 146.



§ 143. From the investigations of the preceding paragraph, the law for the equilibrium of cords stretched by weights, comes out thus:

1. *The horizontal tension is at all points of the cord one and the same, viz.:*

$$H = S_1 \, cos. \, \alpha_1 = S_n \, cos. \, \alpha_n.$$

2. *The vertical tension at any one point is equal to the vertical tension at the other extremity above it, less the sum of the intermediate suspended weights, therefore*

$$V_m = V_1 - (G_1 + G_2 + \dots G_{m-1}).$$

If the angle α_1 be known and the horizontal tension H, the vertical tension at the extremity A is known; $V_1 = H \, . \, tang. \, \alpha_1$, and accordingly that at the extremity B: $V_n = (G_1 + G_2 + \dots + G_n) - V_1$.

If, on the other hand, the angles of inclination α_1 and α_n at both points of suspension A and B are known, the horizontal and vertical tensions are given at the same time, viz.:

$$\frac{V_n}{V_1} = \frac{tang. \, \alpha_n}{tang. \, \alpha_1}, \text{ and, therefore,}$$

$$V_n = \frac{V_1 \, tang. \, \alpha_n}{tang. \, \alpha_1}.$$

The tension T varying constantly and at the same time ~~with~~ with the arc s of the curve, is a function of it, as are also the cosines $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ of the angles which the tangent makes with the axes. The components of the tension considered in the direction in which s is augmente are then, at the two extremities of ds, $T\frac{dx}{ds}, T\frac{dy}{ds}, T\frac{dz}{ds}$, and $T\frac{dx}{ds}+d(T\frac{dx}{ds}), T\frac{dy}{ds} + d(T\frac{dy}{ds}), T\frac{dz}{ds} + d(T\frac{dz}{ds})$. Let us suppose the arcs s counted from the extremity A and let MN be the infinitely small arc ds. The directions of the two fore tangent at the points M and N meet if it is a plane curve, an they may be considered to meet in a curve of double curvature for their shortest distance is infinitely small with respect to ds, th have then a resultant, and therefore all the forces which ac upon the curve MN must have a resultant equal and opposite to the first, and consequently comprised in the plane of the curve. The desired conditions of equilibrium are then those of forces applied at the same poin and they must express that th components parallel to each of t axes are separately null.

$\frac{V_n}{V_n} = \frac{V_1 \tan\alpha_n}{\tan\alpha_1}$ ∴ $V_1 + V_n = V_1\left(1 + \frac{\tan\alpha_n}{\tan\alpha_1}\right) =$

$V_1\left(\frac{\tan\alpha_1 + \tan\alpha_n}{\tan\alpha_1}\right)$ ∴ — . [illegible] $V_n = \frac{(\Sigma V_1) \tan\alpha_n}{[illegible] \tan\alpha_1 + \tan\alpha_n}$

From equation (1) page 16[illegible] we have

$\frac{V_n}{\tan\alpha_n} = \frac{\cot\alpha_1 V_1}{\tan\alpha_1}$ or $V_n \cot\alpha_n = V_1 \cot\alpha_1$

If then we observe the components $T\frac{dx}{ds}$, $T\frac{dy}{ds}$, $T\frac{dz}{ds}$, should change their signs we have the following equations.

$$1\begin{cases} d(T\frac{dx}{ds}) + X\,ds = 0 \\ d(T\frac{dy}{ds}) + Y\,ds = 0 \\ d(T\frac{dz}{ds}) + Z\,ds = 0 \end{cases}$$

If we multiply the first by $\frac{dx}{ds}$, the second $\frac{dy}{ds}$, and the third by $\frac{dz}{ds}$, and add them observing that $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$ and consequently $\frac{dx}{ds} d\frac{dx}{ds} + \frac{dy}{ds} d\frac{dy}{d} + \frac{dz}{ds} d\frac{dz}{ds}$ we will obtain $dT + X\,dx + Y\,dy + Z\,dz = 0$ an equation which can replace any one of equations 1. In the ordinary case $X\,dx + Y\,dy + Z\,dz$ is the differential of a function $\phi(x, y, z)$, and we will then have $T = -\phi(x, y, z) + ($ and if we know the tension T' at the point whose coördinates are $x_1' y_1' z_1'$ we will have $T - T' = \phi(x_1', y_1', z_1') - \phi(x, y, z_1)$ so that we may know the tension at every other point in terms of its coördinates. If we substitute T in two of the equations (1) we will have the [illegible]tion formed by the curve F[illegible]ed.

Since $V_1 + V_n = G_1 + G_2 + \ldots$, *i.e.*

$$\left(\frac{tang.\,\alpha_1 + tang.\,\alpha_n}{tang.\,\alpha_1}\right) V_1 = G_1 + G_2 \ldots, \text{ it follows that:}$$

$$V_1 = \frac{(G_1 + G_2 + \ldots)\, tang.\,\alpha_1}{tang.\,\alpha_1 + tang.\,\alpha_n},$$

$$V_n = \frac{(G_1 + G_2 + \ldots)\, tang.\,\alpha_n}{tang.\,\alpha_1 + tang.\,\alpha_n}, \text{ and from this:}$$

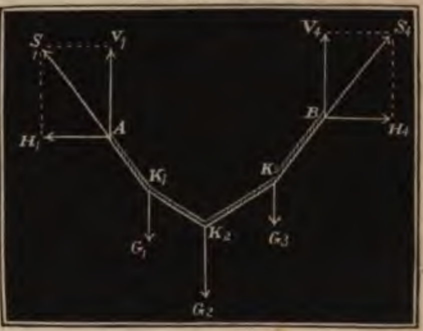
$$H = V_1\, cotg.\,\alpha_1 = V_n\, cotg.\,\alpha_n.$$

If both sides have the same inclination $\alpha_n = \alpha_1$, then $V_1 = V_n = \frac{G_1 + G_2 + \ldots + G_n}{2}$, and the one extremity A supports as much as the other B.

For the rest, these laws hold good also for the funicular polygon, especially when stretched by forces, if the directions of the forces are substituted for the verticals.

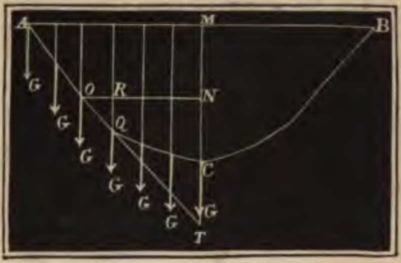
Example. The funicular polygon $AK_1 K_2 K_3 B$, Fig. 147, is stretched by three weights $G_1 = 20$, $G_2 = 30$, and $G_3 = 16$ lbs., as well as by the horizontal force $H_1 = 25$ lbs.; required to find the tensions of the axis and the angles of inclination of the sides, in the hypothesis that the ends of the string have the same inclination. Here the vertical tensions are equal, viz., $V_1 = V_4 = \frac{G_1 + G_2 + G_3}{2} = \frac{20 + 30 + 16}{2} = 33$ lbs. The vertical tension of the second portion of the string is $V_2 = V_1 - G_1 = 33 - 20 = 13$ lbs., that of the third $V_3 = V_4 - G_3$ or $(G_1 + G_2 - V_1) = 33 - 16 = 17$ lbs.; the angles of inclination α_1 and α_4 of the ends are determined by $tang.\,\alpha_1 = tang.\,\alpha_4 = \frac{V_1}{H} = \frac{33}{25} = 1{,}32$; that of the second and third portions by the $tang.\,\alpha_2 = tang.\,\alpha_1 - \frac{G_1}{H} = 1{,}32 - \frac{20}{25} = 0{,}52$, and $tang.\,\alpha_3 = tang.\,\alpha_4 - \frac{G_3}{H} = 1{,}32 - \frac{16}{25} = 0{,}68$; hence $\alpha_1 = \alpha_4 = 52° 51'$; $\alpha_2 = 27° 28'$ $\alpha_3 = 34° 13'$; lastly, the tensions of the axis are $S_1 = S_4 = \sqrt{V_1^2 + H^2} = \sqrt{33^2 + 25^2} = \sqrt{1714} = 41{,}40$ lbs., $S_2 = \sqrt{V_2^2 + H^2} = \sqrt{13^2 + 25^2} = \sqrt{794} = 28{,}18$ lbs., and $S_3 = \sqrt{V_3^2 + H^2} = \sqrt{17^2 + 25^2} = 30{,}23$ lbs.

Fig. 147.



§ 144. *The Parabola as Catenary.*—Let us suppose that the string ACB, Fig. 148, is stretched by equal weights G_1, G_2, &c., suspended at equal horizontal distances from each other. Let us represent by b_1 the horizontal distance AM between the point of suspension A and the lowest C, but the vertical distance CM by a. Let

Fig. 148.



us put further for another point O of the polygon, the corresponding co-ordinates $ON = y$ and $CN = x$. If, now, the vertical tension of A be $= V$, that of O will be $= \frac{y}{b} \cdot V$, and hence for the angle of inclination to the horizon, $NOT = ROQ = \phi$ of the portion of the string OQ, we shall have $tang. \phi = \frac{y}{b} \cdot \frac{V}{H}$, where H is the constant of the horizontal tension.

Hence $QR = OR \cdot tang. \phi = OR \cdot \frac{y}{b} \cdot \frac{V}{H}$ is the vertical distance of two adjacent angles of the funicular polygon. If we substitute for y OR, $2\ OR$, $3\ OR$, &c., the last equation will give the corresponding vertical distances of the first, second, and third angles, &c., reckoned from below upwards; then, if we add together all these values, whose amount may be $= m$, we shall obtain the height CN of the point O vertically above the lowest point C, viz.:

$$x = CN = \frac{V}{H} \cdot \frac{OR}{b} (OR + 2\ OR + 3\ OR + \ldots + m.\ OR)$$

$$\frac{V}{H} \cdot \frac{\overline{OR^2}}{b} (1 + 2 + 3 + \ldots + m) = \frac{V}{H} \cdot \frac{m(m+1)}{1\ .\ 2} \cdot \frac{OR^2}{b},$$

in accordance with the theory of arithmetical series.

Lastly, if OR be put $= \frac{y}{m}$, we shall have:

$$x = \frac{V}{H} \cdot \frac{m(m+1)}{2m^2} \cdot \frac{y^2}{b}.$$

If the number of weights be very great, $m + 1$ may be taken $= m$, whence we shall have:

$$x = \frac{V}{H} \cdot \frac{y^2}{2b}.$$

For $x = a$, $y = b$, hence also:

$a = \frac{V}{H} \cdot \frac{b}{2}$, and more simply:

$\frac{x}{a} = \frac{y^2}{b^2}$, which is the equation to a parabola.

If, therefore, a string devoid of weight be stretched by infinitely many weights applied at equal horizontal distances, the funicular polygon will pass into a parabola.

For the angle of inclination ϕ we have besides:

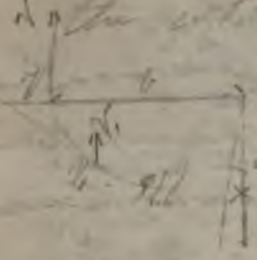
$$tang. \phi = \frac{y}{b} \cdot \frac{2a}{b} = 2y \cdot \frac{a}{b^2} = 2y \cdot \frac{x}{y^2} = \frac{2x}{y}, \text{ as also}$$

$$tang. \alpha = \frac{2a}{b}.$$

Therefore the tangent OT cuts the axis of the abscissæ, so that $CT = CN = x$.

If the chains and rods of a chain bridge, Fig. 149, were without

Shorter demonstration of the whole of page 184.



$V_1 = \frac{y}{h} V$
$\tan\phi = \frac{V_1}{H}$ $\frac{y\,dy}{2r}$
$x = \frac{Vy}{H} = \frac{y^2}{2r}$ (dem)

$V_1 = \frac{y}{h} V_0$ $\tan\phi = \frac{y}{h}\frac{V}{H} = \frac{V_1}{H}$ $dx = dy\tan\phi = \frac{V}{H}$

$x = \frac{V}{H}\frac{y^2}{2h}$

$+ \frac{V(m+1)}{H\,2m} \cdot \frac{y^2}{h}$

$$V_3 = V_2 - G_2 = V_1 - (G_1 + G_2) = S_1 \, sin. \, \alpha_1 - (G_1 + G_2),$$
whilst the horizontal force H_3 remains $= H$. The whole tension of the third portion is

$S_3 = \sqrt{V_3^2 + H^2}$, and for its angle of inclination α_3, we have

$$tang. \, \alpha_3 = \frac{V_3}{H} = \frac{S_1 \, sin. \, \alpha_1 - (G_1 + G_2)}{S_1 \, cos. \, \alpha_1}, \text{ i. e.}$$

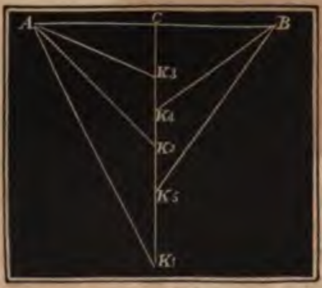
$$tang \, \alpha_3 = tang. \, \alpha_1 - \frac{G_1 + G_2}{H}.$$

For the angle of inclination of the fourth portion of the cord,

$$tang. \, \alpha_4 = tang. \, \alpha_1 - \frac{G_1 + G_2 + G_3}{H}, \text{ \&c.}$$

Besides, the tensions S_1, S_2, S_3, &c., as well as the angles of inclination α_1, α_2, α_3, &c., of the separate portions of the cord may easily be represented geometrically. If we make the horizontal line $CA = CB$, Fig. 146, $=$ the horizontal tension H and the vertical $CK_1 =$ vertical tension V_1 at the point of suspension A, the hypothenuse AK_1 gives the whole tension S_1 and the $\angle \, CAK_1$, also its inclination to the horizon; if now further we apply the weights G_1, G_2, G_3, &c., as parts K_1K_2, K_2K_3, &c., of CK, and draw the transversal lines AK_2, AK_3, &c., we shall have in them the tensions of the successive portions of the cord, and in the angles K_2AC, K_3AC, &c., the angles of inclination α_2, α_3, &c. of these portions.

Fig. 146.



§ 143. From the investigations of the preceding paragraph, the law for the equilibrium of cords stretched by weights, comes out thus:

1. *The horizontal tension is at all points of the cord one and the same, viz.:*

$$H = S_1 \, cos. \, \alpha_1 = S_n \, cos. \, \alpha_n.$$

2. *The vertical tension at any one point is equal to the vertical tension at the other extremity above it, less the sum of the intermediate suspended weights, therefore*

$$V_m = V_1 - (G_1 + G_2 + \ldots G_{m-1}).$$

If the angle α_1 be known and the horizontal tension H, the vertical tension at the extremity A is known; $V_1 = H \, . \, tang. \, \alpha_1$, and accordingly that at the extremity B: $V_n = (G_1 + G_2 + \ldots + G_n) - V_1$.

If, on the other hand, the angles of inclination α_1 and α_n at both points of suspension A and B are known, the horizontal and vertical tensions are given at the same time, viz.:

$$\frac{V_n}{V_1} = \frac{tang. \, \alpha_n}{tang. \, \alpha_1}, \text{ and, therefore,}$$

$$V_n = \frac{V_1 \, tang. \, \alpha_n}{tang. \, \alpha_1}.$$

The tension T varying constantly and at the same time with the arc s of the curve, is a function of it, as are also the cosines $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ of the angles which the tangent makes with the axes. The components of the tension considered in the direction in which s is augmente are then, at the two extremities of ds, $T\frac{dx}{ds}$, $T\frac{dy}{ds}$, $T\frac{dz}{ds}$, and $T\frac{dx}{ds}+d(T\frac{dx}{ds})$, $T\frac{dy}{ds} + d(T\frac{dy}{ds})$, $T\frac{dz}{ds} + d(T\frac{dz}{ds})$. Let us suppose the arcs s counted from the extremity A and let MN the infinitely small arc ds. The directions of the two forc tangent at the points M and meet if it is a plane curve, an they may be considered to meet a curve of double curvature, for their shortest distance is infinitely small with respect to ds, th have then a resultant, and therefore all the forces which ac upon the curve MN must have a resultant equal and opposite to the first, and consequently comprised in the plane of the curve. The desired conditions of equilibrium are then those of forces applied at the same poin and they must express that th components parallel to each of th axes are separately null.

$\therefore \gamma/_n = \frac{\gamma/_1 \tan\alpha_n}{\tan\alpha_1}$; $\gamma/_1 + \gamma/_n = \gamma/_1\left(1 + \frac{\tan\alpha_n}{\tan\alpha_1}\right)$

$\gamma/_1\left(\frac{\tan\alpha_1 + \tan\alpha_n}{\tan\alpha_1}\right)$ [illegible] $= \frac{[illegible]}{\tan\alpha_n + \tan\alpha_1}$

+ From equation (1) page [illegible] we have

$\frac{[illegible]}{\tan\alpha_n} = \frac{\cot\alpha_1}{\tan\alpha_1}$ [illegible] $= \gamma_1\cot\alpha_1$

If then we observe the components $T\frac{dx}{ds}$, $T\frac{dy}{ds}$, $T\frac{dz}{ds}$, should change their signs we have the following equations.

$$1\begin{cases} d(T\frac{dx}{ds}) + X\,ds = 0 \\ d(T\frac{dy}{ds}) + Y\,ds = 0 \\ d(T\frac{dz}{ds}) + Z\,ds = 0 \end{cases}$$

If we multiply the first by $\frac{dx}{ds}$, the second $\frac{dy}{ds}$, and the third by $\frac{dz}{ds}$, and add them observing that $(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 + (\frac{dz}{ds})^2 = 1$ and consequently $\frac{dx}{ds}d\frac{dx}{ds} + \frac{dy}{ds}d\frac{dy}{ds} + \frac{dz}{ds}d\frac{dz}{ds}$ we will obtain $dT + X\,dx + Y\,dy + Z\,dz = 0$ an equation which can replace any one of equations 1. In the ordinary case $X\,dx + Y\,dy + Z\,dz$ is the diffelential of a function $\phi(x, y, z)$, and we will then have $T = -\phi(x, y, z) + ($ and if we know the tension T' at the point whose coördinates are $x_1' y_1' z_1'$ we will have $T - T' = \phi(x_1', y_1', z_1') - \phi(x, y, z_1)$ so that we may know the tension at every other point in terms of its coördinates. If we substitute T in two of the equations (1) we will have the equation formed by the curve of the cord.

Since $V_1 + V_n = G_1 + G_2 + \ldots$, *i.e.*

$$\left(\frac{tang.\,\alpha_1 + tang.\,\alpha_n}{tang.\,\alpha_1}\right) V_1 = G_1 + G_2 \ldots, \text{ it follows that:}$$

$$V_1 = \frac{(G_1 + G_2 + \ldots)\, tang.\,\alpha_1}{tang.\,\alpha_1 + tang.\,\alpha_n},$$

$$V_n = \frac{(G_1 + G_2 + \ldots)\, tang.\,\alpha_n}{tang.\,\alpha_1 + tang.\,\alpha_n}, \text{ and from this:}$$

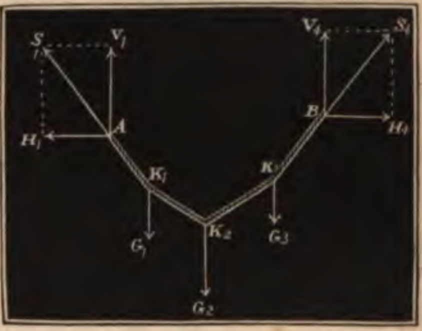
$$H = V_1\, cotg.\,\alpha_1 = V_n\, cotg.\,\alpha_n.$$

If both sides have the same inclination $\alpha_n = \alpha_1$, then $V_1 = V_n = \frac{G_1 + G_2 + \ldots + G_n}{2}$, and the one extremity *A* supports as much as the other *B*.

For the rest, these laws hold good also for the funicular polygon, especially when stretched by forces, if the directions of the forces are substituted for the verticals.

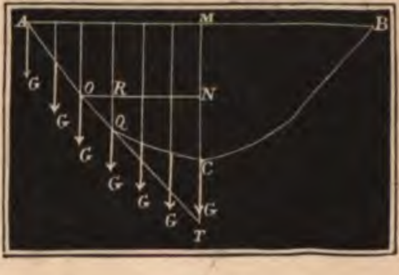
Example. The funicular polygon $AK_1 K_2 K_3 B$, Fig. 147, is stretched by three weights $G_1 = 20$, $G_2 = 30$, and $G_3 = 16$ lbs., as well as by the horizontal force $H_1 = 25$ lbs.; required to find the tensions of the axis and the angles of inclination of the sides, in the hypothesis that the ends of the string have the same inclination. Here the vertical tensions are equal, viz., $V_1 = V_4 = \frac{G_1 + G_2 + G_3}{2} = \frac{20 + 30 + 16}{2} = 33$ lbs. The vertical tension of the second portion of the string is $V_2 = V_1 - G_1 = 33 - 20 = 13$ lbs., that of the third $V_3 = V_4 - G_3$ or $(G_1 + G_2 - V_1) = 33 - 16 = 17$ lbs.; the angles of inclination α_1 and α_4 of the ends are determined by $tang.\,\alpha_1 = tang.\,\alpha_4 = \frac{V_1}{H} = \frac{33}{25} = 1{,}32$; that of the second and third portions by the $tang.\,\alpha_2 = tang.\,\alpha_1 - \frac{G_1}{H} = 1{,}32 - \frac{20}{25} = 0{,}52$, and $tang.\,\alpha_3 = tang.\,\alpha_4 - \frac{G_3}{H} = 1{,}32 - \frac{16}{25} = 0{,}68$; hence $\alpha_1 = \alpha_4 = 52° 51'$; $\alpha_2 = 27° 28'$ $\alpha_3 = 34° 13'$; lastly, the tensions of the axis are $S_1 = S_4 = \sqrt{V_1^2 + H^2} = \sqrt{33^2 + 25^2} = \sqrt{1714} = 41{,}40$ lbs., $S_2 = \sqrt{V_2^2 + H^2} = \sqrt{13^2 + 25^2} = \sqrt{794} = 28{,}18$ lbs., and $S_3 = \sqrt{V_3^2 + H^2} = \sqrt{17^2 + 25^2} = 30{,}23$ lbs.

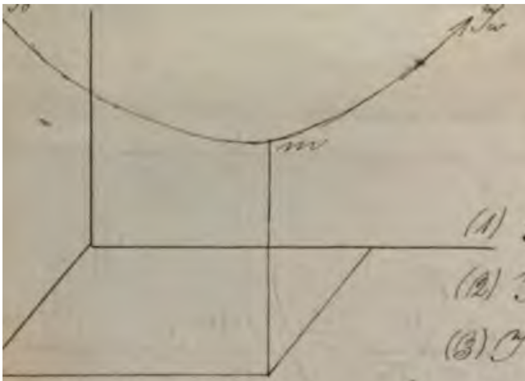
Fig. 147.



§ 144. *The Parabola as Catenary.*—Let us suppose that the string *ACB*, Fig. 148, is stretched by equal weights G_1, G_2, &c., suspended at equal horizontal distances from each other. Let us represent by b_1 the horizontal distance *AM* between the point of suspension *A* and the lowest *C*, but the vertical distance *CM* by a. Let

Fig. 148.





$T_0\ a_0\ b_0\ c_0$

$T_\omega\ a_\omega\ b_\omega\ c_\omega$

$p,\ \alpha,\ \beta,\ \gamma.$

(1) $T\frac{dx}{ds} = \int_s^{s_\omega}(p\,ds\cos\alpha + T_\omega\cos a_\omega)$

(2) $T\frac{dy}{ds} = \int_s^{s_\omega}(p\,ds\cos\beta + T_\omega\cos b_\omega)$

(3) $T\frac{dz}{ds} = \int_s^{s_\omega}(p\,ds\cos\gamma + T_\omega\cos c_\omega)$

T = tension at the point S. Equations (1), (2), (3) must be equal to zero in order to have equilibrium. Equation (4) is the expression for the tension at any point of the curve. Eqs (5) have been obtained by differentiating eqs (1), (2), (3). Eq (6) is obtained by adding eqs (5). Equation (7) is the expression for the elementary increase of tension as we pass along the curve. The second member of eq (6) is p × the cosine of the angle which the force makes with the curve.

$$\frac{dx}{dz} = \frac{(1)}{(3)} \quad ,, \quad \frac{dy}{dz} = \frac{(2)}{(3)}$$

(4) $T = (1)^2 + (2)^2 + (3)^2$

$$(5)\begin{cases} T\,d\frac{dx}{ds} + \frac{dx}{ds}\,dT = -p\,ds\cos\alpha \\ T\,d\frac{dy}{ds} + \frac{dy}{ds}\,dT = p\,ds\cos\beta \\ T\,d\frac{dz}{ds} + \frac{dz}{ds}\,dT = -p\,ds\cos\gamma \end{cases}$$

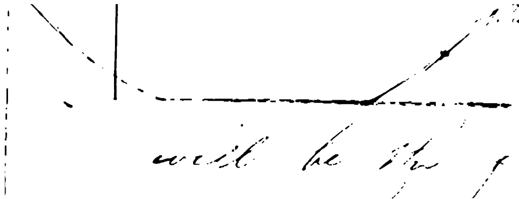
$$(6)\begin{cases} T\left(\frac{dx}{ds}d\frac{dx}{ds} + \frac{dy}{dy}d\frac{dy}{ds} + \frac{dz}{ds}d\frac{dz}{ds}\right) \\ + dT\left(\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right) \\ = -p(\cos\alpha\,dx + \cos\beta\,dy + \cos\gamma\,dz) \end{cases}$$

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1$$

$$2\left(\frac{dx}{ds}d\frac{dx}{ds} + \frac{dy}{ds}d\frac{dy}{ds} + \frac{dz}{ds}d\frac{dz}{ds}\right) = 0$$

(7) $\therefore -dT = p(\cos\alpha\,dx + \cos\beta\,dy + \cos\gamma\,dz) = X\,dx + Y\,dy + Z\,dz$

Equilibrium of a perfectly flexible inelastic cord acted upon by several forces at all of its points or a Funicular Polygon of an infinite number of sides. The cord will be in equilibrium when all of its elements are in equilibrium hence if we find the conditions of equilibrium for an element we have those for the whole curve. Let mn ds be any element it will be in equilibrium if its form is unchanged as we suppose this a perfectly flexible cord if the forces acting upon it have any resultant it will deflect the cord. Let R be the resultant of all the forces acting upon the curve for a unit of length then R ds



[illegible]

will be the force acting upon the element. It is now under the action of three forces the tensions at its two extremities and the force Rds whose components in the directions of the three coordinate axes we will suppose Xds Yds & Zds. Let the curve be attached at ~~either~~ each extremity or acted upon by forces given in intensity and direction; these will cause tensions at the extremities of the arc ds in the directions of the tangents at those points and directed opposite to each other, the tension T varying with the arc s of the curve is a function of it as are also the cosines $\frac{dx}{ds}$ $\frac{dy}{ds}$ & $\frac{dz}{ds}$ of the angles made by T with the three coordinate axes.

The components of the tension considered in the direction in which T is augmented and (at ~~either~~ each extremity of ds) at one $T\frac{dx}{ds}$, $T\frac{dy}{ds}$ and $T\frac{dz}{ds}$, at the other since T has received an increment of tension dT $T\frac{dx}{ds}+d(T\frac{dx}{ds})$, $T\frac{dy}{ds}+d(T\frac{dy}{ds})$, $T\frac{dz}{ds}+d(T\frac{dz}{ds})$.

The directions of the tension at the extremities of the arc ds i.e. the tangents intersect if ds is a plane curve and may be considered to intersect if it is a curve of double curvature since the shortest distance between them is infinitely small [illegible] supposed to [illegible] the arc ds. They have therefore a resultant which in equilibrium is to be held by the force Kds which cannot be unless they are equal & opposite

T_0 a_0 b_0 c_0

In which case the sum of the components parallel to each of the coördinate axes must separately equal zero. R ds the resultant of all the forces acting upon the element has the components X ds Y ds and Z ds. Since T acts contrarily to the direction in which it is augmented we consider it as negative and therefore our equations of equilibrium are $-T\frac{dx}{ds}+T\frac{dx}{ds}+d(T\frac{dx}{ds})+X$ $-T\frac{dy}{ds}+T\frac{dy}{ds}+d(T\frac{dy}{ds})+Y\,ds=0$ and thirdly $-T\frac{dz}{ds}+T\frac{dz}{ds}+d(T\frac{dz}{ds})+Z\,ds=0$ or reducing

$d(T\frac{dx}{ds})+X\,ds=0$ $d(T\frac{dy}{ds})+Y\,ds=0$ & $d(T\frac{dz}{ds})+Z\frac{dz}{ds}=$

Which are our first three equations of equilibrium. In order to place this under a different form let us perform the differentiations indicated and multiply equations (1) by $\frac{dx}{ds}$ " (2) $\frac{dy}{ds}$ and (3) by $\frac{dz}{ds}$.

The components of the tension considered in the direction in which T is augmented are (at each exremity of ds) at one $T\frac{dx}{ds}$, $T\frac{dy}{ds}$ and $T\frac{dz}{ds}$, at the other since T has received an increment of tension dT $T\frac{dx}{ds}+d(T\frac{dx}{ds})$, $T\frac{dy}{ds}+d(T\frac{dy}{ds})$, $T\frac{dz}{ds}+d(T\frac{dz}{ds})$ The directions of the tensions at the extremities of the arc ds i.e. the tangents intersect if it is a plane curve and may be considered to intersect if it is a curve of double curvature since the shortest distance between them is infinitely small as compared to the length of the arc ds. They have therefore a resultant which for equilibrium is to be held by the force Rds which cannot be unless they are equal & opposite

In which case the sum of the components parallel to each of the coordinate axes must separately equal zero. If ds be the resultant of all the forces acting upon the element has the components Xds Yds and Zds. Since T acts contrary to the direction in which it is measured we consider it as negative and therefore the equations of equilibrium are $-T\frac{dx}{ds} + T\frac{dx}{ds} + d(T\frac{dx}{ds}) + X ds$

$-T\frac{dy}{ds} + T\frac{dy}{ds} + d(T\frac{dy}{ds}) + Y ds = 0$ and similarly

$-T\frac{dz}{ds} + T\frac{dz}{ds} + d(T\frac{dz}{ds}) + Z ds = 0$ or reducing

$d(T\frac{dx}{ds}) + X ds = 0$ $d(T\frac{dy}{ds}) + Y ds = 0$ & $d(T\frac{dz}{ds}) + Z\frac{dz}{ds}$

Which are then the equations of equilibrium. In order to transform [illegible] [illegible] [illegible] we perform the differentiations indicated and multiplying equations (1) by $\frac{dx}{ds}$, (2) $\frac{dy}{ds}$ and (3) by $\frac{dz}{ds}$.

Differentiating we obtain

$dT\frac{dx}{ds} + Td\frac{dx}{ds} + Xds = 0$ $dT\frac{dy}{ds} + Td\frac{dy}{ds} + Yds = 0$

and $dT\frac{dz}{ds} + Td\frac{dz}{ds} + Zds = 0$ now multiplying and adding $dT\left(\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right) + T\left(\frac{dx}{ds} d\frac{dx}{ds} + \frac{dy}{ds} d\frac{dy}{ds} + \frac{dz}{ds} d\frac{dz}{ds}\right) + Xdx + Ydy + Zd$

but $\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = \frac{dx^2 + dy^2 + dz^2}{ds^2} = \frac{ds^2}{ds^2} = 1$ Differentiating this expression we have $2\frac{dx}{ds} d\frac{dx}{ds} + 2\frac{dy}{ds} d\frac{dy}{ds} + 2\frac{dz}{ds} d\frac{dz}{ds} = 0$ hence the coefficient of T disappears and our equation reduces to

$dT + Xdx + Ydy + Zdz = 0$. In the ordinary case $Xdx + Ydy + Zdz$ is a differential of some function of x y and z hence integrating $T = -\phi(x, y, z)$. If now we know the tension T_1 at any point whose coördinates are x_1, y_1, & z_1, then T_1 will $= f(x_1, y_1, z_1)$ $\therefore T - T_1 = \phi(xyz) - f(x_1 y_1 z_1)$ So that we may know the tension at any other point in terms of its coördinates. If we substitute T in two of equations (1)(2)(3) we get equation of curve at rest.

Case when the thread is act-
-ed upon by normal forces.

If the force whose components are X Y & Z is constantly normal to the curve we will $Xdx + Ydy + Zdz = 0$

For $X = R\cos\alpha$ $Y = R\cos\beta$ & $Z = R\cos\gamma$. Let us now divide our equation by ds and calling the angles made by the tang or tension a b & c their cosines are $\frac{dx}{ds}$ $\frac{dy}{ds}$ + $\frac{dz}{ds}$ substituting for the[m] values as also for X Y & Z we have

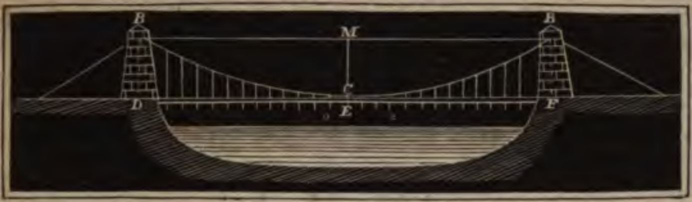
$$R(\cos a\cos\alpha + \cos b\cos\beta + \cos c\cos\gamma) = 0$$

but their coefficient of R is the cosine of the angle included between R and the tension but as R is normal this cosine is 0 and then our equation $Xdx + Ydy + Zdz = 0$ is true hence substituting this value in our equation [illegible] eq we obtain $dT = 0$ or $T = C$ hence when the force is normal the tension is constant throughout.

In this this case then the thread is equally drawn at every point, this would be the case for example when the curve lies upon a plane that produces no friction and then the forces were applied. the tension T being constant our equations (1) become $T d\frac{dx}{ds} = -X ds$ $T d\frac{dy}{ds} = -Y ds$ and $T d\frac{dz}{ds} = -Z ds$ whence squaring and adding we have now $T^2((d\frac{dx}{ds})^2 + (d\frac{dy}{ds})^2 + (d\frac{dz}{ds})^2) = (X^2 + Y^2 + Z^2) ds^2$ but $X^2 + Y^2 + Z^2 =$ the square of the force acting upon the curve which we will term P while the expression $\frac{ds^2}{(d\frac{dx}{ds})^2 + (d\frac{dy}{ds})^2 + \&c}$ is from the Calculus the term for the Radius of Curvature hence calling this radius ρ and substituting we have $T^2 = P^2 \rho^2$ or $P = \frac{T}{\rho}$ so that whatever be the normal force the curve will take such a form that this force shall be inversely as the radius of Curvatu

weight, or light enough in respect to the weight of the loaded bridge *DEF*, which only is to be taken into consideration, then the chain *ACB* would form a parabola.

Fig. 149.



Example. The whole load of a chain-bridge in Fig. 149, $=320000$ lbs.; the span $AB = 2b = 150$ feet, and the height of the arch $CM = a = 15$ feet; to find the tensions and other relations of the chains. The inclinations of the ends of the chain to the horizon is determined by the formula, $tang.\ \alpha = \frac{2a}{b} = \frac{30}{75} = \frac{2}{5} = 0{,}4$, therefore, $\alpha = 21°\ 48'$. The vertical tension at each point of suspension is $V_1 = \frac{1}{2}$ the weight $= 160000$ lbs.; the horizontal, $H = V_1\, cotg.\ \alpha = 160000 \cdot \frac{1}{0{,}4} = 400000$ lbs.; lastly, the whole tension at one end:

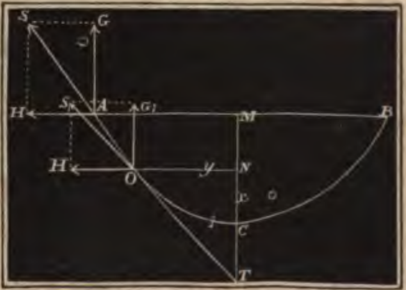
$$S = \sqrt{V^2 + H^2} = V\sqrt{1 + cotg.\ \alpha^2} = 160000 \cdot \sqrt{1 + \left(\frac{1}{0{,}4}\right)^2}$$

$$= 160000\sqrt{\frac{29}{4}} = 80000\sqrt{29} = 430813 \text{ lbs.}$$

§ 145. *Catenary.*—When a perfectly flexible and extensible string suspended from two points, or a chain consisting of short links, is stretched by its own weight, its axis forms a curved line, to which the name of catenary has been given. The imperfectly elastic and extensible cords, ropes, bands, chains, &c., met with in practice, give curved lines which approximate to the catenary only, but may usually be treated as such. From the foregoing, the horizontal tension of the catenary is equally great at all points, on the other hand, the vertical tension is equivalent to the vertical tension of the points of suspension lying above it, less the weight of the portions of the chain above. Since the tension at the vertex, where the catenary is horizontal, is null, the vertical tension, therefore, at the point of suspension, is equivalent to the weight of the chain from that point to the vertex; and the vertical tension at each place also equivalent to the weight of the portion of the rope or chain lying below it.

If equal lengths of the chain be equally heavy, we have then the common catenary, which only we will now consider. If

Fig. 150.

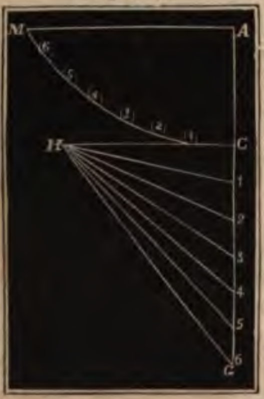


a portion of the rope, or chain one foot in length, weighs γ, and if the arc corresponding to the co-ordinates $CM = a$ and $MA = b$, Fig. 150, $AOC = l$, we then have the weight of the portion of the chain $AOC = l\gamma$; if, on the other hand, the length of the arc (l) corresponding to the co-ordinates ($CN = x$, and $NO = y$) $= s$, we have the weight of this arc $= s\,\gamma$. If we put the length of a similar portion, whose weight $= H$, $= c$, (the horizontal tension,) we have further $H = c\,\gamma$, and, therefore, for the angles of inclination α and ϕ at the points A and O:

$$tang.\ \alpha = tang.\ SAH = \frac{G}{H} = \frac{l\gamma}{c\gamma} = \frac{l}{c}, \text{ and}$$

$$tang.\ \phi = tang.\ NOT = \frac{s\,\gamma}{c\,\gamma} = \frac{s}{c}.$$

Fig. 151.



§ 146. If we make the horizontal line CH, Fig. 151, $=$ the length c of the portion of chain measuring the horizontal tension, and $CG =$ the length l of the arc of the chain on one side, we have, in accordance with § 142, in the hypothenuse GH, the measure and direction of the funicular tension at the point A, for

$$tang.\ CHG = \frac{CG}{CH} = \frac{l}{c} \text{ and}$$

$$\overline{GH} = \sqrt{\overline{CG}^2 + \overline{CH}^2} = \sqrt{l^2 + c^2},$$

$$\text{or } S = \sqrt{G^2 + H^2} = \sqrt{l^2 + c^2}\,.\,\gamma$$

$$= \overline{GH}\,.\,\gamma.$$

If now we divide CG into equal parts and draw from H to the points 1, 2, 3, &c., straight lines, these will give the measure and directions of the tensions of those points of the catenary which we obtain when we divide the length of the catenary arc AC into as many equal parts. So, for example, the line $H3$ gives the measure and direction of the tension or the tangents at the point (3) to the arc AC, because in this point the vertical tension $= C3\,.\,\gamma$, whilst the horizontal tension remains the same $= c\,.\,\gamma$, therefore for this point $tang.\ \phi = \frac{C3\,.\,\gamma}{c\gamma} = \frac{C3}{CH}$, which the figure actually gives.

This peculiarity of the catenary is of use in constructing this curve mechanically, with an approximation to correctness After the given length CG of the catenary arc for construction has been divided into very many equal parts, the line $CH = c$ measuring the horizontal tension is applied to it, and the transversal lines $H1$, $H2$, $H3$, &c., drawn; if a part $C1$ of the arc be placed upon CH, and through the point of division obtained (1) a parallel to $H1$ be drawn, which cuts off from it a part (12); and likewise through the point (2) another line parallel to $H2$ be drawn, and which cuts off from it a point (23) equal to a part of the arc, and again through this (3) another, parallel

The tightly stretched Catenary.

Let s = the length of the cord, sy will represent the weight of the same. V = vertical tension. H = cy From the equation of the circle $y^2 = 2rx - x^2$ whence $r = \frac{y^2}{2x} + \frac{x}{2}$ but $\frac{x}{2}$ is so small compared to $\frac{y^2}{2x}$ that it may be neglected. Hence $r = \frac{y^2}{2x}$, $\tan\varphi = \frac{dx}{dy} = \frac{V}{H} = \frac{sy}{cy} = \frac{s}{c}$, $\sin\varphi = \frac{y}{r}$ (when $y = y_1$ and $x = x_1$) $= \frac{2xy}{y^2} = \frac{2x}{y}$. But $\varphi = \sin\varphi + \sin\varphi^3 \frac{1}{6} + \frac{3}{40} \sin^5\varphi + \ldots$. Regarding only the first two terms and substituting in the above found values of $\sin\varphi$ we have $\varphi = \frac{2x}{y} + \frac{1}{6}\left(\frac{2x}{y}\right)^3 = \frac{2x}{y} + \frac{4}{3}\left(\frac{x}{y}\right)^3$

Now the arc $s = r\varphi = \frac{y^2}{2x}\left[\frac{2x}{y} + \frac{4}{3}\left(\frac{x}{y}\right)^3\right] = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] = y + \frac{2}{3}\frac{x^2}{y}$. This is the equation of the arc s and from this equation we can obtain one containing x and y and into which does not enter. this will be the equation of the Catenary.

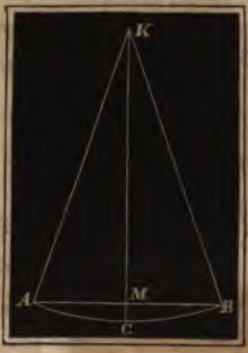
to $H3$, and (34) be made equal to another part of the arc, and we proceed in this manner, we shall obtain a polygon ($C\,1\,2\,3\,4\ldots$); as we have taken these sides very small, we may consider it as a curve and easily find the curve to it, if we connect the middle points of the small sides ($C1$), (12), (23), by a trace or line.

For practical purposes, a finely linked chain suspended against a perpendicular wall enables us to determine accurately enough a catenary answering certain conditions, as those of given length and height, or of given width or length of the arc.

§ 147. In many cases, and also in applications to architecture and to machines, the horizontal tension of the catenary is very great, and the height of the arc small in comparison with the width. Under this supposition, an equation to this curve is obtained in the following manner.

Let s be the length, $x = CM$ the absciss, and $y = AM$ the ordinate of a very compressed arc AC, Fig. 152. If we make $AK = CK$, we may consider this arc as a circular one described from K as a centre. Since from the known equation of the circle $y^2 = x(2r - x)$, it follows that the radius CK of the circle, $r = \frac{y^2}{2x} + \frac{x}{2}$, or more simply, if we neglect $\frac{x}{2}$ as small in comparison with $\frac{y^2}{2x}$, $r = \frac{y^2}{2x}$. For the angle $AKC = \phi^0$, subtended at the centre by AB: $sin.\ \phi = \frac{AM}{AK} = \frac{y}{r} = \frac{2x}{y}$, and the arc $\phi = sin.\ \phi + \frac{1}{6}\, sin.\ \phi^3 + \frac{3}{40}\, sin.\ \phi^5 + \ldots$; if we have regard only to the two first members, it therefore follows that:

Fig. 152.



$$\phi = \frac{2x}{y} + \frac{1}{6}\cdot\left(\frac{2x}{y}\right)^3 = \frac{2x}{y} + \frac{4}{3}\cdot\left(\frac{x}{y}\right)^3.$$

Now the arc $AC = s = r\phi = \frac{y^2}{2x}\cdot\phi$; hence:

$$s = y + \frac{2}{3}\cdot\frac{x^2}{y} = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right].$$

But inversely, $y = \dfrac{s}{1 + \frac{2}{3}\left(\frac{x}{y}\right)^2}$, which may be put:

$$y = s\left[1 - \frac{2}{3}\left(\frac{x}{s}\right)^2\right], \text{ and on the other hand:}$$

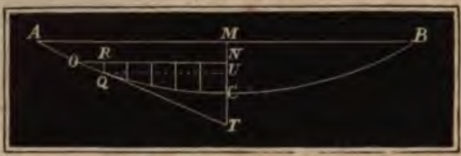
$$x = \sqrt{\frac{3}{2}y(s - y)}.$$

Example. The width of a very compressed arc, whose law for the rest is not known, is $2\,b = 3{,}5$ feet, and the height $a = 0{,}25$ feet; its length, therefore, is:

$$2\,l = 3{,}5\left[1 + \frac{2}{3}\cdot\left(\frac{0{,}25}{1{,}75}\right)^2\right] = 3{,}5\left(1 + \frac{2}{3}\,.\,0{,}143^2\right) = 3{,}5 + 3{,}5\,.\,0{,}0136 = 3{,}548 \text{ ft.}$$

§ 148. We will now apply the formula $s = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$ for

Fig. 153.



the length of a compressed arc to a strongly stretched catenary ACB, Fig. 153, while we put the vertical tension at a point O, $= V = s\gamma$ $= y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] . \gamma$, and therefore for the angle made by the tangent $TON = \phi$, $tang. \phi = \frac{s}{c} = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$.

If we divide the ordinate y into m equal parts, we find the portion $RQ = NU$ of the absciss x corresponding to one such part OR, when we put $RQ = OR . tang. \phi = OR . \frac{y}{c}\left[1 + \left(\frac{x}{y}\right)^2\right]$.

Since x is small in comparison with y, RQ is approximately $= OR . \frac{y}{c}$. If now we put $OR = \frac{y}{m}$ and successively for y: $\frac{y}{m}, \frac{2y}{m}, \frac{3y}{m}$, &c., we obtain by degrees the several parts of x, whose sum, therefore, is $x = \frac{y^2}{cm^2}(1 + 2 + 3 + \ldots + m) = \frac{y^2}{cm^2} . \frac{m(m+1)}{2}$ (§ 144) $= \frac{y^2}{2c}$, and which corresponds with the equation to the parabola.

But if we wish to attain greater accuracy, we must put $QR = OR . \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$, substitute for x its value last found $\frac{y^2}{2c}$, and we shall then obtain:

$$QR = OR . \frac{y}{c}\left(1 + \frac{1}{6} \cdot \frac{y^2}{c^2}\right) = \frac{OR}{c}\left(y + \frac{1}{6} \cdot \frac{y^3}{c^2}\right).$$

Let us again successively put $y = \frac{y}{m}, \frac{2y}{m}, \frac{3y}{m}$, &c., and for OR likewise $\frac{y}{m}$, we shall then find the several values of x, and the sum itself:

$$x = \frac{y}{cm}\left[\frac{y}{m}(1+2+3+\ldots+m) + \frac{1}{6\,c^2}\cdot\left(\frac{y}{m}\right)^3(1^3+2^3+3^3+\ldots+m^3)\right].$$

Now for a very great number of members, the sum of the natural numbers from 1 to $m = \frac{m^2}{2}$, and the sum of their cubes $= \frac{m^4}{4}$, accordingly:

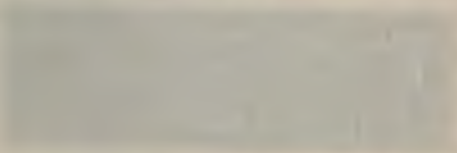
$$x = \frac{y}{c}\left(\frac{y}{2} + \frac{1}{6\,c^2} \cdot \frac{y^3}{4}\right) i.\ e.$$

Tan $\varphi = \frac{s}{c} = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$ as x is very small with respect to y we may neglect the second term in the parenthesis therefore $\tan\varphi = \frac{dx}{dy} = \frac{y}{c}$; hence $dx = \frac{y\,dy}{c}$ or $x = \frac{y^2}{2c}$. To obtain a more correct result substitute the value x in the above equation whence we get

$$\frac{dx}{dy} = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{y}{2c}\right)^2\right] = \frac{y}{c}\left[1 + \frac{1}{6}\left(\frac{y}{c}\right)^2\right]. \quad dx = \frac{y\,dy}{c}\left[1 + \frac{1}{6}\left(\frac{y}{c}\right)^2\right.$$

$$x = \frac{y^2}{2c} + \frac{y^4}{24c^3} = \frac{y^2}{2c}\left[1 + \frac{y^2}{12c^2}\right].$$

[illegible]



[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

1. $x = \frac{y^2}{2c} + \frac{y^4}{24\,c^3} = \frac{y^2}{2c}\left[1 + \frac{1}{12}\cdot\left(\frac{y}{c}\right)^2\right]$, the equation of a strongly stretched catenary.

By inversion it follows that $y^2 = 2\,c\,x - \frac{y^4}{12\,c^2} = 2\,c\,x - \frac{4\,c^2\,x^2}{12\,c^2}$ $= 2\,cx - \frac{x^2}{3}$, therefore:

2. $y = \sqrt{2\,c\,x - \frac{x^2}{3}}$, or approximately $= \sqrt{2\,c\,x}\left(1 - \frac{x}{12c}\right)$.

The measure of the horizontal tension is further given:

$$c = \frac{y^2}{2x} + \frac{y^4}{2\,x\,.\,12\,c^2} = \frac{y^2}{2x} + \frac{y^4}{24\,x}\,.\,\frac{4\,x^2}{y^4},\ i.\ e.$$

3. $c = \frac{y^2}{2x} + \frac{x}{6}$.

The angle of the tangent ϕ is determined by:

$$tang.\ \phi = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] = \frac{y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]}{\frac{y^2}{2x}\left[1 + \frac{1}{3}\left(\frac{x}{y}\right)^2\right]}$$

$$= \frac{2x}{y}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]\left[1 + \frac{1}{3}\left(\frac{x}{y}\right)^2\right],\ i.\ e.$$

4. $tang.\ \phi = \frac{2x}{y}\left[1 - \frac{1}{3}\left(\frac{x}{y}\right)^2\right]$.

Lastly, we must here place the formula of rectification found in the former paragraph:

5. $s = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] = y\left[1 + \frac{1}{6}\left(\frac{y}{c}\right)^2\right]$.

Example.—1. For a span $2\,b = 16$ feet and height of arc $a = 2\frac{1}{2}$ feet, the length $2\,l$ is $= 16\left[1 + \frac{2}{3}\left(\frac{2{,}5}{8}\right)^2\right] = 16 + 16\,.\,0{,}065 = 17{,}04$ feet, the length of the portion of chain which measures the horizontal tension: $c = \frac{b^2}{2a} + \frac{a}{6} = \frac{64}{5} + \frac{5}{12} = 12{,}8 + 0{,}417$ $= 13{,}217$ feet; the tangent of the angle of suspension: $tang.\ \alpha = \frac{2a}{b}\left[1 + \frac{1}{3}\left(\frac{a}{b}\right)^2\right]$ $= \frac{5}{8}\left[1 + \frac{1}{3}\left(\frac{5}{16}\right)^2\right] = \frac{5\,.\,1{,}03255}{8} = 0{,}6453\ldots$, the angle of suspension, therefore, $\alpha = 32°\ 50'$.—2. A chain of 10 feet length and $9\frac{1}{2}$ span, has the height of its arc

$$a = \sqrt{\frac{3}{2}(l - b)\,b} = \sqrt{\frac{3}{2}\,\frac{(10 - 9\frac{1}{2})}{2}\,\frac{9\frac{1}{2}}{2}} = \sqrt{\frac{3}{2}\,.\,\frac{19}{16}} = \sqrt{\frac{57}{32}}$$

$= \sqrt{1{,}7812} = 1{,}335$ feet, and the measure of the horizontal tension:

$$c = \frac{b^2}{2a} + \frac{a}{6} = \frac{4{,}75^2}{2\,.\,1{,}335} + \frac{1{,}335}{6} = 8{,}673 \text{ feet.}$$

3. If a line 30 feet long and weighing 8 lbs., be stretched horizontally by a force of 20 lbs., the vertical tension $V = \frac{1}{2}\,G = 4$ lbs.; the horizontal force $H = \sqrt{S^2 - V^2}$ $= \sqrt{20^2 - 4^2} = \sqrt{384} = 19{,}596$ lbs. The tangent of the angle of suspension: $tang.\ \phi = \frac{V}{H} = \frac{4}{19{,}596} = 0{,}20412$, the angle ϕ itself $= 11°\ 32'$; the measure of

the horizontal tension $c = \frac{H}{\gamma} = H \div \frac{8}{30} = \frac{30}{8} H = 73{,}485$ feet; the span $2\,b$ $= 2\,l\left[1 - \frac{1}{6}\cdot\left(\frac{l}{c}\right)^2\right] = 30 . \left[1 - \frac{1}{6}\cdot\left(\frac{15}{73{,}48}\right)^2\right] = 30 - 0{,}208 = 29{,}792$ ft., and the height of the arc $a = \sqrt{\frac{3}{2}\,b\,(l - b)} = \sqrt{\frac{3}{2}\,\frac{29{,}792\,.\,0{,}208}{2\,.\,2}}$ $= \sqrt{29{,}792\,.\,0{,}078} = 1{,}524$ feet.

§ 149. The higher calculus gives the following general formulæ for the catenary, and which hold good for all tensions.

1. $s = \sqrt{2\,c\,x + x^2}$, and inversely, $x = \sqrt{c^2 + s^2} - c$ and $c = \frac{s^2 - x^2}{2x}$.

2. $s = \frac{c}{2}\left(e^{\frac{y}{c}} - e^{-\frac{y}{c}}\right)$, inversely $y = c\,L\,n\left(\frac{s + \sqrt{c^2+s^2}}{c}\right)$, where e is the base 2,71828 of the natural system of logarithms, and $L\,n$ the logarithm = 2,30258 times the common logarithm.

3. $y = c\,L\,n\left(\frac{c + x + \sqrt{2\,c\,x + x^2}}{c}\right)$, inversely $x = \frac{c}{2}\left(e^{\frac{y}{c}} + e^{-\frac{y}{c}}\right) - c$,

4. $y = \frac{s^2 - x^2}{2x}\,L\,n\left(\frac{s + x}{s - x}\right)$.

The use of these formulæ is very troublesome, especially in complicated problems, where a direct solution is generally not possible.

Example. The two co-ordinates of a catenary are $x = 2$ feet, and $y = 3$ feet; required the horizontal tension c of this curve? Approximately from No. 3 of the former paragraphs $c = \frac{y^2}{2x} + \frac{x}{6} = \frac{9}{4} + \frac{2}{6} = 2{,}58$. From No. 3 of the present paragraphs y is exactly $= c\,Ln\left(\frac{c + x + \sqrt{2\,c\,x + x^2}}{c}\right)$, i. e. $3 = c\,Ln\left(\frac{c + 2 + \sqrt{4\,c + 4}}{c}\right)$. If c be here put $= 2{,}58$, we then have the error $f = 3 - 2{,}58\,Ln\left(\frac{4{,}58 + 2\sqrt{3{,}58}}{2{,}58}\right)$ $= 3 - 2{,}58\,Ln\left(\frac{8{,}3642}{2{,}58}\right) = 3 - 3{,}035 = -0{,}035$; but if c be put $= 2{,}53$, we then have the error $f_1 = 3 - 2{,}53\,Ln\left(\frac{4{,}53 + 2\sqrt{3{,}53}}{2{,}53}\right) = 3 - 2{,}53\,Ln\left(\frac{8{,}2876}{2{,}53}\right)$ $= 3 - 3{,}002 = -0{,}002$. In order now to find the true value of c, if, according to a known rule, we put

$$\frac{c - 2{,}58}{c - 2{,}53} = \frac{f}{f_1} = \frac{0{,}035}{0{,}002} = 17{,}5;\ \text{in this manner it will follow that:}$$

$$16{,}5\,.\,c = 17{,}5\,.\,2{,}53 - 2{,}58 = 41{,}69;\ \text{therefore:}$$

$$c = \frac{41{,}69}{16{,}5} = 2{,}527\ \text{feet.}$$

Remark. Practical applications of the catenary will be given when, in the Second Part, we come to treat of the construction of vaults, chain-bridges, &c.

§ 150. *The Pulley.*—Ropes, cords, &c., are the usual means by which forces are transmitted over the wheel and axle. We will here develop what is most general in the theories of these two arrangements, without, however, taking into account friction and rigidity of cords.

A pulley is a circular disc, *ABC*, Fig. 154 and Fig. 155, turning about an axis on whose circumference lies a cord or string, and whose

Method of obtaining the four formu=
=lae upon page 140. The Horizon=
=tal tension is represented by the
chain whose length is = H. and
V = the vertical tension $\tan\phi = \frac{dx}{dy} = \frac{V}{H}$
$= \frac{sy}{cy} = \frac{s}{c} \cdot \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2+dy^2}} = \frac{dx}{dx\sqrt{1+\frac{dy^2}{dx^2}}}$
$= \frac{dx}{dx\sqrt{1+\left(\frac{c}{s}\right)^2}}$ $\frac{dx}{ds} = \frac{1}{\sqrt{1+\frac{c^2}{s^2}}} = \frac{s}{\sqrt{c^2+s^2}}$ $\frac{s\,ds}{\sqrt{c^2+s^2}} =$
$c^2+s^2=z^2$ $s\,ds = z\,dz$ (a.) $\frac{z\,dz}{z} = dz$, $2\sqrt{c^2+}$
shows the manner of
integration. (a.) shows
that the s of (2) is c^2+s^2.
$\sqrt{c^2+s^2} = x + C$. $x=0$. $s=0$. $C=c$
$x=\sqrt{c^2+x^2}-c$ first formula
given in the text. $\frac{dy}{dx} = \frac{s}{c}$
$\frac{dy^2}{ds^2-dy^2} = \frac{c^2}{s^2}$ $(s^2-c^2)dy^2 = c^2ds^2$ whence $dy = \frac{c\,ds}{\sqrt{s^2+c^2}}$. To
integrate this place $\sqrt{s^2+c^2} = z - s$ ∴ $s^2+c^2 = z^2-2zs+$
$s = \frac{z}{2} - \frac{c^2}{2z}$. differentiating $ds = \frac{dz}{2} + \frac{c^2}{2} \cdot \frac{dz}{z^2}$
$= \frac{(z^2+c^2)dz}{2z^2}$; $\sqrt{c^2+s^2} = z - s$ substitute the value
of $s = \frac{z}{2} + \frac{c^2}{2z} = \frac{z^2+c^2}{2z}$. $\int\frac{c\,ds}{\sqrt{c^2+s^2}} = \int\frac{c\,dz}{z} = c\log z + C$ to
find the value of C. $y=0$. $s=0$. $z=c$
$0 =$ ~~$c(\log z - \log c$~~ $= c\log c + C = -c\log c$. $y = c(\log z$
$- \log c) = c\log\left(\frac{z}{c}\right) = c\log\left(\frac{\sqrt{s^2+c^2}+s}{c}\right)$. 2d equation
$\frac{s+\sqrt{s^2+c^2}}{c} = e^{\frac{y}{c}}$ solving with respect to s
$s+\sqrt{c^2+s^2} = ce^{\frac{y}{c}}$ $c^2+s^2 = c^2e^{\frac{2y}{c}} - 2ce^{\frac{y}{c}}s + s^2$. $s = \frac{c^2e^{\frac{2y}{c}}-c^2}{2ce^{\frac{y}{c}}}$
chang form $= \frac{c^2}{2c}\left(\frac{e^{\frac{2y}{c}}-1}{e^{\frac{y}{c}}}\right)$. $s = \frac{c}{2}\left(e^{\frac{y}{c}} - e^{-\frac{y}{c}}\right)$ = first
form of 2 in the text. To find the second
form of formula $\sqrt{c^2+s^2} = c+s$. $s = \sqrt{2cx+x^2}$
substituting $y = c\log\left(\frac{c+x+\sqrt{2cx+x^2}}{c}\right)$ first
form of 3 passing to numbers $\frac{c+x+\sqrt{2cx+x^2}}{c}$
$\sqrt{2cx+x^2} = ce^{\frac{y}{c}} - (c+x)$. $2cx+x^2 = c^2e^{\frac{2y}{c}} - 2ce^{\frac{y}{c}}(c+x) + c^2 + 2cx + x^2$
$c+x = \frac{c^2e^{\frac{2y}{c}}+1}{2ce^{\frac{y}{c}}} = \frac{c}{2}\left(e^{\frac{y}{c}} + e^{-\frac{y}{c}}\right)$ the form of formula 3.
To find formula 4 take the expression of y from
$y = c\log\left(\frac{\sqrt{s^2+c^2}+s}{c}\right)$. from 1 we have $\sqrt{c^2+x^2} = c+x = \frac{s^2+x^2}{2x} + x$
substit. $c = \frac{s^2+x^2}{2x}$ we have $y = c\log\left(\frac{s+\frac{s^2+x^2}{2x}}{\frac{s^2-x^2}{2x}}\right) = c\log\left(\frac{(s+x)^2}{(s^2-x^2)}\right)$. $y = c\log$
subst. value c, $y = \frac{s^2-x^2}{2x}\log\left(\frac{s+x}{s-x}\right)$. Form (4) of the text.
we have here four unknown quantities in equation 1,
y 1 known, 2d x known, 3 s known, and 4th c known

extremities are stretched by the forces P and Q. In a fixed pulley, the block in which the axis or pivot reposes is immovable; in a free pulley, on the other hand, it is movable.

Fig. 154.

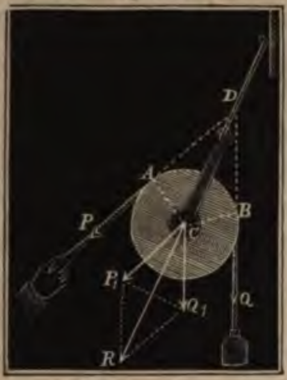
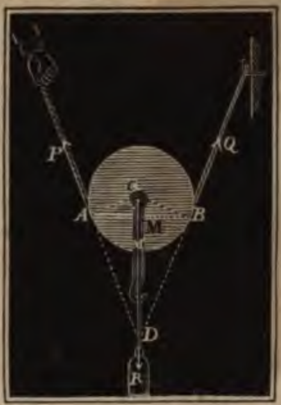


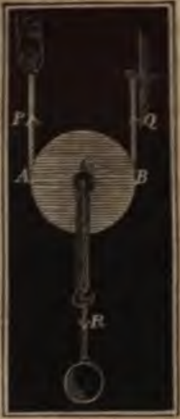
Fig. 155.



In the condition of equilibrium of a pulley, the forces P and Q at the extremities of the string are equal; for every pulley is a bent lever, the arms of which are equal in length, which we may obtain if we let fall perpendiculars CA and CB from the axis C on the directions of the forces, or of the strings DP and DQ. It is clear that the forces P and Q in any revolution about C describe the same space, viz. $r\,\phi$, if r be the radius $CA = CB$ and ϕ° the angle of revolution; and that from this we may infer the equality between P and Q. From the forces P and Q there arises the resultant $CR = R$, which is taken up by the block and is dependent on the angle $ADB = \alpha$, which the directions of the string include; and moreover it gives as the diagonal of the rhomb CP_1RQ_1 constructed from P and α : $R = 2\,P\cos.\frac{\alpha}{2}$.

§ 151. In the fixed pulley, Fig. 154, the force Q consists of the weight to be overcome or raised at one extremity of the string; here, therefore, the force is equal to the weight, and the application of this pulley effects nothing but a change of direction. In the movable pulley, Fig. 155, on the other hand, the weight on the hook R acts at the extremity of the block, whilst the one extremity of the string is fastened to a fixed object; here, thereforce, the force P is to be put $= \frac{R}{2\cos.\frac{\alpha}{2}}$. If we represent the chord AMB, which corresponds to the arc over which the string passes, by a, the radius $CA = CB$, as before $= r$, then $a = 2\,AM = 2\,.\,CA\cos.\,CAM = 2\,CA\cos.\,ADM = 2\,r\cos.\frac{\alpha}{2}$, hence $\frac{r}{a}$ may be put $= \frac{1}{2\cos.\frac{\alpha}{2}}$, and likewise $\frac{P}{R} = \frac{r}{a}$. From this, therefore, *the ~~power~~ force in the movable pulley is to the weight as the radius of the pulley to the chord of the arc over which the string passes.*

Fig. 156.



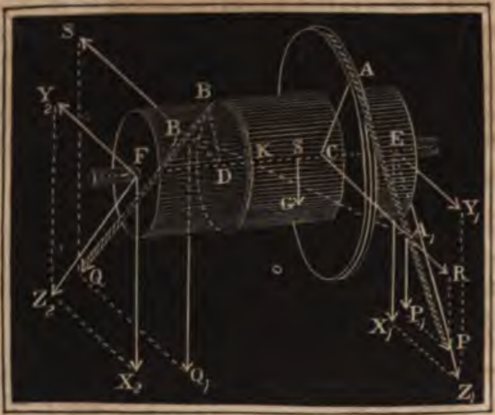
If $a = 2\,r$, the string passes over a semicircle, Fig. 156, the force then is at a minimum; viz. $P = \frac{1}{2}\,R$; if $a = r$, that is 60° of the part of the pulley over which the string passes, we have $P = R$; the smaller, therefore, a becomes, the greater is P, and for a infinitely small, the force P becomes infinitely great. An inverse proportion takes place in the spaces; if s is the space of P, which corresponds to a space $R = h$, we have then $Ps = Rh$, therefore, $\frac{s}{h} = \frac{a}{r}$.

The movable pulley is thus a means of modifying force; for example, a given weight may by this means be raised by a smaller force, but in proportion as there is gain in force, there is loss in space.

Remark. We shall treat of the composition of pulleys and systems of pulleys, as well as of the resistances arising from friction and rigidity, more fully in a subsequent Part.

§ 152. *The Wheel and Axle.*—The wheel and axle is a rigid connection of two fixed pulleys or wheels, capable of revolving about a common axis $ABFE$, Fig. 157. The smaller of these wheels is called the axle, the greater one the wheel. The round extremities E and F, on which this arrangement rests, are called gudgeons. The axis of revolution of the wheel and axle is either horizontal, or vertical, or inclined. Here we shall only speak of the wheel and axle which revolves about a horizontal axis. We shall also here suppose, that the forces P and Q, or the power P and the weight Q act at the extremities of a perfectly flexible string, which passes round the circumference of the wheel and axle. The questions to be answered are, in what relations the powers and weights are to each other, and what pressures the gudgeons E and F have to sustain?

Fig. 157.



Let us imagine a horizontal plane passed through the axis CD and the points of application A and B of the power P, and the weight Q transferred to this plane, and therefore P and Q applied at A_1 and B_1. If the angles AA_1C and BB_1D, which both forces make with the horizon $= \alpha$ and β, these forces may be replaced by the horizontal forces $R = P\ cos.\ \alpha$, $S = Q\ cos.\ \beta$, and by the vertical forces $P_1 = P\ sin.\ \alpha$, $Q_1 = Q\ sin.\ \beta$. The horizontal forces are directed towards the axis, and being applied at C and D, become perfectly counteracted by the axis. The vertical forces P_1 and Q_1, on the other hand, tend to turn the wheel and axle about its axis. If K be the intersection with the axis of the line connecting the points A_1 and B_1, KA_1 and KB_1 are the arms of P_1 and Q_1, and equilibrium subsists about K, and also about CD, if:

$$P_1 \,.\, KA_1 = Q_1 \,.\, KB_1, \text{ or, since } \frac{KA_1}{KB_1} = \frac{CA_1}{DB_1}, \text{ if}$$

$$P_1 \,.\, CA_1 = Q_1 \,.\, DB_1, \text{ or, as } \frac{P_1}{P} = \frac{CA}{CA_1}, \text{ and}$$

$$\frac{Q_1}{Q} = \frac{DB}{DB_1},$$

$$\frac{P\,.\,CA}{CA_1} \,.\, CA_1 = \frac{Q\,.\,DB}{DB_1} \,.\, DB_1, \textit{ i. e.}$$

$$P \,.\, CA = Q \,.\, DB, \text{ or } Pa = Qb,$$

if a and b represent the arms of the power and weight, or the radii of the wheel and axle. In the wheel and axle, therefore, as in every lever, the moment of the power is equivalent to the moment of the weight.

§ 153. The forces P_1 and Q_1 give at K a vertical pressure $P_1 + Q_1$, with which must also be associated the weight G of the whole wheel and axle applied at the centre of gravity S. The supports of the gudgeons at E and F have also to sustain the vertical pressure $P_1 + Q_1 + G = P\ sin.\ \alpha + Q\ sin.\ \beta + G$. If we put the whole length of the wheel and axle measured from E to $F = L$, the part $EC = l_1$, $CD = l$, $DF = l_2$, therefore $L = l + l_1 + l_2$, and the distances ES and FS of the centre of gravity S from the supports d_1 and d_2, therefore also $L = d_1 + d_2$, we shall obtain since

$$\frac{DK}{DC} = \frac{P_1}{P_1 + Q_1}, \text{ as } DK = \frac{P_1 l}{P_1 + Q_1}$$

for the vertical pressure X_1 at the gudgeon E:

$$X_1 \,.\, EF = G \,.\, FS + (P_1 + Q_1)\, FK,$$

$$X_1 = \frac{Gd_2 + (P_1 + Q_1)\left(l_2 + \frac{P_1}{P_1 + Q_1} \,.\, l\right)}{L}, \textit{ i. e.}$$

$$X_1 = \frac{Gd_2 + (P_1 + Q_1)\, l_2 + P_1 l}{L}.$$

On the other hand, for the vertical pressure X_2 at F:

$$X_2 \,.\, EF = G \,.\, ES + (P_1 + Q_1)\, EK, \textit{ i. e.}$$

$$X_2 = \frac{Gd_1 + (P_1 + Q_1)\left(l_1 + \frac{Q_1}{P_1 + Q_1} \,.\, l\right)}{L}, \textit{ i. e.}$$

$$X_2 = \frac{Gd_1 + (P_1 + Q_1)\, l_1 + Q_1 l}{L}.$$

The horizontal forces R and S have the moments about F, $R \,.\, FC = R\,(l + l_2)$, and $S \,.\, FD = S \,.\, l_2$, and about E : $S \,.\, ED = S\,(l + l_1)$, and $R \,.\, EC = Rl_1$; if, therefore, we put the horizontal pressures upon E and F effected by them $=$ Y_1 and Y_2, we shall obtain:

$$Y_1 \,.\, FE = R \,.\, FC - S \,.\, FD, \text{ as}$$

$$Y_1 = \frac{R\,(l + l_2) - Sl_2}{L}; \text{ and}$$

$$Y_2 \,.\, FE = S \,.\, ED - R \,.\, EC, \text{ as}$$

$$Y_2 = \frac{S\,(l + l_1) - Rl_1}{L}.$$

From X_1 and Y_1 the total pressure at E is:

$Z_1 = \sqrt{X_1^2 + Y_1^2}$, and likewise from X_2 and Y_2, the same at F: $Z_2 = \sqrt{X_2^2 + Y_2^2}$.

Lastly, if ϕ and ψ be the angles which the directions of these pressures make with the horizon, we shall then have

$$tang. \,\phi = \frac{X_1}{Y_1} \text{ and } tang\, \psi = \frac{X_2}{Y_2}.$$

Example. The weight Q of a wheel and axle pulls perpendicularly downwards, and amounts to 365 lbs.; the radius of the wheel $a = 1\frac{3}{4}$ ft.; that of the axle $b = \frac{3}{4}$ ft.; the weight of the machine itself is 200 lbs.; its centre of gravity S lies distant from E and F, $d_1 = 1\frac{1}{2}$, and $d_2 = 2\frac{1}{2}$ ft.; the middle of the wheel is about $l_1 = \frac{3}{4}$ ft. from the gudgeon E, and the vertical plane in which the weight acts is about $l_2 = 2$ ft. from the gudgeon F. Now if the force P necessary for restoring the equilibrium at the wheel inclined to the horizon at an angle $50^\circ = \alpha$, pulls downwards, what will this be, and what will be the pressures on the gudgeons? $Q = 365$, $\beta = 90^\circ$, consequently $Q_1 = Q$ *sin.* $\beta = Q$ and $S = Q$ *cos.* $\beta = 0$; further, P being unknown, and $\alpha = 50^\circ$, consequently $P_1 = P$ *sin.* $\alpha = 0{,}7660 \,.\, P$ and $R = P$ *cos.* $\alpha = 0{,}6428 \,.\, P$; but now $a = 1\frac{3}{4} = \frac{7}{4}$ and $b = \frac{3}{4}$, it follows, therefore, $P = \dfrac{b}{a}\, Q = \frac{3}{7} \,.\, 365 = 156{,}4$ lbs., $P_1 = 119{,}8$ and $R = 100{,}5$. Further, because $G = 200$, $d_1 = \frac{3}{2}$, $d_2 = \frac{5}{2}$, $l_1 = \frac{3}{4}$, $l_2 = 2$, $L = \frac{3}{2} + \frac{5}{2} = 4$, and $l = L - (l_1 + l_2) = 4 - \frac{11}{4} = \frac{5}{4}$, so that the vertical pressure at E is:

$$X_1 = \frac{200 \,.\, \frac{5}{2} + (365 + 119{,}8) \,.\, 2 + 119{,}8 \,.\, \frac{5}{4}}{4} = \frac{1619{,}35}{4} = 404{,}8 \text{ lbs.}$$

and that at F:

$$X_2 = \frac{200 \,.\, \frac{3}{2} + (365 + 119{,}8) \,.\, \frac{3}{4} + 365 \,.\, \frac{5}{4}}{4} = \frac{1119{,}85}{4} = 280{,}0 \text{ lbs.}$$

Both of these forces together give:

$$X_1 + X_2 = Q + G + P_1 = 684{,}8 \text{ lbs.}$$

The horizontal force at E is:

$$Y_1 = \frac{100{,}5 \,.\, (\frac{5}{4} + 2) - 0 \,.\, 2}{4} = 81{,}7 \text{ lbs., and that at } F:$$

$$Y_2 = \frac{0 \,.\, (\frac{5}{4} + \frac{3}{4}) - 100{,}5 \,.\, \frac{3}{4}}{4} = -\, 18{,}8 \text{ lbs.}$$

the sum of these is exactly $= R + S = 100{,}5$ lbs.

The pressure at E is inclined at an angle ϕ to the horizon, for which we have:

$$tang. \,\phi = \frac{X_1}{Y_1} = \frac{404{,}8}{81{,}7},\ Log.\ tang.\ \phi = 0{,}69502,\ \phi = 78^\circ\, 35'.$$

The pressure itself: $Z_1 = \dfrac{X_1}{sin.\ \phi} = 413{,}0$ lbs.

On the other hand, for the inclination ψ of the pressure at F:

$$tang. \,\psi = \frac{X_2}{Y_2} = \frac{280{,}0}{18{,}8},\ Log.\ tang.\ \psi = 1{,}17300,\ \psi = 86^\circ,\ 9',\ 5;$$

and the pressure: $Z_2 = \dfrac{Y_2}{cos.\ \psi} = 280{,}6$ lbs.

CHAPTER V.

ON THE RESISTANCES OF FRICTION AND RIGIDITY.

§ 154. We have hitherto assumed that two bodies can only act upon each other by forces at right angles to the plane of contact. If the surfaces at the point of contact were perfectly mathematical, *i. e.* not interrupted by the smallest irregular elevations or depressions, this law would also be fully confirmed by experience; but because everybody possesses a certain degree of elasticity or softness, and because the surface of every body, even if it is smoothed or polished in a high degree, has still some small elevations or indentures, and in consequence of the porosity of matter, no continuity; therefore, by the reciprocal action of two bodies in contact, reciprocal impressions and partial penetration of the parts take place at the point of contact, by which an adhesion of the two bodies is caused, which can only be overcome by a distinct force, whose direction coincides with the plane of contact.

This adhesion, produced by the impression and partial penetration of the bodies in contact, and the resistance on the plane of contact arising from it, has obtained the name of friction. Friction presents itself in the motion of bodies as a passive power or resistance, because it only impedes and checks motion, but never produces nor promotes it. It is introduced into investigations in mechanics as a force which is opposed to every motion, whose direction lies in the plane of contact of two bodies. In whatever direction we move forward a body resting on a horizontal or inclined plane, friction will always act opposite to the direction of motion; for example, it will impede the ascent as much as the descent of a body on an inclined plane. The smallest addition of force produces motion in a system of forces in equilibrium, so long as friction is not called into action; but when the same exerts its effect, a greater addition of force, dependent on the friction, is required to disturb the equilibrium.

§ 155. On overcoming friction, the parts in contact are compressed, and those which protrude, bent down, torn, or broken off, &c. Friction is not only dependent on the roughness or smoothness of the surfaces in contact, but also on the physical properties of the bodies themselves. Hard metals, for instance, cause less friction than soft. We can, however, lay down no general rules *à priori* of the dependence of friction on the physical properties of bodies; it is, on the contrary, necessary to make experiments on friction with bodies of different substances in order to find out the friction which takes place under various circumstances between bodies of the same substance.

The unguents which are applied to the rubbing surfaces exert a particular influence upon the friction and on the abrasions arising from the contact of bodies. The pores are filled up and other asperities diminished, and in general, the further penetration of the bodies prevented by the fluid or semi-fluid unguents, such as oil, tallow, fat, soap, &c., for which reason these occasion a considerable diminution of friction.*

Friction must not, however, be confounded with adhesion, *i. e.* with that holding together of two bodies, which takes place when they come into contact at many points without reciprocal pressure. Adhesion increases with the size of the surface in contact, and is independent of the pressure, whilst the contrary is the case with friction. If the pressure be slight, the adhesion will be considerable in proportion to the friction; but if the pressure be considerable, then it will constitute but a small part of the friction, and, therefore, generally may be neglected. Unguents, like all fluid bodies, increase the adhesion, because they increase the number of the points of contact.

§ 156. *Kinds of Friction.*—Two kinds of friction are distinguishable, viz., the rolling and the sliding. Sliding friction is that kind of resistance which is given out when a body so moves that all its points describe parallel lines. Rolling friction, on the other hand, is that resistance which arises from rolling, *i. e.* that motion of a body which moves progressively and rolls at the same time, and whose point of contact describes as great a space upon the body in motion as upon the body at rest. A body M supporting itself upon the plane HR, Fig. 158, for instance, moves sliding over the plane, and consequently has to overcome sliding friction when its points, A, B, C describe parallel spaces AA_1, BB_1, CC_1, &c., and therefore all these points of the moving body come into contact with others of the support. The body M, Fig. 159, on the other hand, rolls upon the plane HR, and has to overcome rolling friction, when the points A, B, &c., of the surface so move that the space $AB_1 = AB = A_1B_1$, likewise $AD = AE$, and $B_1E = B_1D_1$, &c.

Fig. 158.

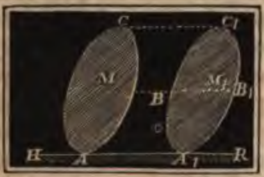
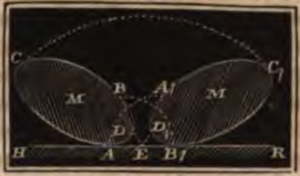


Fig. 159.



The friction of axles is a particular kind of sliding friction, which arises when a cylindrical axle revolves in its bearing. We distinguish two kinds of axles, the gudgeon and the pivot. The gudgeon rubs against its support or envelop, whilst its other points always successively come into contact with the same points of the support.

---

* The "anti-attrition metal," composed of copper 1 lb., antimony 2 lbs., and tin 3 lbs., (afterwards tempered or softened by re-melting with more tin,) now very generally used by machinists, in the United States, performs the same office, and prevents the heating of gudgeons, boxes, &c.—Am. Ed.

The pivot, on the other hand, presses with its circular base against its support, where its points revolve in concentric circles.

Further, particular frictions arise when a body oscillates upon a sharp edge, as in the balance, or when a vibrating body reposes upon a point, as in the magnetic needle.

Lastly, we distinguish the friction of quiescence which is to be overcome, when a body at rest is put into motion, from the friction of motion which opposes itself to the transmission of motion.

§ 157. *Laws of Friction.*—The general laws to which friction is subject, are the following:

1. Friction is proportional to the normal pressure between the rubbing bodies. If a body be pressed against another by a double force, the friction is as great again; three times the pressure gives three times the friction. If in small pressures this law varies from observation, it must be attributed to the proportionately greater effect of adhesion.

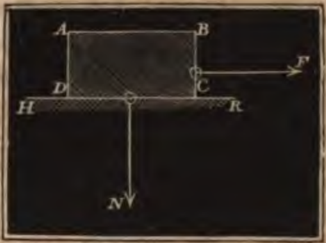
2. Friction is independent of the extent of the surfaces of contact. The greater the surfaces are, the greater is the number of parts which rub against each other; the smaller the pressure, the less the friction of each part; the sum of the frictions of all the parts is the same for a greater as for a less surface, in so far as the pressure and the other circumstances remain the same. If the side surfaces of a parallelopipedical brick are of the same quality, the force necessary to push it along a horizontal plane is the same, whether it rest upon the least, the mean, or the greatest surface. With very large side surfaces and with small pressures, this law has exceptions, in consequence of the effect of adhesion.

3. The friction of quiescence is indeed generally greater than that of motion; the last, however, is independent of the velocity; it is the same in small as in great velocities.

4. The friction of greased surfaces is generally less than that of ungreased, and depends less on the rubbing bodies than on the unguents.

5. The friction of gudgeons revolving on their bearings is less than the common sliding friction; the friction of rolling is in most cases so small, that it need hardly be taken into account in comparison with the sliding friction.

Fig. 160.



§ 158. *Co-efficient of Friction.*—From the first law laid down in the former paragraph, the following may be deduced. A body AC, Fig. 160, presses against its support, first with the force N, and requires to draw it along, *i. e.* to overcome its friction, the exertion of a certain force F, and secondly with the force N_1, and requires the force F_1 to cause it to pass from a state of rest into one of motion. From the foregoing we have:

$$\frac{F}{F_1} = \frac{N}{N_1}, \text{ and therefore } F = \frac{F_1}{N_1} \cdot N.$$

If, by experiment, we have found the friction F_1 corresponding to a certain pressure N_1, we hence find, if the rubbing bodies, and the other circumstances are the same, the friction F corresponding to another pressure N when we multiply this pressure by the ratio $\left(\frac{F_1}{N_1}\right)$ of the values F_1 and N_1 corresponding to the first observation.

The ratio of the friction to the pressure, or the friction for a pressure = unity, a pound, for instance, is called the *co-efficient of friction*, and will in the sequel be expressed by f, wherefore we may generally put $F = f \,.\, N$.

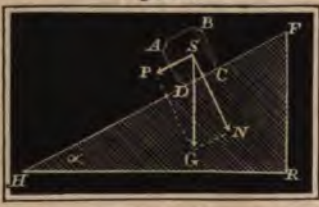
The co-efficient of friction is different for different substances and different conditions of friction, and must therefore be found out by experiment for each particular case.

When a body AC is drawn a distance s over a surface, there is a mechanical effect Fs to perform; the mechanical effect or work required to overcome friction is, therefore, fNs, equal to the product of the co-efficient of friction, the normal pressure, and the distance along the plane of contact. When the plane is also moving, we must then understand by s the relative distance.

Example.—1. If by a pressure of 260 lbs., the friction amounts to 91 lbs., the corresponding co-efficient of friction is $f = \frac{91}{260} = \frac{7}{20} = 0{,}35$.—2. To draw a sledge of 500 lbs. weight along a horizontal and very smooth surface of snow, the co-efficient of friction is $f = 0{,}04$, the required force $F = 0{,}04 \,.\, 500 = 20$ lbs.—3. If the co-efficient of friction of a cart drawn over a paved road is 0,45 and the load amounts to 500 lbs., the mechanical effect required to draw it 480 feet is $= fNs = 0{,}45 \,.\, 500 \,.\, 480 = 108000$ ft. lbs.

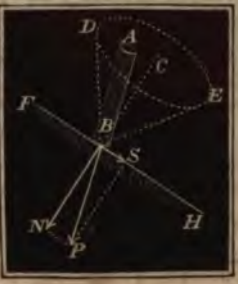
§ 159. *The Angle of Friction and the Cone of Friction.*—A body AC, Fig. 161, lies on an inclined plane FH, whose angle of inclination $FHR = \alpha$, its weight G resolves itself into the normal pressure $N = G \cos. \alpha$ and into the parallel force $P = G \sin. \alpha$. From the first force there arises the friction $F = f\, G \cos. \alpha$, which is opposed to every motion upon the plane, wherefore the force to push it upwards on the plane $= F + P = f\, G \cos. \alpha + G \sin. \alpha = (\sin. \alpha + f \cos. \alpha)\, G$, on the other hand, the force to push it downwards is $= F - P = (f \cos. \alpha - \sin. \alpha)\, G$; the last force is null, *i. e.* the body is sustained by its friction on the plane, when $\sin. \alpha = f \cos. \alpha$, *i. e.* when the $tang. \alpha = f$. As long as the inclined plane has an angle of inclination, whose tangent is less than f, the body remains at rest on the plane, but when the tangent of this angle is a little greater than f, the body immediately begins to slide down. The angle, whose tangent is equal to the co-efficient of friction, is called the angle of friction or the angle of repose. The co-efficient of friction is given by observing the angle of friction ρ (for the friction of repose), when f is put $= tang. \rho$.

Fig. 161.



In consequence of friction, the surface FH, Fig. 162, reacts not only against the normal pressure N of another body AB; but also against its oblique pressure P, when the deviation $NBP = \phi$ of the direction of this pressure from the normal BN does not exeeed the angle of friction, for since the force P gives the normal pressure $BN = P \,.\, cos.\, \phi$, and the lateral or tangential pressure $BS = S = P\, sin.\, \phi$, and there arises from the normal pressure $P\, cos.\, \phi$ the friction $f\, P\, cos.\, \phi$ opposed to every motion in the plane FH, S will therefore be unable to give rise to motion, and will remain in equilibrium so long as $f\, P\, cos.\, \phi > P\, sin.\, \phi$, or $f\, cos.\, \phi > sin.\, \phi$, *i. e. tang.* ϕ is $< f$, or $\phi < \rho$. If the angle of repose $CBD = \rho$ be made to revolve about the normal CB, it will describe a cone, which we may call the cone of friction or resistance. The cone of resistance includes all those directions of force by which a perfect counteraction of the oblique pressure takes place.

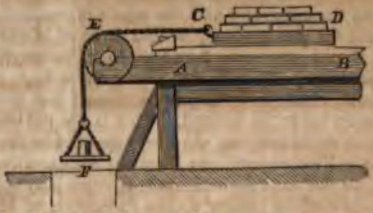
Fig. 162.



Example. To draw a filled cask weighing 200 lbs. up an inclined wooden plane of 50°, the force required with a co-efficient of friction $f = 0{,}48$ is $= P = (f\, cos.\, \alpha + sin.\, \alpha)\, G = (0{,}48\, cos.\, 50^\circ + sin.\, 50^\circ)\,.\,200 = (0{,}308 + 0{,}766)\,.\,200 = 215$ lbs.; to let it down, or to prevent its sliding down, the force required, on the other hand, is: $(P = f\, cos.\, \alpha - sin.\, \alpha)\, G = -(sin.\, 50^\circ - 0{,}48\,.\, cos.\, 50^\circ)\, 200 = -(0{,}766 - 0{,}308)\,.\,200 = -91{,}5$ lbs.

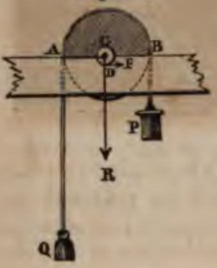
§ 160. *Experiments on Friction.*—Experiments on friction have been made by many philosophers, the most extensive of which, and on the greatest scale, are those of *Coulomb* and *Morin*. To find out the co-efficients of friction for sliding motion, these two made use of a sledge sliding on a horizontal surface, which was pulled forward by a cord, passing over a fixed pulley, from which weights were suspended, as in Fig. 163, where AB represents the way, CD the sledge, E the pulley, and G the weight. To obtain the co-efficient of friction for different substances, the surfaces in contact, not only of the sledge, but also of the way forming the support, were covered with the smoothest possible pieces of the substances under experiment, such as wood, iron, &c. &c. The co-efficients of the friction of repose were given by the weight which was necessary to cause the sledge to pass from a state of rest into one of motion, and the co-efficient of the friction of motion by the time t, which the sledge required to pass over a certain space s. If G be the weight of the sledge, and P the weight required to draw it, we have the friction $= f\, G$, the motive force $= P - f\, G$, and the mass $M = \frac{P + G}{g}$, it therefore follows from § 65, that the acceleration of the uniformly accele-

Fig. 163.



rated motion arising, is : $p = \frac{P - fG}{P + G} g$, and inversely, the co-efficient of friction $f = \frac{P}{G} - \frac{P + G}{G} \cdot \frac{p}{g}$. But $s = \frac{1}{2} pt^2$ (§ 11), therefore, $p = \frac{2s}{t^2}$, and $f = \frac{P}{G} - \frac{P + G}{G} \cdot \frac{2s}{gt^2}$.

Fig. 164.



To measure the co-efficient of friction, for axle friction, a fixed pulley ACB, Fig. 164, is made use of, over which a cord passes, which is stretched by the weights P and Q. From the sum of the weights, the pressure $P + Q$ is given, and from their difference $P - Q$ the force at the circumference of the pulley, which is in equilibrium with the friction of the axle, $F = f(P + Q)$, if now $CA = a$ the radius of the pulley, and $CD = r$ that of the axle, we have from the equality of moments $(P - Q)\, a = Fr = f(P + Q)\, r$, and therefore the friction of repose; $f = \frac{P - Q}{P + Q} \cdot \frac{a}{r}$, on the other hand, for that of motion, if the weight P falls a space s in the time (t), and Q rises as much, $f = \left(\frac{P - Q}{P + Q} - \frac{2s}{gt^2}\right) \cdot \frac{a}{r}$.

Remark. Before Coulomb, Amontons, Camus, Bülffinger, Muschenbroek, Ferguson, Vince, and others turned their attention to and made experiments on friction. The results of all these investigations are of little value in practice, because they were conducted upon too small a scale. The experiments of Ximenes, which were made about the same time as those of Coulomb, also fail in this respect. The results are to be found in a work, "Teoria e Pratica delle resistenze de' solidi ne' loro attriti," Pisa, 1782. The experiments of Coulomb are fully described in his work, "Théorie des Machines simples," 1821. The latest experiments upon friction are those of Rennie and Morin. Rennie used for his experiments partly, a sledge upon a horizontal surface, and partly upon an inclined plane, from which the bodies were allowed to slide down, and by which the amount of the friction was deduced from the angle of friction. Rennie's experiments extend to substances of various kinds met with in practice, as cloth, leather, wood, stones, and metals; they give important results upon the abrasion of bodies, but from the apparatus and the mode of conducting these experiments, we cannot rely upon them for that accuracy which those of Morin appear to have attained. The experiments of Rennie are to be found in the "Philosophical Transactions" of 1818. The most extensive experiments, and promising a high degree of accuracy, have been completed by Morin, although it cannot be denied that they leave some doubts and uncertainties, and somewhat to be desired. This is not the place to describe the methods and apparatus of these experiments; we can only refer to the author's writings, "Nouvelles Expériences sur le Frottement," par Morin. An excellent article on Friction, and a full description of all the experiments upon it, especially those of Morin, is given by Brix in the "Transactions of the Society for the Promotion of Manufacturing Industry in Prussia," 16 and 17 Jahrgang, Berlin, 1837–8.[*]

---

[*] A series of experiments on the resistance of friction, particularly as applied to railway cars, will be found in "Wood's Treatise on Railways," 2d ed., 1832, chap. 6; Vid. Smith's Am. ed., pp. 171–228.—AM. ED.

G = weight producing friction

fG = friction arising therefrom

P = applied force then P less the friction

P − fG = moving force, P + G = the weights moved

$\frac{P+G}{g}$ = mass moved; and the equation $w = \frac{F}{M}$ gives

$f = \frac{P - fG}{P+G} g$ $\quad f(P+G) = Pg - fG$ $\quad \therefore fG = Pg - f(P+G)$ [illegible]

and from "first principles" $f = \frac{2s}{t^2}$ $\therefore f = \frac{P}{G} - \frac{P+G}{gG} \cdot \frac{2s}{t^2}$.

P + Q = weight producing friction

f(P + Q) = friction arising therefrom

P − Q = applied force then this less the friction

$(P - Q) - \frac{ar}{R} f(P+G)$ = moving force [illegible] the friction is [illegible] that of the [illegible] a and in order to subtract the friction from the applied force we must reduce it to the same arm that is find the force S, which applied at the circumference of the wheel would equal the force f(P+G) at the axle i.e. $S.R = f(P+G)r$ or $S = \frac{r}{R} f(P+G)$. Hence [illegible] above equation for the moving force, but the mass moved $\frac{P+G}{g}$ [illegible]

$p = \frac{(P-Q)\ddot{a} + (P+Q)}{P+Q}$ [illegible] $p(P+Q) = (P-Q)$ [illegible] $- \ddot{a}\,(P+Q)g$

$\therefore \frac{r}{a} f (P+Q) g = -p(P+Q) + (P-Q) g \quad \therefore f = \frac{a}{r}\frac{P-Q}{P+Q} - \frac{p}{g}\cdot\frac{a}{r} = \frac{a}{r}\left(\frac{P-Q}{P+Q} - \frac{p}{g}\right)$

§ 161. The following tables contain a condensed summary of the *co-efficients of friction* the most useful in practice.

TABLE I.

CO-EFFICIENTS OF THE FRICTION OF REPOSE.

NAMES OF BODIES.		Nature of the surfaces and unguents.							
		Dry.	Damped with water.	With olive oil.	Lard.	Tallow.	Dry soap.	Polished and greasy.	Greasy and wetted.
Wood upon wood	least,	0,30	0,65	—	—	0,14	0,22	0,30	—
	mean,	0,50	0,68	—	0,21	0,19	0,36	0,35	—
	greatest,	0,70	0,71	—	—	0,25	0,44	0,40	—
	values,								
Metal upon metal	least,	0,15	—	0,11					
	mean,	0,18	—	0,12	0,10	0,11	—	0,15	—
	greatest,	0,24	—	0,16					
	values,								
Wood upon metal		0,60	0,65	0,10	0,12	0,12	—	0,10	—
Hempen ropes, twisted or matted, upon wood	least,	0,50							
	mean,	0,63	0,87						
	greatest,	0,80							
	values,								
Thick sole leather, upon wood or iron	high at the edges,	0,43	0,62	0,12	—	—	—	—	0,27
	flat or smooth,	0,62	0,80	0,13					
Black strap leather, upon pulleys	of wood,	0,47	—	—	—	—	—	0,28	0,38
	of iron,	0,54							
Stones or bricks upon stones or bricks, smooth worked	least,	0,67							
	greatest,	0,75							
	value,								
Stones and wrought iron	least,	0,42							
	greatest,	0,49							
	value,								
Oak upon muschekalk		0,64							

TABLE II.

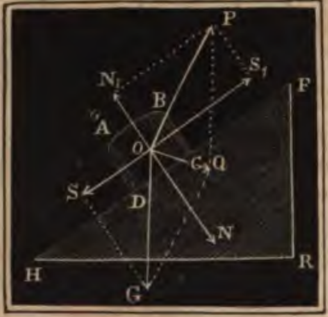
CO-EFFICIENTS OF THE FRICTION OF MOTION.

NAMES OF BODIES.		Nature of the surfaces and unguents.								
		Dry.	Water.	Olive oil.	Lard.	Tallow.	Lard and black lead.	Polished and greasy.	Dry soap.	Greasy and wetted.
Wood upon wood -	least,	0,20	—	—	0,06	0,06	—	—	0,14	0,08
	mean,	0,36	0,25	—	0,07	0,07	—	—	0,15	0,12
	greatest, value,	0,48	—	—	0,07	0,08	—	—	0,16	0,15
Metal upon metal -	least,	0,15	—	0,06	0,07	0,07	0,06	0,12	—	0,11
	mean,	0,18	0,31	0,07	0,09	0,09	0,08	0,15	0,20	0,13
	greatest, value,	0,24	—	0,08	0,11	0,11	0,09	0,17	—	0,17
Wood upon metal -	least,	0,20	—	0,05	0,07	0,06	—	—	—	0,10
	mean,	0,42	0,24	0,06	0,07	0,08	0,08	0,10	0,20	0,14
	greatest, value,	0,62	—	0,08	0,08	0,10	—	—	—	0,16
Hemp, cords, twists, &c.	on wood,	0,45	0,33							
	on iron,	—	—	0,15	—	0,19				
Sole leather, smooth, upon wood or metal	raw,	0,54	0,36	0,16	—	0,20				
	compressed,	0,30	—							
	greasy,	—	0,25							
The same, high at the edges, &c.	dry,	0,34	0,31	0,14	—	0,14				
	greasy,	—	0,24							

Remark. The co-efficients of friction for porous masses will be given in the Second Part, in the theory of the pressure of earth.

§ 162. *Inclined Plane.*—The theory of sliding friction has its chief application in the investigation of the equilibrium of a body AC, on an inclined plane FH, Fig. 165. If in accordance with § 135, $FHR = \alpha$, the angle of inclination of the inclined plane, and $POS_1 = \beta$, the angle which the force P makes with the inclined plane, we have the normal force arising from the weight G of the body $N = G \cos. \alpha$, on the other hand, the force for sliding down $= S = G \sin. \alpha$, further the force N_1 with which P strives to draw the body down the plane is $= P \sin. \beta$,

Fig. 165.



from

and the force S_1 with which it pushes the body up the plane $= P \cos. \beta$. The remaining normal pressure is: $N - N_1 = G \cos. \alpha - P \sin. \beta$, consequently the friction $F = f(G \cos. \alpha - P \sin. \beta)$. If it be required to find the force P drawing the body up the plane, then there will be friction to overcome, and it must therefore be $S_1 = S + F$, *i. e.*

$$P \cos. \beta = G \sin. \alpha + f(G \cos. \alpha - P \sin. \beta).$$

But if the force, which is to prevent the body from sliding down is to be determined, then friction comes to its assistance, and the force is:

$$S_1 + F = S, \text{ i. e. } P \cos. \beta + f(G \cos. \alpha - P \sin. \beta) = G \sin. \alpha.$$

From this the force may be determined:

$$\text{For the first case: } P = \frac{\sin. \alpha + f \cos. \alpha}{\cos. \beta + f \sin. \beta} . G,$$

$$\text{For the second: } P = \frac{\sin. \alpha - f \cos. \alpha}{\cos. \beta - f \sin. \beta} . G.$$

If the angle of friction ρ be introduced, whilst we put $f = tang. \rho = \frac{\sin. \rho}{\cos. \rho}$, we shall obtain $P = \frac{\sin. \alpha . \cos. \rho \pm \cos. \alpha . \sin. \rho}{\sin. \beta . \cos. \rho \pm \cos. \beta . \sin. \rho} . G$, or from the known rules of trigonometry: $P = \frac{\sin. (\alpha \pm \rho)}{\cos. (\beta \mp \rho)} . G$, and the upper signs are to be taken, when motion is to be brought about; the lower, on the other hand, when motion is to be impeded.

The last formula is found by a simple application of the parallelogram of forces. Since a body counteracts that force of another body, which deviates by the angle of friction ρ from the normal to its surface (§ 159), equilibrium in the foregoing case can subsist if the resultant $OQ = Q$ of the components P and G makes with the normal ON the angle $NOQ = \rho$. If now we put in the general formula $\frac{P}{G} = \frac{\sin. GOQ}{\sin. POQ}$, $GOK = GON + NOQ = \alpha + \rho$, and $POQ = POS_1 + S_1OQ = \beta + 90^\circ - \rho$, we then have $\frac{P}{G} = \frac{\sin. (\alpha + \rho)}{\sin. \left(\beta - \rho + \frac{\pi}{2}\right)} = \frac{\sin. (\alpha + \rho)}{\cos. (\beta - \rho)}$, and for a negative value of ρ: $\frac{P}{G} = \frac{\sin. (\alpha - \rho)}{\cos. (\beta + \rho)}$, quite in accordance with the above.

If the body reposes on a horizontal plane $\alpha = 0$, therefore, the force to push P forward is: $P = \frac{f G}{\cos. \beta + f \sin. \beta} = \frac{f G \sin. \rho}{\cos. (\beta - \rho)}$.

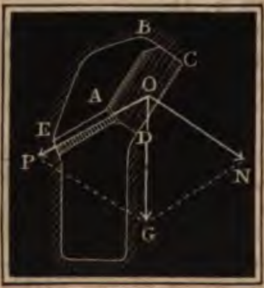
If the force acts parallel to the inclined plane, then $\beta = 0$, and therefore, $P = (\sin. \alpha \pm f \cos. \alpha) G, = \frac{(\sin, \alpha \pm \rho)}{\cos. \rho} . G.$ (compare § 159). If the force acts horizontally $\beta = - \alpha$; $\cos. \beta = \cos$

α and $sin.\ \beta = -\ sin\ \alpha$, therefore, $P = \frac{(sin.\ \alpha \pm f\ cos.\ \alpha}{cos.\ \alpha \mp f\ sin.\ \alpha} .\ G = \frac{tang.\ \alpha \pm f}{1 \mp f\ tang.\ \alpha} .\ G$, also $= tang.\ (\alpha \pm \rho)\ G$.

Again, the force to push a body upwards is least when the denominator $cos.\ (\beta - \rho)$ is greatest, viz. $= 1$, therefore, $\beta - \rho = 0$, *i.e.* $\beta = \rho$. When, therefore the direction of force deviates by the angle of friction from the inclined plane, the force itself is the least and $= sin.\ (\alpha + \rho)\ .\ G$.

Example. What pressure on the axis has the prop *AE*, Fig. 166, to sustain, in order to prevent a block of stone (a wall) *ABCD*, of 5000 lbs. weight from slipping down the inclined plane *CD*, supposing the angle of the prop to the horizon to be 35°, that of the inclined plane *CD*, 50°, and the coefficient of friction $f = 0{,}75$? Here $G = 5000$, $\alpha = 50°$, $\beta\ 35° - 50° = -15°$, and $f = 0{,}75$; therefore the formula gives:

Fig. 166.

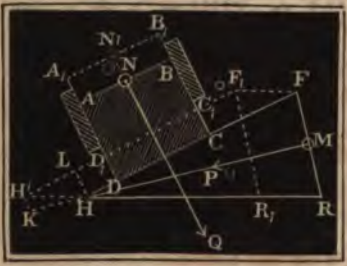


$$P = \frac{sin.\ \alpha - f\ cos.\ \alpha}{cos.\ \beta - f\ sin.\ \beta} .\ G = \frac{sin.\ 50° - 0{,}75\ cos.\ 50°}{cos.\ 15° + 0{,}75\ sin.\ 15°} .5000$$
$$= \frac{0{,}766 - 0{,}482}{0{,}966 + 0{,}194} .\ 5000 = \frac{1420}{1{,}160} = 1224 \text{ lbs.}$$

If the prop were horizontal, we should have $\beta = -50°$, and $tang.\ \rho = 0{,}75$; hence $\rho = 36°\ 52'$; lastly, $P = G\ tang.\ (\alpha - \rho) = 5000\ tang.\ (50° - 36°\ 52') = 5000\ tang.\ 13°\ 8' = 5000\ .\ 0{,}2333 = 1166$ lbs. To push up the same wall upon the supporting one by a horizontal force, under otherwise similar circumstances, a force *P* would be necessary $= G\ tang.\ (\alpha + \rho) = 5000\ tang.\ 86°\ 52' = 5000\ .\ 18{,}2676 = 91338$ lbs.

§ 163. *Wedge.*—In the wedge, friction exerts a considerable influence upon the statical relations. The section of a wedge forms an isosceles triangle *FHR*, Fig. 167, with the edge angle *FHR* $= \alpha$, the force *P* acts at right angles to the back (?) and the weight *Q* at right angles to (?) the side *FH*. If we drive the wedge upon the base *HR* a space $s = FF_1 = HH_1 = RR_1$, the weight *Q* is raised through a space $CC_1 = DD_1 = HL = HH_1\ .\ sin.\ HH_1L = s\ sin.\ \alpha$, and force passes over $HK = HH_1\ .\ cos.\ H_1HK = s\ cos.\ \frac{\alpha}{2}$; according to the principle of virtual velocities, and without regard to friction, $P\ .\ HK = Q\ .\ DD_1$, *i. e.* $P\,s\ cos.\ \frac{\alpha}{2} = Qs\ sin.\ \alpha$, therefore $P = \frac{Q\ sin.\ \alpha}{cos.\ \frac{\alpha}{2}} = \frac{2\ Q\ sin.\ \frac{\alpha}{2}\ .\ cos.\ \frac{\alpha}{2}}{cos.\ \frac{\alpha}{2}} = 2$

Fig. 167.



The Wedge as used for splitting
Page 155.

In the Wedge as it is used for splitting asunder and compressing bodies the force P normal to the back is decomposed into the components Q, Q_1, normal to the two faces of the Wedge and giving rise to the frictions F, F_1. The component Q can be decomposed into the components V vertical and H horizontal, and Q_1 into the vertical V_1 and the horizontal H_1, but H and H_1 destroy each other and leave V and V_1 acting vertically. The frictions $F F_1$ can be decomposed into the two horizontal h and h_1 and the two vertical v and v_1 but h and h_1 destroy each other leaving v and v_1 acting vertically. $P = V + V_1 + v + v_1$ for equilibrium. $V = Q \sin \frac{\alpha}{2}$ $V_1 = Q_1 \sin \frac{\alpha}{2}$, but Q being $= Q_1$, $V = V_1$, and $V + V_1 = 2Q \sin \frac{\alpha}{2}$. $F = fQ = F_1 = fQ_1$. $v = F \cos \frac{\alpha}{2}$ $v_1 = F_1 \cos \frac{\alpha}{2}$, $v = fQ \cos \frac{\alpha}{2}$, $v_1 = fQ_1 \cos \frac{\alpha}{2}$. $v + v_1 = 2fQ \sin \frac{\alpha}{2}$. $P = V + V_1 + v + v_1 = 2Q \sin \frac{\alpha}{2} + 2fQ \cos \frac{\alpha}{2}$ $= 2Q(\sin \frac{\alpha}{2} + 2f \cos \frac{\alpha}{2})$ Neatly, very [illegible]

$Q \sin. \frac{\alpha}{2}$, which also follows from the formula in § 137, if we put in it $\sin. \beta = 1$, and $\cos. (\alpha - \delta) = \cos. \frac{\alpha}{2}$.

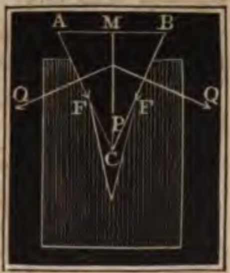
There are now, however, three frictions which come into play, viz., the friction against the sides HF and HR, and the friction of the body $ABCD$ in its constrained motion. As the directions of the force on both sides of the wedge deviate equally, the pressure against both is equal, namely $= Q$, and the friction arising $= f\, Q$. The spaces of these frictions, however, are different. For the friction upon $HR : s = HH_1$, for that upon $HF = H_1L = s \cos. \alpha$; accordingly the mechanical effects of both frictions are: $= f\, Q\, s + f\, Q\, s \cos. \alpha = f\, Q\, s\, (1 + \cos. \alpha) = 2 f\, Q\, s \left(\cos. \frac{\alpha}{2}\right)^2$. Lastly, the friction between CD and FH presses upon the body $ABCD$ at right angles to its direction, and there produces the friction $f_1 . fQ$, if f_1 represent the co-efficient of friction for its constrained motion. This friction, however, has the same space as the weight Q, viz., $DD_1 = s \sin. \alpha$; and to it corresponds the mechanical effect $f_1\, f\, Q\, s \sin. \alpha$. In order now to find the extreme limits of the condition of equilibrium, we must put the mechanical effect of the force P equal to that of the weight Q, plus the mechanical effects of the friction, therefore,

$$Ps \cos. \frac{\alpha}{2} = Qs \sin. \alpha + 2\, Q f s \left(\cos. \frac{\alpha}{2}\right)^2 + f f_1\, Qs\, .\, \sin. \alpha,$$

and we obtain the force:

$$P = 2\, Q \left(\sin. \frac{\alpha}{2} + f \cos. \frac{\alpha}{2} + f f_1 \sin. \frac{\alpha}{2}\right).$$

Fig. 168.



In a wedge ABC, Fig. 168, as it is used for the splitting asunder and compression of bodies, the force at the back corresponding to the normal pressure Q against the sides AC and BC, is $P = 2\, Q \left(\sin. \frac{\alpha}{2} + f \cos. \frac{\alpha}{2}\right)$, which is given if we put the sum of the vertical components of Q and $F = f\, Q$, *i. e.* $2\, V_1 = 2\, Q \sin. \frac{\alpha}{2}$ and $2\, V_2 = 2\, f\, Q \cos. \frac{\alpha}{2}$ equivalent to the force P.

Example. The load of the wedge Q in Fig. 167 $= 650$ lbs., the edge $\alpha = 25°$, the co-efficient of friction $f_1 = f = 0{,}36$. Required, the mechanical effect necessary to move the load Q forward about ½ a foot.

The force is $P = 2\, .\, 650\, [\sin. 12\tfrac{1}{2}° + 0{,}36 \cos. 12\tfrac{1}{2}° + (0{,}36)^2 \sin. 12\tfrac{1}{2}°]$
$= 1300\, .\, (0{,}2164 + 0{,}36\, .\, 0{,}9763 + 0{,}1296\, .\, 0{,}2164)$
$= 1300\, .\, (0{,}2164 + 0{,}3515 + 0{,}0281) = 1300\, .\, 0{,}5960 = 774{,}8$ lbs. For, to the space of the load $CC_1 = \frac{1}{2}$ foot, corresponds the space of the force $HK = s = \frac{CC_1}{\sin. \alpha}\, .\, \cos. \frac{\alpha}{2} = \frac{CC_1}{2 \sin. \frac{\alpha}{2}} = \frac{1}{4\, .\, 0{,}2164} = 1{,}155$ feet; therefore the mechanical effect and weight is

Ps = 774,8 . 1,155 = 895 ft. lbs. Without regard to friction, it would only be 650 . $\frac{1}{2}$ = 325 ft. lbs. In consequence of friction, the mechanical effect expended would be nearly tripled.

§ 164. *Axle Friction.*—In axles, the friction of motion only is of importance, on which account experiments on this only exist.

From the following table very important results for practice may be drawn, with axles of wrought or cast iron, moving in bearings of cast iron or brass, coated with oil, tallow or hog's lard, the co-efficient of friction is

By continuous greasing = 0,054,
In the usual manner = 0,070 to 0,080.

The values found by *Coulomb* vary partially from the annexed.

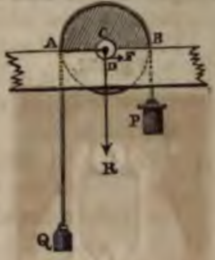
TABLE III.

CO-EFFICIENTS OF AXLE FRICTION, FROM MORIN.

NAMES OF THE BODIES.	Nature of the surfaces and unguents.							
	Dry or a little greasy.	Greasy and wetted with water.	Greased and wetted with water.	Oil, tallow, or lard.		Very soft and purified carriage grease.	Hog's lard with plumbago.	Fatty.
				In the usual way.	Continuously.			
Bell metal on the same	—	—	—	0,097	—	—	—	—
Cast iron upon bell metal	—	—	—	—	0,049			
Wrought iron upon bell metal	0,251	0,189	—	0,075	0,054	0,090	0,111	—
Wrought iron upon cast iron	—	—	—	0,075	0,054	—	—	—
Cast iron upon cast iron	—	0,137	0,079	0,075	0,054	—	—	0,137
Cast iron upon bell metal	0,194	0,161	—	0,075	0,054	0,065	—	0,166
Wrought iron upon lignum vitæ	0,188	—	—	0,125				
Cast iron upon lignum vitæ	0,185	—	—	0,100	0,092	—	0,109	0,140
Lignum vitæ upon cast iron	—	—	—	0,116	—	—	—	0,153
Lignum vitæ upon lignum vitæ	—	—	—	—	0,070			

§ 165. If we know the pressure *R* between an axle and its bearing, and if further the radius *r* of the axle, Fig. 169, be given, the mechanical effect which the friction of the axle counteracts in every revolution may be calculated. The friction $F = fR$, the space corresponding to it, the circumference $2 \pi r$ of the axle; it therefore follows that the mechanical effect lost by friction in each revolution is $= f R \cdot 2\pi r = 2\pi f R r$. If the axle makes one revolution per minute u, the mechanical effect expended in each second

Fig. 169.

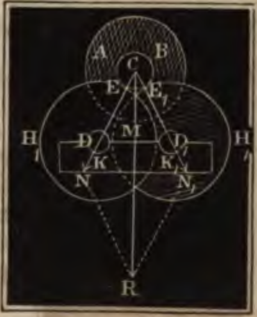


$$= 2\pi f R r \cdot \frac{u}{60} = \frac{\pi u f R r}{30} = 0{,}105 \cdot u f R r.$$

The mechanical effect consumed by friction increases, therefore, with the pressure on the axle, in proportion to the radius of the axle and the number of revolutions. It is, therefore, a rule in practice, not to augment unnecessarily the pressure on the axis in rotating machines by heavy weights, to make the axles no stronger than the solidity required for durability, and likewise not to make a great many revolutions in a minute, at least, not unless other circumstances require it.

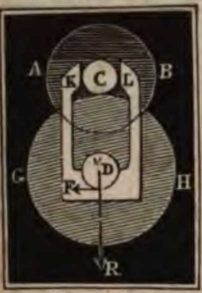
By the application of friction wheels, which are substituted for the bearings, the mechanical effect of friction is much diminished. In Fig. 170, AB is a wheel which reposes by its axle CEE_1 on the circumferences EH, $E_1 H_1$ lying close to each other of the friction wheels revolving about D and D_1. From the given pressure R of the wheel, there follow the pressures $\mathcal{N} = \mathcal{N}_1 = \frac{R}{2\cos.\frac{\alpha}{2}}$, if α be the angle DCD_1 which the central lines, or *lines of pressure*, CD and CD_1 make between them. From the rolling friction between the axle C and the circumferences of the wheels, these latter revolve with the axle, and there arise at the bearings D and D_1 the frictions $f\mathcal{N}$ and $f\mathcal{N}_1$, which together amount to $\frac{fR}{\cos.\frac{\alpha}{2}}$. If the radius of the wheel $DE = D_1 E_1$ be represented by a_1, and that of the axle $DK = D_1 K_1$ by r_1, we shall have the force at the circumference of the wheels, or at the circumference of the axle C resting upon these, which is requisite to overcome $\frac{fR}{\cos.\frac{\alpha}{2}}$: $F_1 = \frac{r_1}{a_1} \cdot \frac{fR}{\cos.\frac{\alpha}{2}}$, whilst it will be $= fR$, if the axle C rest immediately in a socket. If we disregard the weights of the friction wheels, the mechanical effect of the friction by the application of these wheels is $\frac{r_1}{a_1 \cos.\frac{\alpha}{2}}$ times as great as without them.

Fig. 170.



If we oppose to the pressure of the axle R a single friction wheel GH, Fig. 171, and prevent any accidental lateral forces, by the fixed cheeks K and L, $\alpha = 0$, $cos.\ \frac{a}{2} = 1$, and the above relations $= \frac{r_1}{a_1}$.

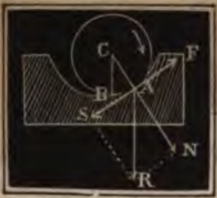
Fig. 171.



Example. A wheel weighs 30000 lbs., its radius $a = 16$ ft. and that of its axle $r = 5$ inches, what is the amount of force at the circumference of the wheel necessary to overcome the friction of the axle, and to maintain it in uniform motion, and what is the corresponding expenditure of mechanical effect if it makes 5 revolutions a minute? We may assume the co-efficient of friction f here $= 0{,}075$, wherefore the friction $f R = 0{,}075 \,.\, 30000 = 2250$ lbs. Since the diameter of the wheel is $\frac{16 \,.\, 12}{5} = \frac{192}{5} = 38{,}4$ times as great as the diameter of the axle or the arm of the friction, the axle friction reduced to the circumference of the wheel $= \frac{f R}{38{,}4} = \frac{2250}{38{,}4} = 58{,}59$ lbs. The circumference of the axle is $\frac{2 \,.\, 5 \,.\, \pi}{12} = 2{,}618$ feet; consequently the path of the friction in one second $= \frac{2{,}618 \,.\, 5}{60} = 0{,}2182$ feet, and its mechanical effect during one second $= 0{,}2182 \,.\, f R = 0{,}2182 . 2250 = 491$ ft. lbs. If the axles of this wheel rest upon friction
?/ wheels whose radii are 5 times as great as those of the axle, and therefore $\frac{r_1}{a_1} = \frac{1}{5}$, the power expended, referred to the circumference of the wheel, will only be $\frac{1}{5} \,.\, 38{,}4 = 7{,}68$ ft. lbs., and the mechanical effect of friction expended only $\frac{491}{5} = 98{,}2$ ft. lbs.

§ 166. The friction of an axle ACB, Fig. 172, which presses on its bearing in one point A only, is less than that of a new axle resting on all points of the bearing. If no revolution takes place, the axle then presses on the point B, through which passes the direction of the mean pressure R; but if revolution begins in the direction AB, the axle by its friction will rise just so high in its bearing until the sliding force comes into equilibrium with the friction. The mean pressure R is decomposed into a normal pressure N and a tangential S; N passes into the bearing and gives rise to $F = f N$, acting tangentially; S puts itself in equilibrium with F; S is therefore $= f N$. According to the Pythagorean doctrine, R^2 is $= N^2 + S^2$, therefore R^2 is here $= (1 + f^2) N^2$; inversely the normal pressure $N = \frac{R}{\sqrt{1 + f^2}}$, and the friction $F = \frac{f R}{\sqrt{1 + f^2}}$; or, if the angle of friction ρ be introduced, $f = tang.\ \rho$:

Fig. 172.

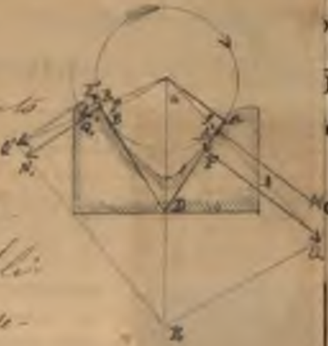


$$F = \frac{tang.\ \rho}{\sqrt{1 + tang. \rho^2}} \,.\, R = tang.\ \rho\ cos.\ \rho\ R = R\ sin.\ \rho.$$

If the axle begins to move, the point of pressure B moves forward in the bearing by the angle $ACB = \rho$ in the opposite direction.

If no forward motion took place, F would be $= fR = R\,tang.\,\rho =$

Prismatic bearing.



§ 167 Q and Q_1 components
f mean pressure R, making
ngles ρ with radii passing
hrough the points of appli-
cation AB. Q and Q_1 are re-
solved into N and F. N_1 &
F_1. N and N_1 being normal are taken up
by the bearing, while F and F_1 represent
in quantity the friction at these points
vide § 166). $F = Q \sin\rho$. $F_1 = Q_1 \sin\rho$ ∴ $F + F_1 = (Q_1 + Q)\sin\rho$
$\frac{Q}{R} = \frac{\sin QRC}{\sin CQR} = \frac{\sin(\alpha+\rho)}{\sin 2\alpha}$ ∴ $Q = \frac{R\sin(\alpha+\rho)}{\sin 2\alpha}$; moreover
$\frac{Q_1}{R} = \frac{\sin CRQ_1}{\sin 2\alpha} = \frac{\sin(\alpha-\rho)}{\sin 2\alpha}$ ∴ $Q_1 = \frac{R\sin(\alpha-\rho)}{\sin 2\alpha}$. Hence we
get $F + F_1 = (Q + Q_1)\sin\rho = (\sin(\alpha+\rho) + \sin(\alpha-\rho))\frac{R\sin(\alpha-\rho)}{\sin 2\alpha}$
But $\sin(\alpha+\rho) + \sin(\alpha-\rho) = \sin\alpha\cos\rho + \cos\alpha\sin\rho +$
$\sin\alpha\cos\rho - \cos\alpha\sin\rho = 2\sin\alpha\cos\rho$, and also
$\sin 2\alpha = 2\sin\alpha\cos\alpha$] ∴ $F + F_1 = \frac{2\sin\alpha\, R\cos\rho\sin\rho}{2\sin\alpha\cos\alpha}$ =
$\frac{R\frac{1}{2}\sin 2\rho}{\cos\alpha}$ = (since ρ is so small that it is not
essentially different from its sine) $\frac{R\sin\rho}{\cos\alpha}$

If we oppose to the pressure of the axle *R* a single friction wheel
G.

ch

se

0,

wi

th

7,

a
a
fo
th

—

ρ

in

If no forward motion took place, F would be $= fR = R\,tang.\,\rho = \frac{R\,sin.\,\rho}{cos.\,\rho}$; consequently the friction is the $cos.\,\rho$ times as great after moving forward as before the motion. Generally, $f = tang.\,\rho$ not quite $\frac{1}{10}$ and $cos.\,\rho > 0{,}995$, therefore the difference is not quite $\frac{5}{1000} = \frac{1}{200}$; therefore, in ordinary cases of application, we need have no regard to the effect of this motion.

If the wheel AB revolves in a nave or eye, Fig. 173, about a fixed axis AC, the friction is the same as if the axis moves in a roomy nave, only the arm of the friction is the arm of the nave, not that of the fixed axle.

Fig. 173.

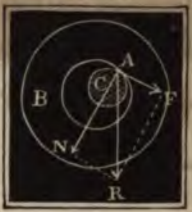
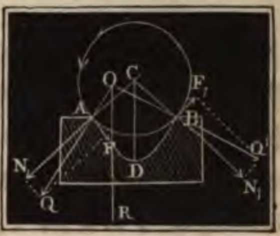


Fig. 174.



§ 167. If the axle lies in a prismatic bearing, there is greater pressure, and consequently more friction, than in a round bearing. If the bearing ADB, Fig. 174, is triangular, the axle lies on two points A and B, and at each there is the same friction to overcome, the mean pressure R is decomposed into two lateral forces Q and Q_1, and each of these gives a normal pressure N and N_1, and each a tangential force $F = fN$ and $F_1 = fN_1$ equal to the friction. According to the former §, these frictions may be put $= Q\,sin.\,\rho$ and $= Q_1\,sin.\,\rho$, we have then the whole friction $= (Q + Q_1)\,sin.\,\rho$. The forces Q and Q_1 are given by the resolution of a parallelogram constructed from Q and Q_1 with the aid of the mean pressure R, the angle of friction ρ, and the angle $ACB = 2\,\alpha$, which corresponds to the arc AB, lying in the bearing. If $QOR = ACD - CAO = \alpha - \rho$, $Q_1OR = BCD + CBO = \alpha + \rho$; lastly, $QOQ_1 = \alpha - \rho + \alpha + \rho = 2\,\alpha$. The application of the formula § 75, gives:

$$Q = \frac{sin.\,(\alpha - \rho)}{sin.\,2\,\alpha}\,.\,R \text{ and } Q_1 = \frac{sin.\,(\alpha + \rho)}{sin.\,2\,\alpha}\,.\,R;$$

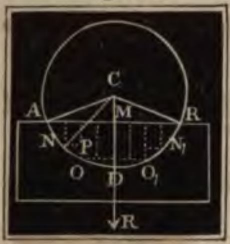
hence the friction sought is:

$$F + F_1 = (Q + Q_1)\,sin.\,\rho = (sin.\,[\alpha - \rho] + sin.\,[\alpha + \rho])\,\frac{R\,sin.\,\rho}{sin.\,2\,\alpha}.$$

But the $sin.\,(\alpha - \rho) + sin.\,(\alpha + \rho) = 2\,sin.\,\alpha\,cos.\,\rho$ and $sin.\,2\,\alpha = 2\,sin.\,\alpha\,cos.\,\alpha$, therefore $F + F_1 = \frac{2\,sin.\,\alpha\,R\,sin.\,\rho\,cos.\,\rho}{2\,sin.\,\alpha\,cos.\,\alpha} = \frac{R\,sin.\,2\,\rho}{2\,cos.\,\alpha}$, which from the smallness of ρ may be put $= \frac{R\,sin.\,\rho}{cos.\,\alpha}$. The friction of a triangular bearing is from this $\frac{1}{cos.\,\alpha}$ times as great as that of a cylindrical one. If, for example, $ADB = 60^\circ$, ACB

$= 180^\circ - 60^\circ = 120^\circ$, and $ACD = \alpha = 60^\circ$, we then have: $\frac{1}{cos.\ 60^\circ}$ = twice the friction of that of a cylindrical bearing.

Fig. 175.



§ 168. With the aid of the last formula, the friction may be found for a new round bearing which the axle touches at all points.* Let ADB, Fig. 175, be such a bearing. Let us divide the arc ADB, in which lie the axle and the bearing, into many parts, such as AN, NO, &c., which have equal projections on the chord AB, and let us suppose that each of these parts supports an equal amount of the whole pressure R, viz., $= \frac{R}{n}$, (n being the number of parts) of the axle on the bearing. According to the former §, the friction of two opposite parts NO and $N_1O_1 = \frac{R}{n} \cdot \frac{sin.\ 2\,\rho}{cos.\ NCD}$. But $cos.\ NCD = cos.\ ONP = \frac{NP}{NO}$, where NP represents the projection of the part NO, and $NP = \frac{chord.\ AB}{n}$. It therefore follows that the friction corresponding to the parts NO and $N_1O_1 = \frac{R\ sin.\ 2\,\rho}{n} \cdot \frac{n \cdot NO}{chord} = \frac{R\ sin.\ 2\,\rho}{chord} \cdot NO$. In order now to find the friction for the whole arc ADB instead of NO, we must put in the arc $AD = \frac{1}{2} ADB$, because the sum of all the frictions is $\frac{R\ sin.\ 2\,\rho}{chord}$ times the sum of all the parts of the arc, it follows that the friction in a new bearing is: $F = R\ sin.\ 2\,\rho \cdot \frac{arc\ AD}{chord\ AB}$, or if we put the angle ACB subtended at the centre by AB, which corresponds to the arc of the bearing, $= 2\,\alpha^\circ$, therefore the chord $AB = 2\,AC \cdot sin.\ \alpha$.

$$F = \frac{R\ sin.\ 2\,\rho}{2} \cdot \frac{\alpha}{sin.\ \alpha},\ \text{or } sin.\ 2\,\rho = 2\ sin.\ \rho$$

taken approximately

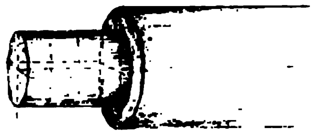
$$F = R\,sin.\,\rho \cdot \frac{\alpha}{sin.\ \alpha}.$$

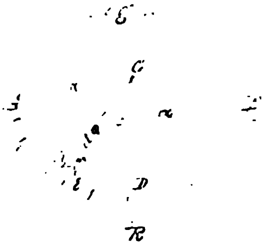
---

* If a bearing be of an acute wedge-shape it is conceivable that an axle pressed into such wedge-shaped cavity would create an additional friction; but as friction depends on *weight*, not on extent of surface, the demonstrations of this section relative to bearings in cylindrical cavities in which there is no wedging or tightening, but only a distribution of the *weight* over the several parts of the *surface* of contact, the effect of the moving force in a "*new* bearing" will be to relieve the pressure on one part and transfer it to another at the same distance from the centre of motion, or centre of the axle. To a certain extent, that is, to the depth of the semi-diameter of the axle, this surface increases, but if there be a distribution of pressure over a greater number of elements or units of surface, so is there a less amount of pressure on each point rubbed. This law of friction, according to pressure, and not according to surface rubbed, agrees with all experiments.—AM. ED.

[illegible]

Friction [illegible] Let R be the pressure [illegible] of [illegible] on a gudgeon AB Fig (1) [illegible] the pressure [illegible] ([illegible]) of the gudgeon [illegible] ABB





[illegible] divided by chord $AB = \frac{R}{2r\sin\alpha}$ the elementary arc $ds = r\,d\varphi$ and its horizontal projection $dn = r\,d\varphi\cos\varphi$

Hence the vertical pressure on the elementary arc ds will be $\frac{R}{2r\sin\alpha}\,r\,d\varphi\cos\varphi$ [illegible]

[illegible] x

$\frac{S\,2R}{\ldots}$ (Cos x [illegible] $\cos\varphi$ [illegible]

$F = \frac{R}{2r\sin\alpha}\int_{-\alpha}^{\alpha} r\,d\varphi\cos\varphi$ [illegible]

$\frac{R}{2\sin\alpha}\cdot 2x$ [illegible] $\frac{\sin 2\alpha}{2}$ [illegible] $\frac{R\sin 2\varphi}{\sin\alpha}$ [illegible]

$= 120°$, and $ACD =$ [illegible] $= 60°$, we then have

From this the friction is the greater the deeper the axle lies in its bearing. If, for instance, the bearing is half the circumference of the axle, α is then $= \frac{1}{2}\pi$ and $sin.\ \alpha = 1$, we then have $F = \frac{\pi}{2} \cdot R\ sin.\ \rho$, and because $\frac{\pi}{2} = 1{,}57$, therefore 1,57 times as great as that of the free bearing. In an axle which does not rest deep in its bearing, α is small, therefore the $sin.\ \alpha$ may be put $= \alpha - \frac{\alpha^3}{6} = \alpha\left(1 - \frac{\alpha^2}{6}\right)$, whence it follows that:

$F = \left(1 + \frac{\alpha^2}{6}\right) R\ sin.\ \rho$, or $= R\ sin.\ \rho$, if α be very small.

§ 169. The axle pressure R is given generally as the resultant of two forces P and Q, directed at right angles to each other, and is therefore $= \sqrt{P^2 + Q^2}$. Provided we require it only for the determination of the friction $f\,R = f\sqrt{P^2 + Q^2}$, we may be satisfied with an approximate value of it, partly because the co-efficient f can never be so accurately determined and depends upon so many accidental circumstances, and partly because the whole product of the friction $f\ R$ is mostly only a small part of the remaining forces of the machine resting upon the axle bearing, as the lever, pulley, wheel and axle, &c. The doctrine which teaches us to find an approximate expression of $\sqrt{P^2 + Q^2}$ is known under the name of Poncelet's theorem, and may be developed in the following manner:

$\sqrt{P^2 + Q^2} = P\sqrt{1 + \left(\frac{Q}{P}\right)^2} = P\sqrt{1 + x^2}$, whereby $x = \left(\frac{P}{Q}\right)$, $\left(\frac{Q}{P}\right)$ which supposes that P is the smaller force, therefore, x is a mere fraction. We may now put:

$\sqrt{1 + x^2} = \mu + \nu x$, and determine the co-efficients μ and ν, answering certain conditions. The relative error is:

$$y = \frac{\sqrt{1 + x^2} - \mu - \nu x}{\sqrt{1 + x^2}} = 1 - \frac{\mu + \nu x}{\sqrt{1 + x^2}}.$$

For the smallest value of x, viz., $x = 0$, $y = 1 - \mu$, and for the greatest, viz. $x = 1$, we have $y = 1 - \frac{\mu + \nu}{\sqrt{2}}$. If we make these errors, corresponding to the limits of x, equal, we then obtain an equation of condition $\mu = \frac{\mu + \nu}{\sqrt{2}}$, or $\nu = \mu\sqrt{2} - \mu = 0{,}414 \cdot \mu$. If we take $x = \frac{\nu}{\mu}$, the result is, that $y = 1 - \sqrt{\mu^2 + \nu^2} = -(\sqrt{\mu^2 + \nu^2} - 1)$, as a negative error, is greater than any other which arises by assuming $x = \frac{\nu}{\mu} \pm \Delta$, that is, a little greater, or a little less than $\frac{\nu}{\mu}$; for in the latter case we have

$$y = -\left(\frac{\mu + \nu\left(\frac{\nu}{\mu} \pm \Delta\right)}{\sqrt{1 + \left(\frac{\nu}{\mu} \pm \Delta\right)^2}} - 1\right)$$

$$= -\left(\frac{\mu^2 + \nu^2 \pm \mu\nu\Delta}{\sqrt{\mu^2 + \nu^2 \pm 2\mu\nu\Delta + \mu^2\Delta^2}} - 1\right)$$

$$= -\left(\sqrt{\frac{(\mu^2 + \nu^2 \pm \mu\nu\Delta)^2}{\mu^2 + \nu^2 \pm 2\mu\nu\Delta + \mu^2\Delta^2}} - 1\right)$$

$$= -\left(\sqrt{\mu^2 + \nu^2 - \frac{\mu^4\Delta^2}{\mu^2 + \nu^2} \ldots} - 1\right).$$

If now we make this greatest negative error equal to the greatest positive error, we shall then obtain the following second equation of condition:

$$\sqrt{\mu^2 + \nu^2} - 1 = 1 - \mu; \text{ or } \mu + \sqrt{\mu^2 + \nu^2} = 2.$$

But the first equation gives $\nu = 0{,}414\,\mu$; it, therefore, follows that

$$\mu(1 + \sqrt{1 + 0{,}414^2}) = 2, \text{ i. e.}$$

$$\mu = \frac{2}{1 + \sqrt{1{,}1714}} = 0{,}96 \text{ and } \nu = 0{,}414 \,.\, 0{,}96 = 0{,}40.$$

We may, therefore, put approximately $\sqrt{1 + x^2} = 0{,}96 + 0{,}40 \,.\, x$, and in like manner the resultant $R = 0{,}96\,P + 0{,}40\,Q$, knowing that we thereby commit, at most, the error $\pm y = 1 - \mu = 1 - 0{,}96 = 0{,}04$ = four per cent. of the true value.

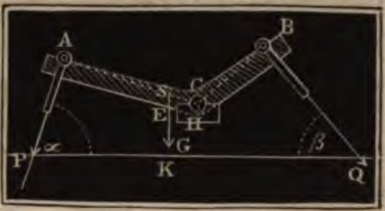
This determination supposes that we know which is the greater of the forces; but if this be unknown, we may assume $\sqrt{1 + x^2} = \mu(1 + x)$, and so obtain $y = 1 - \frac{\mu(1 + x)}{\sqrt{1 + x^2}}$. Here not only the limit $x = 0$ gives the error $= 1 - \mu$, but also the limit $x = \infty$, the same error $= 1 - \frac{\mu x}{x} = 1 - \mu$; but if we put $x = \frac{\nu}{\mu} = 1$, we then obtain the greatest negative error $= -\left(\frac{2\mu}{\sqrt{2}} - 1\right) = -(\mu\sqrt{2} - 1,)$ and by making these errors equal: $1 - \mu = \mu\sqrt{2} - 1$, therefore, $\mu = \frac{2}{1 + \sqrt{2}} = \frac{2}{2{,}414} = \frac{1}{1{,}212} = 0{,}825$, for which 0,83 may be put. In the case where we do not know which is the greater of the forces, R may be put $= 0{,}83\,(P + Q)$, and we know that the greatest error committed will be $\pm\, y = 1 - 0{,}83 = 0{,}17$ per cent. $= \frac{1}{6}$ of the true value.

If, lastly, we know that x does not exceed 0,2, we may disregard it altogether, and write $\sqrt{P^2 + Q^2} = P$, but if x exceeds 0,2, then

$\sqrt{P^2 + Q^2}$ is more accurately $= 0{,}888 \,.\, P + 0{,}490 \,.\, Q$; in both cases the greatest error is about two per cent.*

§ 170. *Lever.*—The theory of friction above developed finds its application in the material lever, the wheel and axle, and in other machines. Let us, in the first place, treat of the lever, and take the general case, viz., that of the bent lever ACB, Fig. 176. Let us represent as before (§ 127) the arm CA of the power P by a, the arm CB of the weight Q by b, and the radius of the axle CH by r, let us put the weight of the lever $= G$, its arm $CE = s$, and the angles APK and BQK, by which the directions of the forces deviate from the horizon $= \alpha$ and β. The power P gives the vertical pressure $P \sin. \alpha$, and the weight $= Q \sin. \beta$; the whole vertical pressure is, therefore, $V = G + P \sin. \alpha + Q \sin. \beta$. The power P gives further the horizontal pressure $P \cos. \alpha$, and the weight Q an opposite pressure $Q \cos. \beta$: since there remains for the horizontal pressure, $H = P \cos. \alpha - Q \cos. \beta$, we may put the whole pressure on the axle: $R = \mu\, V + \nu\, H = \mu\,(G + P \sin. \alpha + Q \sin. \beta) + \nu (P \cos. \alpha - Q \cos. \beta)$, of which the second part $\nu\,(P \cos. \alpha - Q \cos. \beta)$ must never be taken negative, and, therefore, in the case where $Q \cos. \beta$ is $> P \cos. \alpha$, the sign must be changed, or rather $P \cos. \alpha$ must be subtracted from $Q \cos. \beta$. In order to find that value of the power which corresponds to unstable equilibrium, so that the smallest addition of force produces motion, we must put the moment of power equal to the moment of weight, plus or minus the moment of weight of the machine (§ 127) plus the moment of friction, therefore,

Fig. 176.



$$Pa = Qb \pm Gs + f\,Rr$$

$$= Qb \mp Gs + f(\mu\, V + \nu\, H)\,r, \text{ from which follows}$$

$$P = \frac{Qb \pm Gs + f[\mu\,(G + Q \sin. \beta) \mp Q \cos. \beta]\,r}{a - \mu f r \sin. \alpha \mp \nu f r \cos. \alpha}.$$

If P and Q act vertically, R is simply $= P + Q + G$, therefore, $Pa = Qb \pm Gs + f\,(P + Q + G)\,r$. If the lever is one-armed, P and Q act opposite to each other, then $R = P - Q + G$, and consequently the friction is less. Besides R must be put constantly positive in the calculation, because the friction $f\,R$ only impedes, but does not produce motion. From this we see that a one-armed, is mechanically more perfect than a two-armed lever.

Example. If the arms of a bent lever, Fig. 176, are: $a = 6$ ft., $b = 4$ ft., $s = \frac{1}{2}$ ft. and $r = 1\frac{1}{2}$ inches, the angle of inclination $\alpha = 70^0, \beta = 50^0$, and further the weight $Q = 5600$ lbs., and that of the lever $G = 900$ lbs., the power required to restore the unstable equilibrium is the following. Without regard to friction $Pa + Gs = Qb$, therefore,

---

* Polytechnische Mittheilungen, Heft 1, 1844.

$P = \frac{Qb - Gs}{a} = \frac{5600 . 4 - 900 . \frac{1}{2}}{6} = 3658$ lbs. If we put $\mu = 0{,}96$ and $\nu = 0{,}40$, we obtain $\mu\ (G + Q\ sin.\ \beta) = 0{,}96\ (900 + 5600\ sin.\ 50^0) = 4982$ lbs., $\nu\ Q\ cos.\ \beta = 0{,}40 . 5600.\ cos.\ 50^0 = 1440$ lbs.; $\mu\ sin.\ \alpha = 0{,}96 .\ sin.\ 70^0 = 0{,}902$, $\nu\ cos.\ \alpha = 0{,}40.\ cos.\ 70^0 = 0{,}137$. It is easy to see, that here $P\ cos.\ \alpha$ is less than $Q\ cos.\ \beta$, for since approximately $P = 3658$, we have $P\ cos.\ \alpha = 1251$ lbs., and $Q\ cos.\ \beta = 3600$ lbs., let us therefore take for $\nu\ Q\ cos.\ \beta$, and $\nu\ cos.\ \alpha$ the lower sign and put $P = \frac{5600 . 4 - 900 . \frac{1}{2} + fr\ (4982 + 1440)}{6 - fr\ (0{,}902 - 0{,}173)}$. Let us further take the co-efficient of friction $f = 0{,}075$, and we shall have $fr = 0{,}075 . \frac{3}{24} = 0{,}009375$, and the power sought, $P = \frac{22400 - 450 + 60}{6 - 0{,}00683} = \frac{22010}{5{,}9932} = 3673$ lbs. Here the vertical pressure, if we substitute the value $P = 3658$ lbs., and neglect friction, is $V = 3658 .\ sin.\ 70^0 + 5600\ sin.\ 50^0 + 900 = 3437 + 4290 + 900 = 8627$ lbs., on the other hand, the horizontal pressure:

$$H = 5600\ cos.\ 50 - 3658\ cos.\ 70^\circ = 3600 - 1251 = 2349 \text{ lbs.}$$

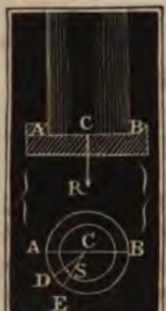
Here H is $> 0{,}2\ V$, therefore, more correctly:

$$R = 0{,}888 . H + 0{,}490\ V = 0{,}888 . 8627 + 0{,}490 . 2349 = 8811,$$

and it follows that the moment of friction $= f r R = 0{,}009375 . 8811 = 82{,}6$ ft. lbs., and lastly, the power $P = \frac{22400 - 450 + 82{,}6}{6} = 3672$ lbs., which value varies little from the above.

§ 171. *Pivot Friction.*—When in the wheel and axle a pressure takes place in the direction of the axis, as in the case, for example, of upright axles, in consequence of their weight, there is a friction on the base of the one axle. Because pressure is there exerted on points between the pivot and its step, this friction approximates to the simple sliding friction, and to the axle friction which we have hitherto considered, and we must put for it the co-efficients of friction given in Table II. To find the mechanical effect absorbed by this friction, we must know the mean space which the base AB, Fig. 177, of such an upright axle describes in a revolution. Let us assume that the pressure R is equally distributed over the whole surface, let us also suppose that on equal parts of the bases the frictions are equal. Let us further divide the base by radii CD, CE, &c., into equal sectors or triangles DCE; to these will correspond not only equal amounts of friction, but also equal moments, therefore, it will be necessary only to find the moment of friction of one of these triangles. The frictions of such a triangle may be regarded as parallel forces, for they all act tangentially, *i. e.* at right angles to the radius CD, and since the centre of gravity of a body or a surface is nothing more than the point of application of the resultant of the parallel forces equally distributed over this body or surface, accordingly the centre of gravity S of the sector or triangle DCE is here the point of application of the resultant arising from its different frictions. If now the pressure on this sector $= \frac{R}{n}$, and the radius $CD = CE$, the base $= r$, it follows that (from § 104) the moment of friction of this sector $= CS . \frac{fR}{n}$

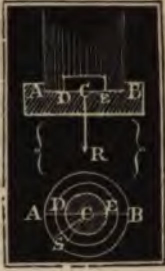
Fig. 177.



$= \frac{2}{3} r \,.\, \frac{f R}{n}$, and lastly, the moment of the entire friction of the axle $= n \,.\, \frac{2}{3} r \frac{f R}{n} = \frac{2}{3} f R r$.

Sometimes the rubbing surface is a ring *ABED*, Fig. 178. If its radii are $CA = r_1$ and $CD = r_2$, we have then to determine the centre of gravity S of a portion of a ring, and from § 109, obtain the arm $CS = \frac{2}{3} \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}$, therefore, the moment of friction $= \frac{2}{3} f R \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$. If we introduce the mean radius $\frac{r_1 + r_2}{2} = r$, and the breadth of the ring $r_1 - r_2 = b$, we obtain this moment of friction also $= f R \left(r + \frac{b^2}{12 r}\right)$.

Fig. 178.

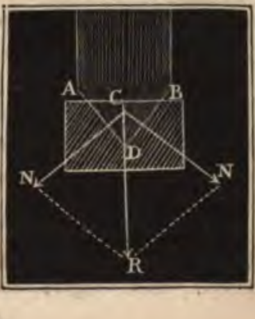


The mechanical effect of friction for a revolution of the axle is in the second case $= 2\,\pi \,.\, \frac{2}{3} f R r = \frac{4}{3} \pi f R r$, and in the first $\frac{4}{3} \pi f R \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$. Here we easily see that to diminish this loss of mechanical effect, the upright axle or shaft must be made as light as possible, and that a greater loss of mechanical effect would arise if, under otherwise similar circumstances, the friction were to take place in a ring instead of a complete circle.

Example. In a turbine making 100 revolutions a minute, and 1800 lbs. weight, the size of the pivot at the base, is $\frac{1}{2}$ inch; how much mechanical effect does the friction of this pivot consume in one second? The co-efficient of friction being taken $= 0{,}1$ we have the friction $f R = 0{,}1 \,.\, 1800 = 180$ lbs., the space per revolution $= \frac{4}{3} \pi r = \frac{4}{3} \,.\, 3{,}14 \,.\, \frac{1}{24} = 0{,}1745$ ft. lbs., hence the mechanical effect per revolution $= 180 \,.\, 0{,}1745 = 31{,}41$ ft. lbs. But now this machine makes in a second $\frac{100}{60} = \frac{5}{3}$ of a revolution, hence it follows that the loss of mechanical effect sought $= \frac{314{,}1}{6} = 52{,}3$ ft. lbs.

§ 172. *Pointed Axles.*—If the axle *ABD*, Fig. 179, has conical ends, the friction comes out greater than if it has plane ends, because the pressure of the axle R is resolved into the normal forces N, N_1, which produce the friction, and which together are greater than R alone. If the half of the convergent angle $ADC = BDC = a$, we have $2\,N = \frac{R}{\sin.\, a}$, ✗ and consequently the friction of the pointed axle $= f \frac{R}{\sin.\, a}$. Let the radius of the axle $CA = CB$ at the entrance into the step be re-

Fig. 179.



$\times \frac{1}{2} R = N \sin a$, $\frac{R}{\sin a} = 2N$

presented by r_1, we shall then have as before the moment of friction $= \frac{2}{3} f \frac{Rr_1}{sin.\, \alpha}$. Let this axle dip a little only into the step, the mechanical effect of this axle will be less than that of an axle with a plane base, and on this account the application of the pointed axle is of service. When, for example, $\frac{r_1}{sin.\, \alpha} = \frac{r}{2}$, therefore, $r_1 = \frac{1}{2}\, r\, sin.\, \alpha$, the pointed axle of the radius r_1 causes only half the loss of mechanical effect through friction which the truncated axis of the radius r does.

If the pivot forms a truncated cone, Fig. 180, friction takes place as well at the envelop as the truncated surface, and the moment of friction comes out $= \left(r_1^3 + \frac{r^3 - r_1^3}{sin.\, \alpha}\right) \frac{2}{3} \frac{f R}{r^2}$, if r be the radius of the place of entrance into the step, and $r_1 = DE$ that of the base, and $\alpha°$ the half of the convergent angle.

Fig. 180.

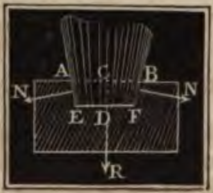


Fig. 181.

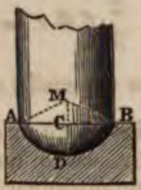
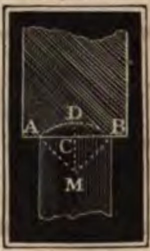


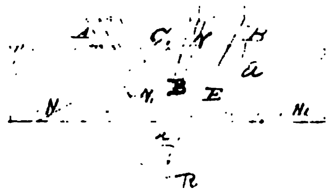
Fig. 182.



Lastly, the pivot or upright axles (Figs. 181, 182) are very often rounded. Although by this rounding, the friction itself is by no means diminished, there arises nevertheless a diminution of the moment of friction, from the extremity not dipping far into the step. If we suppose a spherical rounding, we obtain by the aid of the higher calculus for a semi-spherical step, the moment of friction $= \frac{f\pi}{2} . Rr$; but for that of a step having a less segment $= \frac{2}{3}\left[1 + 0,3 \left(\frac{r_1}{r}\right)^2\right] Rr_1$, r being the radius of the sphere $MA = MB$, r_1 the radius of the step $CA = CB$.

Example. If the weight of the armed axle of a horse capstan $R = 6000$ lbs., the radius of the conical pivot $= r = 1$ inch, and the angle of convergence of the cone $2\,\alpha = 90°$, then the moment of friction of this pivot $= \frac{2}{3} . f . \frac{Rr}{sin.\, \alpha} = \frac{2}{3} . 0,1 . \frac{6000}{sin.\, 45°} . \frac{1}{12} = \frac{100}{3\sqrt{\frac{1}{2}}} = 47,1$ ft. lbs. This axle makes during the lifting up of a ton from a shaft or mine $= u = 24$ revolutions, then the mechanical effect which is expended at the pivot in this time by friction $= 2\,\pi\,u . \frac{2}{3} f \frac{R\,r}{sin.\, \alpha}, = 2\,\pi . 24 . 47,1 = 7103$ ft. lbs.

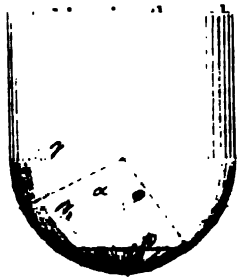
§ 173. *Points and Knife Edges.*—To avoid as much as possible the friction of the axle, rotatory bodies are supported on pointed pivots, knife edges, &c. If we had only to do with perfectly rigid



Friction of a Truncated Cone.

Let CB = r and BE = $R r_1$. πr^2 = the horizontal surface; $\frac{R}{\pi r^2}$ = pressure on unit. $\rho\, d\varphi\, d\rho$ = elementary surface. $\frac{R}{\pi r^2}$ [illegible] $d\rho$ element [illegible] pressed is not [illegible] horizontal [illegible] for pressure [illegible] $\frac{R}{\pi r^2}$ [illegible] $\rho\, d\varphi\, d\rho$ [illegible] pressure [illegible] ABB. [illegible] $\frac{R}{\sin a}$ [illegible] Hence $\frac{R}{\pi r^2 \sin a}$ [illegible] $\rho^2\, d\varphi\, d\rho$ = [illegible] on unit [illegible] $F = \frac{R}{\pi r^2}$ [illegible] $\rho^2\, d\varphi\, d\rho$ + $\frac{R}{\pi r^2 \sin a} \int_{r_1} \int_{0} \rho^2\, d\varphi\, d\rho$. $F = \frac{R}{\pi r^2} \cdot \frac{r_1^3}{3}$ [illegible] + $\frac{R}{\pi r^2 \sin a}\left(\frac{r^3}{3} - \frac{r_1^3}{3}\right)$

$= \frac{2}{3} \cdot \frac{R}{r^2}\left(r_1^3 + \frac{r^3 - r_1^3}{\sin a}\right)$



[illegible] Obj.

πr_1^2 = surface pressed

$\frac{R}{\pi r_1^2}$ = pressure on a horizontal unit of surface.

$2\pi\rho\,d\rho$ = Elementary ring. Element of normal pressure is equal to elementary vertical pressure divided by $\cos\phi$ hence $f\frac{R}{\pi r_1^2}\frac{2\pi\rho\,d\rho}{\cos\phi}$, Ele. moment of friction

$\rho = r\sin\phi$ ∴ $d\rho = r\cos\phi\,d\phi$, substituting

d moment $= f\frac{R}{r_1^2}\frac{2r^3\sin^2\phi\cos\phi\,d\phi}{\cos\phi} = f\frac{R}{r_1^2}r^3\,2\sin^2\phi\,d\phi$

But $2\sin^2\phi = 1-\cos 2\phi$ hence

Moment $= f\frac{R r^3}{r_1^2}\int_0^{\alpha}(1-\cos 2\phi)\,d\phi = f\frac{R r^3}{r_1^2}\left(\alpha - \frac{\sin 2\alpha}{2}\right)$

$r_1 = r\sin\alpha$ ∴ $\alpha = \sin^{-1}\left(\frac{r_1}{r}\right)$, $\frac{\sin 2\alpha}{2} = \sin\alpha\cos\alpha$ $= \sin\alpha\sqrt{1-\sin^2\alpha} = \frac{r_1}{r}\sqrt{1-\left(\frac{r_1}{r}\right)^2}$ hence

Moment $= f\frac{R r^3}{r_1^2}\left[\sin^{-1}\left(\frac{r_1}{r}\right) - \frac{r_1}{r}\sqrt{1-\left(\frac{r_1}{r}\right)^2}\right]$. Developing

$\sin^{-1}\frac{r_1}{r} = \frac{r_1}{r} + \frac{1}{6}\frac{r_1^3}{r^3} + \frac{3}{40}\frac{r_1^5}{r^5} + \text{etc}$

$\frac{r_1}{r}(r^2 - r_1^2)^{\frac{1}{2}} = \frac{r_1}{r}\left(1 - \frac{1}{2}\frac{r_1^2}{r^2} - \frac{1}{8}\frac{r_1^4}{r^4} - \text{etc}\right) = \frac{r_1}{r} - \frac{1}{2}\frac{r_1^3}{r^3} - \frac{1}{8}\frac{r_1^5}{r^5}$

$$\sin^{-1}\left(\frac{r_1}{r}\right) - \frac{r_1}{r}\sqrt{r^2 - r_1^2} = \frac{r_1}{r} + \frac{1}{6}\frac{r_1^3}{r^3} + \frac{3}{40}\frac{r_1^5}{r^5}$$
$$- \frac{r_1}{r} - \frac{1}{2}\frac{r_1^3}{r^3} - \frac{1}{8}\frac{r_1^5}{r^5}$$

Moment $= f\frac{R r^3}{r_1^2}\left(\frac{2}{3}\frac{r_1^3}{r^3} + \frac{1}{5}\frac{r_1^5}{r^5}\right)$

$= f R\left(\frac{2}{3}r_1 + \frac{1}{5}\frac{r_1^3}{r^2}\right) = f R r_1\left(\frac{2}{3} + \frac{1}{5}\frac{r_1^2}{r^2}\right)$

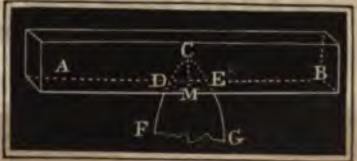
$= f R r_1\left(\frac{2}{3} + \frac{2}{3}\cdot\frac{3}{10}\frac{r_1^2}{r^2}\right) = f R r_1\frac{2}{3}\left(1 + 0.3\frac{r_1^2}{r^2}\right)$

[illegible]

and inelastic bodies, no loss of labor would arise through friction by this method of support or suspension, because no measurable space here is described by the friction; but since every body possesses a certain degree of elasticity, by the resting of such a body on a point or knife edge, a slight penetration takes place, and a rubbing surface is thereby caused, upon which a space is described by the friction which gives rise to a certain loss of labor, although very small. In rotations and vibrations long sustained, bodies supported in this manner, present similar surfaces of friction arising from the abrasion of their points or knife edges, and the friction must then be estimated by what has been already mentioned. For these reasons this mode of support is applicable only to such instruments as the compass, the balance, &c., where it is of importance to diminish the amount of friction, and where motion is only allowed from time to time.

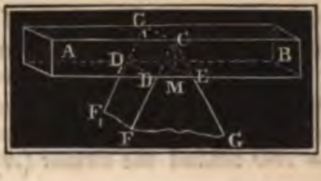
Experiments on the friction of a body resting upon a hard steel point, and revolving about it, have been made by *Coulomb*. From these, it results that the friction increases somewhat more than the pressure, and varies with the thickness of the supporting pivot. It is least for a granite surface, greater for one of agate and of rock crystal, greater still for a glass surface, and greatest of all for a steel one. For a very small pressure, as in the magnetic needle, the pivot may be pointed to 10° or 12° of convergence. But if the pressure is great, we must then apply a far greater angle of convergence, viz. 30° to 45°. The friction is less when the body having a plane surface reposes upon a point than when it lies in a conical or spherical step. Similar relations take place in the knife edge as applied to the balance, and the beams of balances, that are intended for heavy loads, require sharp axes of 90° convergence, while an edge of 30° is sufficient for the lighter ones.

Fig. 183.



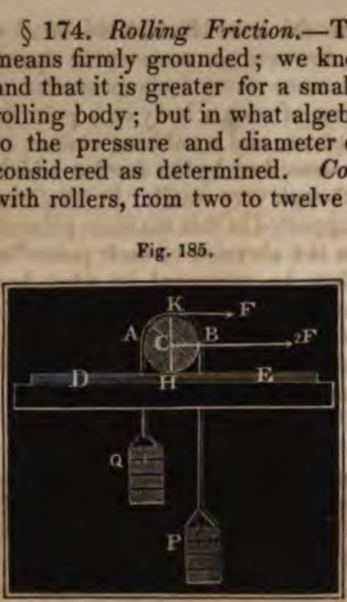
Remark. If we assume that the needle AB, Fig. 183, rests on the point DCE of the pivot FCG, of the height $CM = h$, and radius $DM = r$, and suppose that the volume $\frac{1}{3}\pi r^3 h$ is proportionate to the pressure R, the amount of friction may be found in the following manner. If we put $\frac{1}{3}\pi r^3 h = \mu R$, where μ is a number resulting from experience, and introduce the angle of convergence $DCE = 2\alpha$, and, therefore, put $h = r\ cotg.\ \alpha$, we obtain the radius of the base $r = \sqrt[3]{\frac{3\mu R\ tang.\ \alpha}{\pi}}$, and $f r R = \sqrt[3]{\frac{3\mu R^4\ tang.\ \alpha}{\pi}} = \sqrt[3]{\frac{3\mu}{\pi}} . R\ \frac{4}{3}\ (tang.\ \alpha)\ \frac{1}{3}$. We must, therefore, assume that the friction on the pivot increases equally with the cube root of the fourth power of the pressure, and the cube root of the tangent of half the angle of convergence. The amount of friction of a beam AB, Fig. 184, which oscillates on a sharp edge CC_1, may be found in like manner. If α be half the angle of convergence DCM, l the length CC_1 of the edge, and R the pressure, this amount is given $= \sqrt{\frac{(P\ tang.\ \alpha)^3}{l}}$.

Fig. 184.



§ 174. *Rolling Friction.*—The theory of rolling friction is by no means firmly grounded; we know that it increases with the pressure, and that it is greater for a smaller than for a larger diameter of the rolling body; but in what algebraical dependence this friction stands to the pressure and diameter of the rolling body, cannot as yet be considered as determined. *Coulomb* made only a few experiments with rollers, from two to twelve inches thick, of lignum vitæ and elm, which were rolled along a surface of oak, by means of a thin thread passing over the roller AB, whose extremities were stretched by unequal weights P and Q, Fig. 185. From the results of these experiments, rolling friction appears to increase directly with the pressure, and inversely with the diameter of the roller, so that the force necessary to overcome this friction may be expressed by $F = f \cdot \frac{R}{r}$, if R be the pressure, r the radius of the roller, and f the co-efficient of friction derived from experiment. If r be given in inches, then from these experiments

Fig. 185.

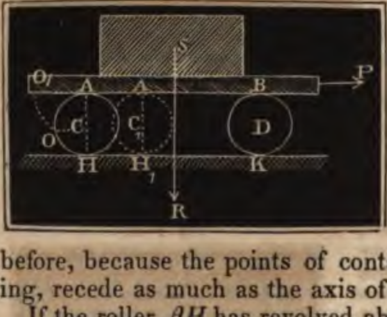


For rolling upon compressed wood $f = 0,0189$,
" " " " $f = 0,0310$.

These formulas suppose that the force F acts at the circumference of the roller, but if the force be applied to the axis C of the rolling bodies, by which, as in every description of carriage, axle friction ensues, the required force is $2\,F$, because here the arm CH is only half that of KH with respect to the point of application H.

A body ABS is moved forwards, Fig. 186, lying on the rollers C and D, the required force P here comes out very small, because two rolling frictions only, viz., that between AB and the rollers, and that between the rollers and the way HR are to be overcome. The progressive space of the rollers is only half that of the load Q, and on this account for farther progression, the rollers must be replaced under it from before, because the points of contact A and B, by virtue of the rolling, recede as much as the axis of the rollers advances.

Fig. 186.



If the roller AH has revolved about an arc AO, the roller has then moved over a space AA_1 equal to this arc, and O comes into contact with O_1, the new point of contact O_1 has, therefore, receded by $AO_1 = AO$ behind the former (A). If the co-efficients of friction are f

and f_1, the power necessary to draw the load R forward is $P = (f + f_1)\frac{R}{r}$.

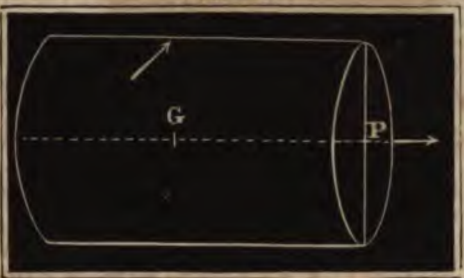
Remark. The extended experiments of *Morin* on the resistance of carriages upon roads, accord with the law by which this resistance increases equally with the pressure, and inversely with the thickness of the roller. Another French engineer, *Dupuit*, on the contrary, deduces from his experiments, that rolling friction increases indeed directly with the pressure, but for the rest, only inversely proportional to the square root of the radius of the roller. Particular theoretical views upon rolling friction may be found in *Gerstner's* Mechanics, vol. i. § 537, and developed in *Brix's* Treatise upon Friction, art. 6.*

---

* The following demonstrations are applicable to wheel carriages in general, and especially to railway cars and locomotives. They bring into view the relation between rolling and dragging friction, as well as the resistance of fixed obstacles to rolling bodies.

These two kinds of friction may be illustrated by the motion of a cylinder, (Fig. 186_1), moved over any plane surface by a force applied, first, opposite to its centre of gravity, and at *right angles* to its axis, and secondly, opposite to the same centre, but in the *same direction* with its axis. The former force will, if both the cylinder and the plane be *perfectly smooth*, unyielding, and free from foreign matter, produce a progressive motion only in the cylinder. But in every practical case such an application of force produces likewise a rotation, and, in proportion as the roughness of the surfaces prevents or resists the sliding of one over the other, in the same proportion will the *rotary*, sooner or later, correspond to the *progressive* motion. We may easily conceive, however, that while a body is moving forward with accelerated velocity, that is, while its centre of gravity advances with increasing rapidity, the revolution on its axis shall be uniform, and the motion of any point on the periphery may, at any given moment, be either greater or less than that of the centre of gravity. Should a cylinder, revolving under such circumstances, come to apply its periphery suddenly to a part of the plane, where, from the roughness of surface, it should be compelled to coincide in its revolving velocity with the motion of progression, while the two motions were coming to an equality, a portion of rubbing must take place, and the extent of surface rubbed must be equal to the difference of motion between the centre of gravity of the cylinder and the assumed point of the periphery. Thus, if while the centre traverses a straight line of four feet, the circumference revolved through five, *in the same direction*, then the extent actually rubbed over would be one foot. If it revolved in the opposite direction to that which friction would of itself produce, then we may conceive that each point of the plane passes over some one point of the cylinder, and, therefore, that there is from this cause a friction through four feet of space due to the progressive motion; and again, that each point of the cylinder rubs against some point of the plane, and produces a friction through five feet due to the rotation, and consequently, that the united effects of these opposite motions would be to change the existing rotary motion into one in the opposite direction, by a quantity equal to a direct friction through *nine* feet; that is, through the sum of the two motions. To illustrate the preceding remarks, we may easily suppose a wheel or cylinder to receive a sudden percussion, which shall cause a rapid progressive motion commencing from a state of rest; this motion may generate a degree of rotary motion, which may or may not be equal to the progressive velocity, according to the nature of the surface over which it moves. If, after a short time, the surface of the rolling body ceases to touch the plane surface and traverses free space, the rotary motion will continue nearly uniform, while the progressive motion may be greatly retarded, or may entirely cease. If in this case the body comes again in contact with the plane, the uniform rotary motion will generate a friction, which will increase the rectilinear motion, by communicating to the centre

Fig. 186_1.



[illegible]

[illegible]

[illegible]

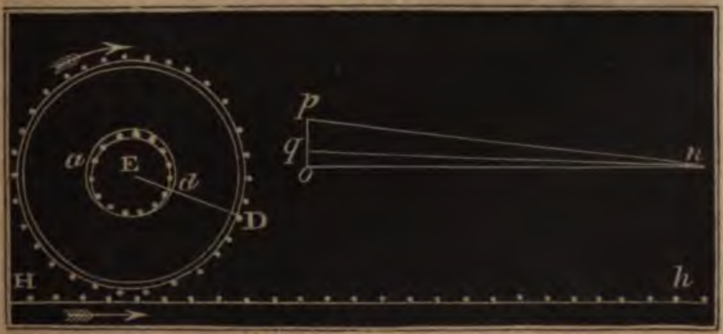
consequently the friction:

$$F = f\sqrt{P^2 + Q^2 - 2\,PQ\cos.\alpha}.$$

If further we put $P = Q + F$ and P^2 approximately $= Q^2 + 2\,QF$,

$fW = P = fW$ as f is but a small
$fP = P_1 = f^2W$ fraction this may
$fP_1 = P_2 = f^3W$ be carried on until
f^n is | as near zero as we please.
$fP_{n-1} = P_n = f^nW_n$ Adding these terms
$P_1 + P_2 + P_3 + \dots P_n = (f^2 + f^3 + \dots f^{n})W = (f + f^2 \dots f_{n-1})fW$
$S - P = S - fW = Sf \therefore S = \frac{fW}{1-f}$ Which indeed
being interpreted is his expression

Fig. 186_3.



The exterior of the wheel *ED* is likewise represented as furnished with equal prominences at equal distances. When, therefore, the wheel is compelled to make one revolution without advancing, as many prominences would be broken from its periphery as would be dislodged from the plane surface while it advanced, without revolving through a space equal to its circumference. This applies to a locomotive slipping its consequently the friction:

$$F = f\sqrt{P^2 + Q^2 - 2\,PQ\cos.\,a.}$$

If further we put $P = Q + F$ and P^2 approximately $= Q^2 + 2\,QF$, we then obtain:

$$F = f\sqrt{Q^2 + 2\,QF + Q^2 - 2\,Q^2\cos.\,a - 2\,FQ\cos.\,a}$$

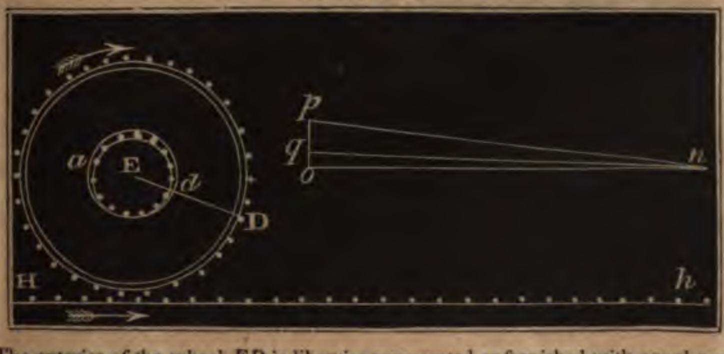
---

of which the cylinder and the bed, *EHC*, are composed; this allowance would again cause a new pressure and friction, and thus a decreasing geometrical series of weights must be added at the point *C*, having for the first term such a part of *W* as is expressed by the relation of pressure to friction, in the case of the given materials, and for a common ratio of the progression, the fraction expressing the same relation. The sum of all the terms, continued to *zero*, will be the actual amount of *P* at the moment when motion commences. The sum of all the terms *following the first*, will be found by *multiplying together the first and second terms of the series*, and *dividing the product by their difference. The quotient added to the first term gives the sum of the series required.* The applicability of a similar method of computation to the friction on the gudgeons of water wheels, moved by the gravity of water, is too obvious to require demonstration.

If instead of applying a weight at *P* only, we should apply, as above supposed, two equal forces, one in the direction of *CP*, and the other in that of *EM*, the amount of friction caused by the former would be relieved by the latter, and consequently, there would remain only the friction of the cylinder. The same would be true if the forces were to take either the directions *NI* and *HT*, or *KO* and *DS* respectively. Supposing the cylinder to be placed on an axis smaller in any given proportion than its own diameter, as *GY*, then the whole effect of gravity would be transferred to this axis, and if this were to be caused to revolve by a force applied tangentially to the axis itself, it must be of the same magnitude as that which had before been applied to the cylinder when placed in the groove. But if applied to the exterior of the cylinder, it must be as much less than before as the diameter of the axis is less than that of the cylinder. In other words, the difficulty of overcoming friction at the axle, is to that of overcoming the same at the outer periphery, when, confined in a bearing, as the diameter of the axle is to that of the cylinder. If *D* be the diameter of the cylinder, *d* of the axle, and *F* the relation of weight to friction, we shall have the proportion $D : d = F : \frac{dF}{D} =$, the force required to overcome the friction on the axle.

This subject may be still further illustrated by Fig. 186_3, where the horizontal plane *Hh*, is represented as furnished at equal distances, with small balls or prominences so attached to its surface as to present equal obstructions to the dragging of heavy bodies along that surface.

Fig. 186_3.



The exterior of the wheel *ED* is likewise represented as furnished with equal prominences at equal distances. When, therefore, the wheel is compelled to make one revolution without advancing, as many prominences would be broken from its periphery as would be dislodged from the plane surface while it advanced, without revolving through a space equal to its circumference. This applies to a locomotive slipping its

$= f\sqrt{2(1-cos.\ \alpha)(Q^2+QF)} = 2f\ sin.\ \frac{\alpha}{2}\ \sqrt{Q^2+QF}$, which again may be put $= 2f\ sin.\ \frac{\alpha}{2}\ (Q+\frac{1}{2}F)$, if we have regard to the two

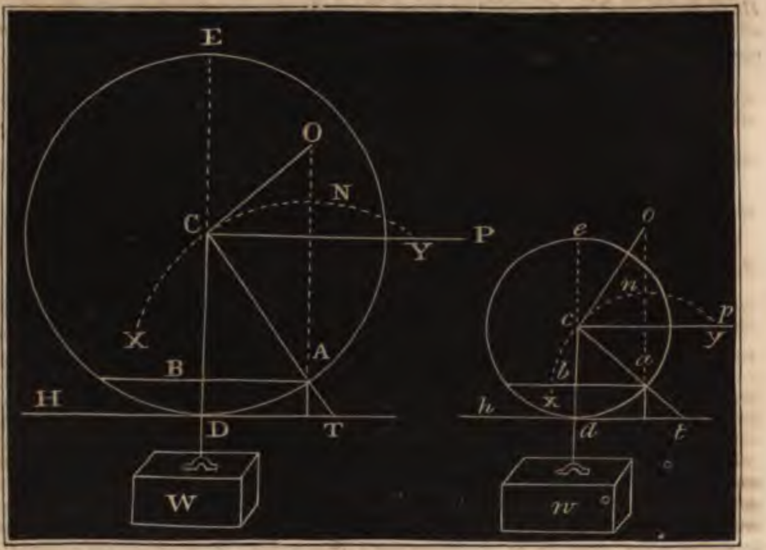
---

wheels on the rails in the one case, and sliding with wheels clogged or engine reversed in the other. But when the wheel is allowed both to revolve and to advance in such a manner as to apply its periphery to a length of plane just equal to the space traversed by the centre, the prominences will be geared together like the teeth of a rack and pinion. But in the latter case the prominences may all remain unbroken. But when the wheel rests with its whole weight on an axle, as *ad*, the number of prominences which can be disposed at the same distances as before, on the circumference of the axle, will be diminished, in proportion as the radius *Ed* of the axle is smaller than *ED*, that of the wheel.

When a wheel or cylinder rolls on a surface as nearly *plane* as it is possible for art to produce, the amount of friction, being no more than is due to the moment of inertia, is extremely small compared with that of dragging, but the observations already made, and the examples cited, will be sufficient to show that the actual advancement tends, by a force equal to that which produces the rotation, to break down the prominences of the surface, for if we consider the cylinder rolled forward by a fine thread unrolled from its upper side, we may consider also the plane beneath to oppose a force tending to draw it in the opposite direction, and this force is friction.

In experimenting with wheel carriages, or cars descending by their own gravity along inclined planes, to ascertain the ratio of *friction* to *weight*, we have to determine separately the rolling friction of wheels and axles with various weights and diameters, and then their influence combined with that of the insistent weight of cars and loads, which latter can alone produce sliding friction at the axle. The weight of the wheels resisted only by the slight amount of rolling friction at the periphery, tends to accelerate the velocity of the car and load. If we suppose the wheels and axle only to be placed on a plane *n q o*, Fig. 186_3, so little inclined as just to continue their rolling motion, and afterwards on another *n p o* so much inclined as to allow a car to descend with all the friction at its axles, we shall readily conceive that over the latter plane the wheels would, by themselves, have descended with a constantly accelerated motion, and consequently, that they

Fig. 186 .



first members of the square root only. Now if $F - f\,F\,sin.\,\frac{a}{2}$ is given $= 2 f\,Q\,sin.\,\frac{a}{2}$, then the friction sought is

---

would, to the extent of their accelerating force, overcome a portion of the resistance which friction opposes to the motion of the car.

Thus, in every case where we would compute the effect of friction by comparing the actual distance passed over by a carriage, with the theoretical descent as caused by the inclination of the plane, we must consider the weight of the car and load as the cause of friction on the axle, and the gravitating power of the wheels (and that of the axles when they revolve with the wheels), as aiding to overcome the friction occasioned by the load.

To compute the effect of any obstacle of given height which a rolling body is compelled to surmount, as dependent on the diameter of the wheel, we may take two wheels EAD and ead, Fig. 186_4, of different heights, intended to surmount the equal obstacles TA and ta. Let the weights W and w be the same for both wheels, and the powers P and p be such as to produce an equilibrium in the wheels DAE and dae respectively. Then since (§ 75 and 139) three forces are in equilibrium, where each is represented by the *sine of the angle comprehended between the directions of the other two*, the direction of gravity and that of the horizontal line of traction being at right angles to each other, the sine of the angle comprehended between their directions is equal to *radius*, and is, therefore, represented by CA or ca. Again, the horizontal force is represented by the sine of the angle BCA or bca, which is the line BA or ba, while the vertical force or gravitating power of W is represented in the two cases by the sines of BAC and bac respectively, which are the lines BC and bc. These two forces multiplied respectively by the distance at which they act, in a perpendicular direction from the point A or a, about which the wheel must revolve in order to surmount the obstacle, ought to give equal moments to those of W multiplied by BA and ba. Hence $BA \times W = BC \times P$, or $P = \frac{BA \times W}{BC}$ and $ba \times W = bc \times p$, or $p = \frac{ba \times W}{bc}$. In order, therefore, to know the absolute values of P and p, we must determine the actual lengths of BA and BC, of ba and bc. BC is easily found by subtracting the perpendicular height of the obstacle from the radius, and BA is found by subtracting the square of BC from that of AC, and extracting the square root of the remainder. Thus $\overline{CA}^2 - \overline{BC}^2 = \overline{BA}^2$. But $BC = CA - BD$, therefore $\overline{BC}^2 = \overline{CA}^2 - 2\,CA \times BD + \overline{BD}^2$. Substituting this value of $\overline{BC}^2$ in the equation $\overline{CA}^2 - \overline{BC}^2 = \overline{BA}^2$, we obtain $2\,CA \times BD - \overline{BD}^2 = \overline{BA}^2$; hence $BA = \sqrt{2\,CA \times BD - \overline{BD}^2}$. The value of P, therefore, must be $\frac{W \times \sqrt{2\,CA \times BD - \overline{BD}^2}}{BC, \text{ or } (CA - BD)}$, which is now expressed in terms of the radius and versed sine. Substituting R for the radius of the larger wheel, and r for that of the smaller, as also h for the perpendicular height of the obstacle in both cases, the above expression of the value of P becomes $\frac{W \times \sqrt{2\,Rh - h^2}}{R - h}$, and by a course of reasoning precisely similar, we obtain $p = \frac{W \times \sqrt{2\,rh - h^2}}{r - h}$. Hence

$$P : p = \frac{W \times \sqrt{2\,Rh - h^2}}{R - h} : \frac{W \times \sqrt{2\,rh - h^2}}{r - h}, \text{ or } = \frac{\sqrt{2\,Rh - h^2}}{R - h} : \frac{\sqrt{2\,rh - h^2}}{r - h},$$

or, as the sines of the angles DCA, dca, divided by the cosines of the same angles. And since the relation of sine to cosine is the same as that of tangent to radius, we may substitute the proportion $P : p = \frac{tang.\ DCA}{R} : \frac{tang.\ dca}{r}$. We may be assured that this statement is true, when we consider that the centre of gravity of the wheel, where we suppose the force P or p, to be applied, must describe the curve CNY or cny, and that it must commence its motion in the direction CO or co, Fig. 186_4, and that the greatest effort will be required when the centre is at C and c respectively. But the tangent CO, which expresses the length of the inclined plane which the weight begins to surmount, is of the same magnitude as the tangent DT of the angle DCA, and the successive planes will be more nearly coincident with the horizontal line. Having, on the foregoing principles,

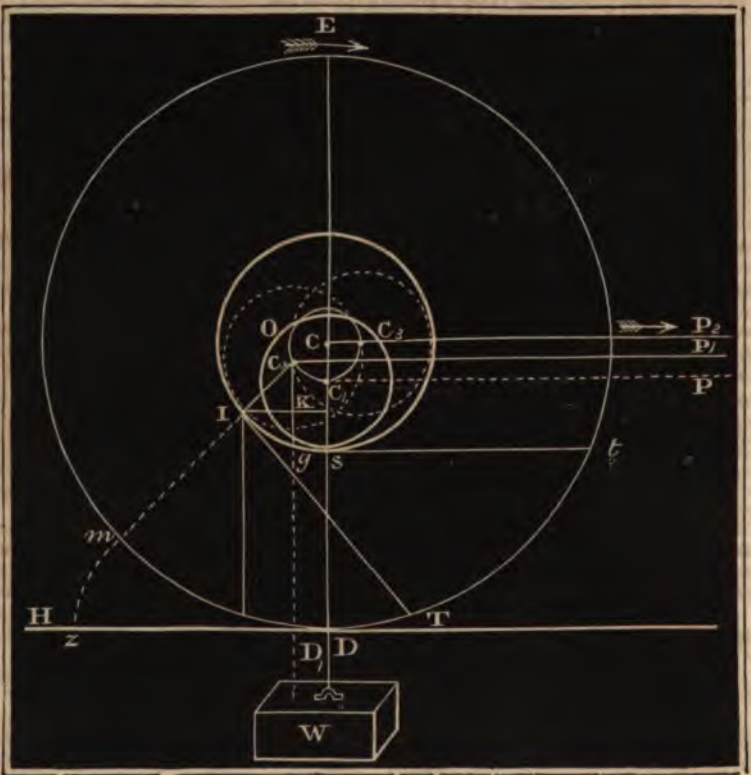
$$F = \frac{2 f\, Q\, sin. \frac{\alpha}{2}}{1 - f\, sin. \frac{\alpha}{2}}, \text{ which generally is } = 2 f\, Q\, sin. \frac{\alpha}{2}\left(1 + f\, sin. \frac{\alpha}{2}\right)$$

and indeed very often $= 2 f\, Q\, sin. \frac{\alpha}{2}$. In order to draw the cord

---

determined the effect of obstacles which oppose the rolling of a wheel, we may proceed to consider the influence of that resistance on the quantity of friction at the axle.

Fig. 186$_5$.



As carriage wheels are ordinarily constructed, and as roads, and especially railroads, are commonly made, the resistance of obstacles at the circumference is much more easily overcome than that of friction at the nave. Thus, in Fig. 186$_5$, where the wheel *mDTE* turns on its axle, *sO*, the line or spoke *sD* becomes the proper representative of a suspended lever, impelled at the upper end in the direction of *st* or C_1P by the resistance of friction at *s*, and at the lower by the resistance of opposing obstacles, (the amount of which resistance has just been stated,) in the direction *DH*. It generally happens, however, that, except on very rough roads, the resistance from the latter cause is less than that from the former. Hence, the wheel commences its forward motion sooner than the axle begins to slide in the box; so that its sliding motion is not (except in cases of great resistance at the periphery), in the direction of the tangent *st*, but in that of some other over the edge, a force $$P = Q + F = \left(1 + \frac{2 f \sin. \frac{a}{2}}{1 - f \sin. \frac{a}{2}}\right) Q$$ is necessary; and inversely, to prevent the descent of the weight Q by the cord, a

---

line, as IT. The axle, then, ought to be found bearing not on the bottom, but on a part *back of the lowest line of the cylinder*. Having been led to this conclusion from the theory now developed, the writer was induced to inquire of several wheelwrights, coach-makers, carriage-smiths, and keepers of livery stables, whether they had ever noticed the fact, or whether they supposed it to be true, that the axle *did* rest in its box elsewhere than on its lowest part; all, after a moment's reflection, answered, that as a force was applied to draw it forward, it must press and be most worn against its front and lower side; but upon examining the old axles in their possession, they have uniformly found the above views to be confirmed by evidence which they could not doubt. In moving a carriage, then, the animal exerts his strength to bring the axle into such a position that it will descend by the gravity of the load along an inclined plane as if to follow some direction, as IT. If the axle be smaller than the box, so as to leave considerable space between them, the centre may retreat from the vertical $C_1 D$, ascending at the same time from C_1 to C_2, where it exercises a gravitating force due to the weight, and in the direction $C_2 D_2$, acting of course on the arm of the lever equal in length to D_1 and D, and which would, if the force P were relaxed, cause the wheel to retreat and again depress the point m towards z, describing the portion mz of a cycloidal curve. This effect is often observed to take place. This position of the axle likewise accounts for the *retrograde rotation* of a wheel which is sometimes observed to take place through a portion of a revolution, when a heavily loaded car first passes from rough ground to smooth ice. The gravitating force, when the centre takes the position C_2, may be resolved into $C_1 I$, perpendicular to the side of the box, and the inclined tangent Ig; then the force which presses the surface causes the friction and opposes motion, is less than when it lies on the horizontal plane st, and the friction is diminished in proportion as $C_2 I$ is less than $C_2 g$. Again, as the force P *now* acts in the direction $C_2 P_1$, it tends to relieve even the remaining portion $C_2 I$, in the ratio of that line to $C_2 K$. When the force necessary to surmount the obstacle becomes infinite, the centre of the axle will take the position C_3, but this can happen only when the height of the obstacle is equal to the radius of the wheel. The tangent of the angle formed at the centre will then be infinite also, and the expression before given, viz. $P = \frac{W \text{ tang. } DCA}{R}$; will be as applicable to this extreme case as to any other where the height of obstacle is less.

The conclusions drawn from the foregoing remarks, are, that the friction of a roller, moving over a horizontal surface, depends on the relation between the velocity of the periphery and that of the centre of gravity; also, that this relation between the tangential velocity and that of transportation, will depend on *the moment of inertia of the cylinder*.

Again, the advantage of a wheel over a sledge, where the same materials are employed to slide over each other as those which compose the box and axle, is *as the radius of the wheel to that of the axle*.

If friction wheels be employed, the ratio just stated must be multiplied by the relation between their diameter and that of their axles.

The amount of friction at the axle to be overcome by the moving force will be proportional to the weight of the load, but will depend also on the obstacles which oppose the progressive rotation, and will attain its *maximum* when the height of the obstacle is equal to the radius of the wheel, at which moment the advantage of the wheel to surmount the obstacle is a *minimum* or *zero*.

The advantage of a wheel to overcome any obstacle of a given height when the plane over which the wheel moves, and the line of draught, are both horizontal, will be *as the tangent of the angle formed by a vertical drawn from the centre of the wheel downwards, and another line drawn from the same point to the top of the obstacle, divided by the radius of the wheel*.

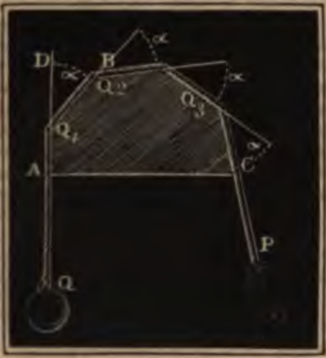
For a more full exposition of the views of the writer of this note on the friction of carriage wheels, see *Journal of the Franklin Institute*, vol. v. p. 57.—Am. Ed.

force $P_1 = Q \div \left(1 + \dfrac{2f\sin.\frac{\alpha}{2}}{1 - f\sin.\frac{\alpha}{2}}\right)$ is requisite; therefore approximately $P = \left[1 + 2f\sin.\frac{\alpha}{2}\left(1 + f\sin.\frac{\alpha}{2}\right)\right] Q$, or more simply, we may put $P = \left(1 + 2f\sin.\frac{\alpha}{2}\right) Q$, and

$$P_1 = \frac{Q}{1 + 2f\sin. = \frac{\alpha}{2}\left(1 + f\sin.\frac{\alpha}{2}\right)},$$ or more simply

$$P_1 = \frac{Q}{1 + 2f\sin.\frac{\alpha}{2}} = \left(1 - 2f\sin.\frac{\alpha}{2}\right) Q.$$

Fig. 188.



If the cord passes over several edges, the forces P and P_1 at the other extremity of the cord may be in like manner calculated by the repeated application of these formulæ. Let us take the simple case of a cord ABC, Fig. 188, passing over a body of n edges, and at each edge making the same small angle α. The tension of the first portion of the cord will be $Q_1 = \left(1 + 2f\sin.\frac{\alpha}{2}\right) Q$, that of the extremity be $= Q$, that of the second

$$Q_2 = \left(1 + 2f\sin.\frac{\alpha}{2}\right) Q_1$$

$= \left(1 + 2f\sin.\frac{\alpha}{2}\right)^2 Q$; that of the third $Q_3 = \left(1 + 2f\sin.\frac{\alpha}{2}\right) Q_2 = \left(1 + 2f\sin.\frac{\alpha}{2}\right)^3 Q$; therefore, the force at the remaining extremity $P = \left(1 + 2f\sin.\frac{\alpha}{2}\right)^n Q$, in so far as motion takes place in the direction of the force P. If we change P into Q, and Q into P, we obtain $P_1 = \dfrac{Q}{\left(1 + 2f\sin.\frac{\alpha}{2}\right)^n}$, provided only a motion in the direction of Q is to be prevented.

The friction $F = P - Q$ is in the first case $= \left[\left(1 + 2f\sin.\frac{\alpha}{2}\right)^n - 1\right] Q$, and in the second $= Q - P_1 = \left[\left(1 + 2f\sin.\frac{\alpha}{2}\right)^n - 1\right] P_1 = \left[1 - \left(1 + 2f\sin.\frac{\alpha}{2}\right)^{-n}\right] Q.$

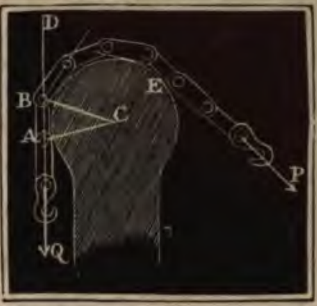
To obtain the expression $P = Q(1+2f\sin\frac{\alpha}{2})$

Vide Page opposite Page 170

No 176.

along a unit of length of cord
p = tension, $\frac{p}{r}$ = Normal pressure, $\frac{p\,ds}{r}$ =
= elementary Normal pressure ∴
$ds = r\,d\varphi$, $\frac{p\,r\,d\varphi}{r} = p\,d\varphi$ = Elementary Nor=
=mal pressure. $f\,p\,d\varphi = dp$, $f\,d\varphi = \frac{dp}{p}$
$f\varphi = \log P + C$, $\varphi = 0$ $\log P = \log Q$, $C = -\log Q$
$\varphi = \alpha$, $f\alpha = \log P - \log Q = \log \frac{P}{Q}$. Passing to
numbers $\frac{P}{Q} = \varepsilon^{f\alpha}$, ε being the base of
the Naperian or Hyperbolic sys=
=tem of logarithms.

Fig. 189.



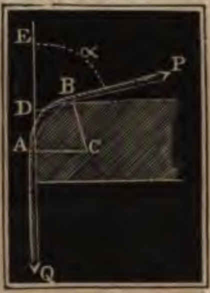
The same formulæ are applicable to a body winding round a cylinder, and consisting of members, as, for instance, a chain *ABE*, Fig. 189, where n is the number of links in contact, the length *AB* of a link $= l$, and the distance *CA* of the axis *A* of a link from the centre of the arc covered $= r$, we then have $sin. \frac{\alpha}{2} = \frac{l}{2r}$.

Example. What is the amount of friction at the circumference of a wheel 4 feet in diameter, if twenty links of a chain, five inches long and one inch thick, pass over it, one end of which is fixed, and the other stretched by a force of 50 lbs.? Here $P_1 = 50$ lbs. $n = 20$, $sin. \frac{\alpha}{2} = \frac{5}{48+1} = \frac{5}{49}$, let us now put for f the mean value 0,35, we then obtain the friction with which the chain acts against the wheel in its revolution:

$$F = \left[\left(1 + 2 \,.\, 035 \,.\, \frac{5}{49}\right)^{20} - 1\right] . \,50 = \left[\left(1 + \frac{35}{490}\right)^{20} - 1\right] . \,50$$

$$= \left[\left(\frac{15}{14}\right)^{20} - 1\right] . \,50 = 2{,}974.50 = 149 \text{ lbs.}$$

Fig. 190.



§ 176. A stretched cord *AB*, Fig. 190, lies about a fixed and cylindrically rounded body *ACB*, the friction may be likewise found from the rule of the former paragraph. Here the angle of deviation $EDB = \alpha^\circ =$ the angle *ACB* at the centre subtended by the arc of the cord *AB*; if we divide this into equal parts, and consider the arc *AB* as consisting of n straight lines, we have then n corners, each with a deviation of $\frac{a^\circ}{n}$, and consequently the equation between the power and weight, as in the former §:

$$P = \left(1 + 2 f sin. \frac{\alpha}{2n}\right)^n Q.$$

From the smallness of $\frac{\alpha}{2n}$ we may put the $sin. \frac{\alpha}{2n} = \frac{\alpha}{2n}$, whence $P = \left(1 + \frac{f\alpha}{n}\right)^n Q$. If further we make use of the binomial series, we obtain:

$$P = \left(1 + n\frac{f\alpha}{n} + \frac{n\,(n-1)}{1\,.\,2}\,\frac{(f\alpha)^2}{n^2} + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,\frac{(f\alpha)^3}{n^3} + \dots\right) Q,$$

but as n is very great, therefore $n-1 = n-2 = n-3 \dots = n$ it may be put:

$$P = \left(1 + f\alpha + \frac{1}{1\,.\,2}(f\alpha)^2 + \frac{1}{1\,.\,2\,.\,3} \cdot (f\alpha)^3 + \dots\right) Q.$$

But now $1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \dots = e^x$, where e denotes the base 2,71828 . . . of the hyperbolic system of logarithms, therefore, it may also be put:

$$P = e^{f\alpha} . Q, \text{ as also } Q = Pe^{-f\alpha}, \text{ lastly } \alpha = \frac{1}{f}\, hyp. Log. \frac{P}{Q}$$

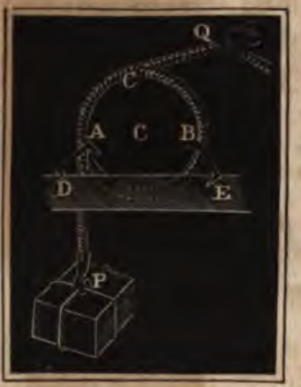
$$= \frac{2,3026}{f} (Log. P - Log. Q).$$

If the arc of the cord is not given in parts of π, but in degrees, we have then to substitute $\alpha = \frac{\alpha^0}{180^0} . \pi$; if lastly, it be expressed by the number of coils u, we have then to put $\alpha = 2 \pi u$.

The formula $P = e^{f\alpha} . Q$ expresses that the friction of the cord $F = P - Q$ upon a fixed cylinder is not dependent on the diameter of the same, but on the number of coils of the cord, and moreover shows that it may very easily be increased, almost to infinity. If we put $f = \frac{1}{3}$, we have:

For $\frac{1}{4}$ of a winding $P = 1,69\ Q$
" $\frac{1}{2}$ " $P = 2,85\ Q$
" 1 " $P = 8,12\ Q$
" 2 " $P = 65,94\ Q$
" 4 " $P = 4348,56\ Q$, &c.

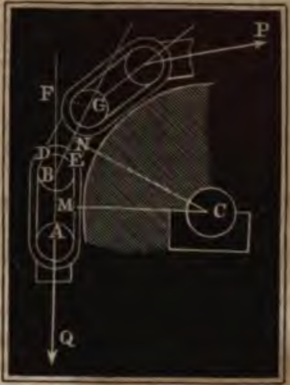
Fig. 191.



Example. To let down a shaft a load P of 1200 lbs. from a certain height, the rope to which this weight is attached is wrapped $1\frac{3}{8}$ times about a round firmly clamped holder AB, Fig. 191, and the other extremity of the rope is held by the hand. With what force must this extremity be stretched, that the load may slowly and uniformly descend? If we put $f = 0,3$ we obtain this power $Q = Pe^{-f\alpha}$ $= 1200 . e^{-0,3 . \frac{11}{8} . 2\pi} = 1200 . e^{-\frac{33}{40}\pi}$, therefore, $hyp. Log. Q = hyp. Log. 1200 - \frac{33}{40}\pi = 7,0901 - 2,5918 = 4,4983 . Log. Q = 1,9536, Q = 89,9$ lbs.

§ 177. *Rigidity of Chains.*—If ropes, or other similar bodies, &c., are placed over a pulley, or on the circumference of other cylinders revolving about an axis, the cord or chain friction considered in the foregoing paragraph ceases, because the circumference of the wheel has the same velocity as the rope; but now the force of bending by the winding of the rope about the pulley, and also that of unbending by the unwinding, becomes perceptible. If it is a chain which winds round a drum, there arises the resistance of the winding and unwinding manifested in a friction of the chain pins, while these last are revolving through a certain angle. If AB, Fig. 192, is one link, and BG the one lying next, if, further, C is the axis of revolution of the wheel on which the chain stretched by the weight Q winds itself, if, lastly, CM and CN are let fall perpendicularly to the longer axes of the links AB and BG, $MCN = \alpha^\circ$ is the angle through which

Fig. 192.



the wheel revolves whilst a fresh link is laid on, $FBG = 180^\circ - ABE$ is the angle by which the link BG with its bolt BD revolves about the link AB. If now $BD = BE = r_1$ is the radius of the bolt, the point of friction or pressure D describes an arc $DE = r_1\alpha$, and the mechanical effect of friction $f_1 Q$ hereby produced at the point B is $f_1 Q \,.\, r_1\alpha$. The force P_1 expended in overcoming this friction, acting in the direction of the longer axis BG, describes the simultaneous space $s = CN$ times the arc of the angle $MCN = CN \,.\, \alpha$, and the mechanical effect $= P_1 \,.\, CN \,.\, \alpha$; by equating both labors we have $P_1 \,.\, CN \,.\, \alpha = f_1 \,.\, Qr_1\alpha$, and the required force, if a represent the radius of the drum CN increased by half the thickness of the chain: $P_1 = f_1 Q \,.\, \frac{r_1}{a}$.

Without regard to friction, the force for a revolution of the wheel would be $P = Q$, having regard to the friction in the winding up of the chain $P = Q + P_1 = \left(1 + f_1 \frac{r_1}{a}\right) Q$. If the chain unwinds itself from the drum, an equal resistance takes place; if, therefore, a winding on one side, and an unwinding on the other take place, the force $P = \left(1 + f_1 \frac{r_1}{a}\right)^2 Q$, or approximately:

$$= \left(1 + 2f_1 \frac{r_1}{a}\right) Q.$$

Lastly, if the pressure on the axle $= R$, and its radius $= r$, it follows that the force, taking into account all resistances, is:

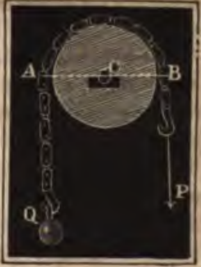
$$P = \left(1 + 2f_1 \frac{r_1}{a}\right) Q + f \frac{r}{a} R.$$

Example. What is the magnitude of a force P at the extremity of a chain passing over a pulley ACB, Fig. 193, if the weight Q drawing vertically downwards $= 110$ lbs., the weight of the pulley with the chain 50 lbs., the radius of the pulley measured to the middle of the chain $= 7$ in., that of the axle C $\frac{5}{8}$ inch, and that the chain bolts $= \frac{3}{8}$ in.? The co-efficients of friction $f = 0{,}075$ and $f_1 = 0{,}15$, therefore from the last formula we obtain the force:

$$P = \left(1 + 2 \,.\, 0{,}15 \,.\, \frac{3}{8 \,.\, 7}\right) . \, 110 + 0{,}075 \,.\, \frac{5}{8 \,.\, 7}(110 + 50 + P),$$

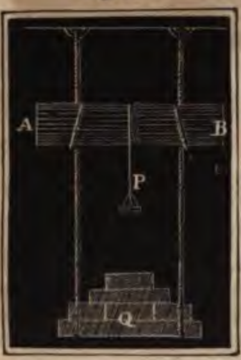
or, if we assume P on the right hand nearly $= 110$ lbs. $P = 1{,}016 \,.\, 110 + 0{,}0067 \,.\, 270 = 111{,}76 + 1{,}81 + 113{,}6$ lbs.

Fig. 193.



§ 178. *Rigidity of Cords.*—In bending a cord over a pulley or wheel, rigidity comes in as a resistance opposed to motion. The same takes place, but in a far less degree, in the unrolling from cylinders. Amontons and Coulomb set about measuring the amount of this resistance by experiment. The results obtained by them are by no means satisfactory; partly because they are not in sufficient accordance with each other, and partly because they have not that extension so desirable for practical application. The experiments of Coulomb, which are those only of which we shall speak, were mostly made with hempen cords, of $\frac{1}{4}$ to $\frac{3}{4}$ inch thick, and with pulleys of from 1 to 4 inches diameter. Other experiments must be made before we can know what is the resistance of rigidity of a hempen rope of from 2 to 3 inches thick, when wrapped round a drum of from 1 to 6 feet in height; and also what is the amount of this resistance in the case of the wire-ropes, now come generally into use.*

Fig. 194.



Coulomb made his experiments in two ways; at one time with the apparatus of Amontons, Fig. 194, where AB is a roller, with two cords winding round it, the tension is effected by a weight Q, and the rolling down of the cylinder by a second one P, which pulls, by means of a thin string at this roller; at another time, with a cylinder, which was allowed to roll upon a horizontal line, and round which a cord was wound, and from the difference of the weights suspended at both extremities, which effected a slow rolling forward, and after abstraction of the rolling friction, the resistance of the rigidity was deduced.

It results from the experiments of Coulomb, that the rigidity increases equally with the tension of the winding cord; that it consists, moreover, of a constant part K, which is no more than might be expected, because a certain force is necessary to bend an unstretched cord. It also appears that this resistance increases inversely as the diameter of the pulley; that it is, therefore, with twice the diameter of the pulley, only half as great; with three times the diameter, one-third, &c. The relation between the thickness and the rigidity of the cord is only approximately given from these experiments, since the rigidity depends upon the quality of the materials, the twisting of the strings, &c. For new ropes, the rigidity was found proportional to the power $d^{1,7}$, for old $d^{1,4}$, d being the diameter of the rope. It is, therefore, only an approximation, when some assume that this resistance increases proportionally with the thickness, others with the square of the thickness of the rope.

§ 179. The rigidity of cords may be therefore expressed by the formulæ:

$S = \frac{d^n}{a} (K + \nu\, Q)$, where d is the thickness of the cord, a the radius of the pulley measured to the axis of the cord, n, K and ν, numbers from experiment. *Prony* deduced from *Coulomb's* experiments that for new cords

---

* See Appendix.

$$S = \frac{d^{1,7}}{a}(2,45 + 0,053\ Q),\text{ and for old}$$

$$S_1 = \frac{d^{1,4}}{a}(2,45 + 0,053\ Q),$$ a and d being expressed in lines, Q and S in pounds. These expressions refer to the Paris measure; expressed in Prussian inches and pounds, they become,

$$S = \frac{d^{1,7}}{a}(14,23 + 0,295\ Q)\text{ and }S_1 = \frac{d^{1,4}}{a}(6,83 + 0,141\ Q).$$

As these complicated formulæ do not always give the results in accordance with experiment, we may, until other experiments supersede them, put with *Eytelwein*

$$S = \nu\,.\,\frac{d^2}{a}Q = \frac{d^2\,Q}{3500\,a},$$ provided that a be expressed in Prussian feet, and d in Prussian lines, Q and S in the same weight, which, however, may be arbitrary. For the metrical standard $S = 18,6\,.\,\frac{d^2\,Q}{a}$. This formula, as might be expected, will give satisfactory approximative results only for great tensions, as they generally occur in practice.

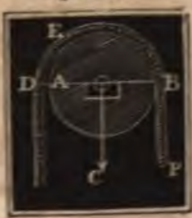
The rigidity of tarred ropes is found to be about $\frac{1}{6}$th greater than that of untarred; for wetted ropes, however, there is no determinate relation of this kind.

Example. With a tension of 350 lbs., and a radius of the pulley of $2\frac{1}{2}$ inches, the rigidity of a new rope of 9 lines $= 0,78$ (English) inches, according to *Prony*, is: $S = \frac{2}{5}\,.\,(\frac{3}{4})^{1,7}\,.\,14,23 + 0,295\,.\,350) = 0,613\,.\,47,0 = 28,8$ lbs.; (according to *Eytelwein*) $S = \frac{9^2\,.\,350\,.\,24}{3500\,.\,5} = 38,9$ lbs. Were the tension Q only 150 lbs., we should have from *Prony*, $S = 0,613\,.\,23,4 = 14,34$ lbs.; from *Eytelwein* : $= \frac{81\,.\,24\,.\,3}{350} = 16,7$ lbs., therefore, here a better accordance. We see from these examples, how little reliance is to be placed on the formula.

Remark. A farther extension of this subject, viz. in respect to the rigidity of wire ropes, will be given under the article, windlass and capstan.

§ 180. Let us now apply the formula given for the rigidity of cords, to the theory of pulleys. The radius CA of a fixed pulley $= a$, Fig. 195, the radius of the axle $= r$, the thickness of rope $= d$, the weight Q at one extremity of the cord, (whose weight $= G$,) and the power which must be applied to the other extremity to draw it slowly up $= P$. Without friction on the axle, and without rigidity, P would be $= Q$, but because the axle exerts a pressure $P + Q + G$ against its bearing, there arises a friction $f\,(P + Q + G)$ which, since it acts at the radius r, makes an increase of power $\frac{f\,r}{a}(P + Q + G)$ necessary; since the rigidity of the rope must be added to this, which manifests

Fig. 195.



itself in this, that the cord does not at once take the curvature of the circumference of the pulley, but lays itself upon the pulley with an increasing curvature, and in this manner causes an extension of the arm of Q; the arm, therefore, of the weight Q is not CA but CD, and the force at the arm CB

$$= CA = a, P = \frac{CD}{CA} \cdot Q = \left(1 + \frac{AD}{CA}\right) Q = Q + S = Q + \frac{d^n}{a} (K + \nu\, Q).$$

The complete equation between the power and the weight is now

$$P = Q + \frac{d^n}{a} (K + \nu\, Q) + \frac{f\,r}{a} (P + Q + G).$$

In the wheel and axle the power P acts at a different arm a to that of the weight, whose arm $= b$, therefore,

$$Pa = Qb + d^n (K + \nu\, Q) + fr\, (P + Q + G), \text{ and}$$

$$\left(P = \frac{b}{a} Q + \frac{d^n}{n} (K + \nu\, Q) + \frac{f\,r}{a} (P + Q + G).\right)$$

Hence the force

$$P = \frac{(b + \nu\, d^n + fr)\; Q + d^n \,.\, K + fr\; G}{a - fr}.$$

Example.—A weight $Q = 200$ lbs. is to be raised with the wheel and axle by a power $P = 50$ lbs.; suppose the wheel to be 1½ feet, and the pivot ½ inch radius, and the rope applied ½ an inch thick, and the weight of the whole machine 70 lbs., what radius must we give to the axle? It must be:

$b = [Pa - d^n\, (K + \nu\, Q) - fr\, (P + Q + G)] \div Q,$

therefore, in numbers if we put $f = 0{,}075$,

$b = [50 \,.\, 18 - (\tfrac{1}{2})^{1,7} \,.\, (14{,}23 + 0{,}295 \,.\, 200) - 0{,}075 \,.\, \tfrac{1}{2} \,.\, 320] \div 200$

$= [900 - 0{,}308 \,.\, 73{,}23 - 12] \div 200 = 865{,}4 \div 200 = 4{,}327$ inches.

Without additional resistances b would be

$= Pa \div Q = 75 \div 200 = 0{,}375$ feet $= 4\tfrac{1}{2}$ inches.

CHAPTER VI.

ELASTICITY AND RIGIDITY.

§ 181. *Elasticity.*—The parts of a rigid body adhere to each other with a certain force, which is called *cohesion*, and which must be overcome when bodies are changed in their figure and extension, or broken. The first effect which forces produce in a body, is a change in the position of their parts relatively to each other, and a resulting change of form or volume of the body. If the forces acting upon a body exceed certain limits, a separation of the parts, and a breaking of the whole body ultimately take place. The capability of bodies, which suffer a change of form by the action of forces, to resume perfectly their former state after the withdrawal of the forces, is called *elasticity*. The elasticity of every body has a certain limit. If the change of form or volume exceeds a certain amount, the body retains an alteration of its volume, even when the forces which have effected it cease to act. The limit of elasticity is different for different bodies.

The Modulus of Elasticity (E) is the force which is necessary to elongate the body its length, which has a transverse section of unity, or to compress it that amount. Properly, as is evident, there is no real limit of elasticity; the depressions or elongations caused by the action of forces are directly proportional to the ~~force~~ forces themselves.

Let a heavy body, a beam for instance be suspended and acted upon by a force P, besides which there is the weight W of the beam which must not be neglected for it may many times be greater than the force applied, Wherefore the total force acting upon the beam is $P_1 = P + \frac{W}{2}$, the truth of which equation we will now proceed to prove. Let l = length of the body, and x the distance of the section F from the point of suspension, it is evident that only the weight of that portion of the body below the section can produce a strain upon the section, wherefore we now only consider that portion of the body lying below the plane, whose weight equals $F(l-x)\gamma$. hence

the small length dx will be elong'd with a elongation dλ ∴ our equation

$$\frac{P+F(l-x)\gamma}{FE} = \frac{d\lambda}{dx} \therefore d\lambda FE = [P+F(l-x)\gamma]dx$$

$$\int d\lambda FE = \int_0^l [P+F(l-x)\gamma]dx \therefore \lambda FE = Pl + \frac{Fl^2\gamma}{2} \therefore \frac{\lambda}{l}FE = P + \frac{Fl\gamma}{2}$$

But $\frac{\lambda}{l}FE$ = total force acting. $Fl\gamma$ = weight

Hence total force acting $= P + \frac{W}{2}$.

Bodies which suffer a considerable change of form before this limit is attained, are called *perfectly elastic*. Those, on the other hand, in which there is scarcely any appreciable change of form preceding the limit, are called *inelastic*, although in reality there exist no bodies of this kind.

It is an important rule in building and in machinery never to load the materials to such an extent, that any alteration of their form should attain, much less exceed, the limits of elasticity.

§ 182. *Elasticity and Strength.*—Different bodies present different phenomena when their form is changed beyond the limits of elasticity. If a body be brittle, it flies into pieces. If it be ductile, as many of the metals, it will admit of alterations of form beyond the limits of elasticity, without suffering a separation of its parts. Many bodies are hard, others soft; the one opposes a great resistance to a separation of their parts, whilst the others easily allow of this to be brought about.*

In the restricted sense of the word, we understand by ***elasticity***, the resistances which a body opposes to a change of form; on the other hand, by ***strength***, the resistance which a body opposes to a separation of its parts. We will accordingly, in what follows, consider each of these separately.

According to the way in which external forces act upon a body, and change their form and dimensions, we distinguish the elasticity and strength of bodies, into:

1. The *absolute resistance*, Resistance to extension
2. The *relative resistance*, Resistance to trans strain.
3. The *resistance to compression*, and
4. The *resistance to torsion*.

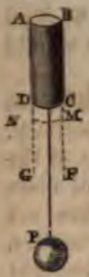
If two external forces act by tension in the direction of the axis of a body, it resists by its *absolute elasticity* and *strength* any extension or rupture. If, on the other hand, these forces act at right angles to the axis of a body, the body will resist by its relative elasticity and strength any bending or fracture. If, further, two forces act in the direction of the axis of a body by compression, so that the body becomes either compressed or crushed, then there is the *elasticity* and *strength of compression* to be overcome. If, lastly, forces strive to turn a body in opposite directions about an axis, or which do not act in the same plane normal to the axis, then there is the *elasticity* and *strength of torsion* to be overcome.

§ 183. *Modulus of Elasticity.*—The change of volume within the limits of elasticity, *i. e.* the extension or compression of a body, is pretty nearly proportional to the force exerted, but if this change exceeds that limit, this proportionality ceases, and the change goes on rapidly to that of rupture or crushing. As a measure of the elasticity, the *modulus of elasticity E*, is that which expresses the force which is necessary to elongate a prismatic body of a transverse section, unity = *i. e.* a *square foot*, to double, ~~or to compress it to one-half of~~ its original length. A different modulus corresponds to different mate- [illegible] rials: for each substance it must be determined by experiment. For the rest, we must bear in mind that the modulus of elasticity only holds good for extensions and compressions within the limit of elasticity, and its measure is one, not of observation, but of hypothesis and calculation, because it is not easy to find a body which, without exceeding the limit of elasticity, allows so great a change of form as the modulus of elasticity supposes.*

---

* See Appendix.

Fig. 196.



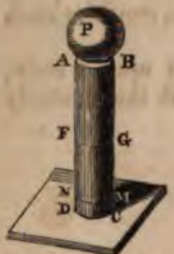
A body AC, Fig. 196, which has the initial length $AD = BC = l$, and the transverse section 1, requires for its extension $DG = l$, the force E, if, however, its transverse section is F, that is, if it consists of F contiguous prisms, this force is then $F \cdot E$. If, on the other hand, this body is to be extended a length $DN = CM = \lambda$, then for the force P

$P : F \cdot E = \lambda : l$, it therefore follows

1. That $P = \frac{\lambda}{l} F \cdot E$, and inversely, 2. $\lambda = \frac{P}{F \cdot E} \cdot l$.

The same formulæ are also applicable to a body AC, Fig. 197, of the length $AD = l$, and the transverse section $AB = F$, if it become shortened a length λ by the compression of a force P.

Fig. 197.



By the aid of these formulæ we may calculate from the change of volume (λ) the corresponding force P, or from the force P the quantity of the extension or compression.

Example. If the modulus of elasticity of brass wire amounts to 14625000 lbs., what force is necessary to stretch $\frac{1}{12}$ inch a wire 5 feet in length and $\frac{1}{6}$ inch in thickness? $l = 5 \cdot 12 = 60$ inches, $\lambda = \frac{1}{12}$ inch consequently $\frac{\lambda}{l} = \frac{1}{720}$; further $F = \frac{\pi d^2}{4} = 0{,}7854 \cdot \left(\frac{1}{6}\right)^2 = 0{,}0218$ square inches, the required force accordingly is $P = \frac{1}{720} \cdot 0{,}0218 \cdot 14625000 = 442$ lbs. — 2. The modulus of elasticity of iron wire is 263250000 lbs.; if an iron chain, 60 feet long and 0,2 inches thick, be stretched by a force of 150 lbs., the same will be increased by a length $\lambda = \frac{150 \cdot}{0{,}7854 \cdot (0{,}2)^2} \cdot \frac{60 \cdot 12}{263250000} = \frac{108000}{31416 \cdot 263{,}25} = 0{,}013$ inches $= 0{,}156$ lines.

§ 184. *Modulus of Working Load and Strength.*—The force T, which a body of the transverse section unity accumulates when its extension attains the limit of elasticity, is easily determined from the modulus of elasticity E and the elongation λ corresponding to this limit, for $T : E = \lambda : 1$, therefore, $T = \lambda E$. This is the strain beyond which materials used in construction and machinery must not be loaded if they are to maintain sufficient safety together with durability. If the transverse section of a body, which has to sustain a tensile strain P be $= F$, we have then

$$1.\ P = FT, \text{ and } 2.\ F = \frac{P}{T}.$$

The force T by which we judge of the working load of bodies, may be introduced into calculations under the name of *modulus of working load.*

---

* See Appendix.

[illegible]

[illegible]

If the section of the body be a circle of the diameter [illegible], we have [illegible]

[illegible]

and hence, both the [illegible] and the modulus of [illegible] the body will not be [illegible] beyond the limit of elasticity.

[illegible]

[illegible]

[illegible]

Strongest form of Body

If we have a prismatic body with a transverse section at the point of suspension large enough to bear the weight of the body + be suspended; therefore the body is too large at every other section consequently there is lost a large amount of material ~~unnecessary weight~~ If on the other hand we have a body with a transverse section taken at any point only large enough to support that portion of the weight of the body lying below it then it is evident that we do not lose any material and no unnecessary weight is in the beam. It is therefore of advantage to find the strongest form of body that is a body whose section at any point shall just be equal to the support of that portion of the body whose weight exercises a strain upon it. Let I_n be the section at the point of suspension I_1 for section at point of suspension of weight P. I = section at distance x from I_1 or lower point of beam. $I\,dx\gamma$ = weight of elementary portion. $W = \int I\,dx\gamma$ ∴ $P_1 = P + \int I\,dx\gamma = TI$. Differentiating $I\,dx\gamma = T\,dI$ ∴ $\frac{dx\gamma}{T} = \frac{dI}{I}$ ∴ $\frac{x\gamma}{T} = \log I + C$. To determine the constant

The *modulus of strength* K, which expresses the force by which a body of the transverse section unity becomes ruptured, is entirely different from this modulus. If the transverse section of a prismatic body, or its least section $= F$, it follows that the force, for the rupture of this body, is:

$$1.\ P_1 = FK, \text{ and inversely, } 2.\ F = \frac{P_1}{K}.$$

Generally the strength of materials of construction and parts of machines are calculated by the co-efficient K, which is divided for security's sake, by one of the numbers 3, 4 to 10. This makes little difference in the result, as we may see from a comparison of the values found in the succeeding table, but the supposition is incorrect, or to be justified only in so far as the modulus of strength is from 3, 4 to 10 times that of the modulus of tenacity, or generally bears a constant relation to it.

If the section of the body be a circle of the diameter d, we have therefore,

$$\frac{\pi d^2}{4} = F, \text{ so that } d = \sqrt{\frac{4\,F}{\pi}} = 1{,}128\sqrt{F} = 1{,}128\sqrt{\frac{P}{T}}, ?$$

and hence, from the load or strain P on a body, and the modulus of tenacity T of its material, the ~~strength~~ section may be found, for which the body will not be strained beyond the limit of elasticity.

Example. What load will a column of fir sustain, if it be 5 inches in breadth and 4 inches in thickness? The modulus of tenacity being taken at 3000 lbs. and the section F being $= 5 \,.\, 4 = 20$ square inches, we obtain $P = 20 \,.\, 3000 = 60000$ lbs. for the power of tenacity of this column. But if we take the modulus of strength $K = 12000$ lbs., and assume a triple security, we obtain $P = 20 \,.\, \frac{12000}{3} = 80000$ lbs.; but to maintain security for a long period, we must only take one-tenth of K, and we shall then have $P = 20 \,.\, 1200 = 24000$ lbs.—2. A round and wrought-iron pump-rod is to sustain a weight of 4500 lbs.; what diameter ought it to have? Here $T = 20000$ lbs., therefore, $d = 1{,}128\sqrt{\frac{4500}{20000}} = 1{,}128 \,.\, \sqrt{\frac{9}{40}} = 0{,}535$ ~~feet~~ inches. The modulus of strength for wrought iron of the medium kind $= 58000$ lbs., and if we take one-sixth for the security, we then obtain $K = 10000$ lbs., and $d = 1{,}128\sqrt{\frac{45000}{10000}} = 0.756$ inch, the requisite thickness of the rod.

§ 185. *Strongest Form of Body.*—If a vertically suspended prismatic body, for example, a pole or cord, is very long, its weight G must be added to the force of rupture, and, therefore, $P + G$ must be put $= FT$. If now l be the length of the body, and γ the weight of a cubic inch of its mass, we have then $G = Fl\gamma$, and, therefore, $P = F(T - l\gamma)$, as inversely $F = \frac{P}{T - l\gamma}$. If a body $ABC \,.\,.\, G$, Fig. 198, consists of equal portions, each of the length l, its successive transverse sections are as follows. The section of the first portion is as before $F_1 = \frac{P}{T - l\gamma}$. For the second portion, whose section is F_2

and weight $l\gamma$, $P + F_1 l\gamma + F_2 l\gamma = F_2 T$, hence $F_2 = \frac{P + F_1 l\gamma}{T - l\gamma} = F_1 + \frac{F_1 l\gamma}{T - l\gamma} = F_1 \left(1 + \frac{l\gamma}{T - l\gamma}\right)$. For the third portion it follows that $F_3 = F_2 \left(1 + \frac{l\gamma}{T - l\gamma}\right) = F_1 \left(1 + \frac{l\gamma}{T - l\gamma}\right)^2$, for the fourth $F_4 = F_3 \left(1 + \frac{l\gamma}{T - l\gamma}\right) = F_1 \left(1 + \frac{l\gamma}{T - l\gamma}\right)^3$, and generally for the nth portion; $F_n = F_1 . \left(1 + \frac{l\gamma}{T - l\gamma}\right)^{n-1}$ or $F_n = \frac{P}{T - l\gamma}\left(1 + \frac{l\gamma}{T - l\gamma}\right)^{n-1}$, the corresponding section.

If l is very small, the portions therefore very short, we may then put:

$$F_n = \frac{P}{T}\left(1 + \frac{l\gamma}{T}\right)^{n-1}.$$

If the number of portions is very great, or if the thickness of the body AG, Fig. 199, increases uniformly from below upwards, we may then (from the reasons in § 175,) put the cross section

$$F_n = \frac{P}{T} . e^{\frac{(n-1) l\gamma}{T}} = \frac{P}{T} . e^{\frac{n l\gamma}{T}} = \frac{P}{T} . e^{\frac{L\gamma}{T}}$$

where e represents the base 2,71828 . . . of the Naperien logarithms, and L the entire length of the body.

Fig. 198.

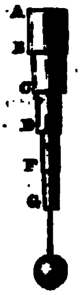
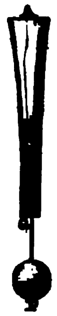


Fig. 199.



A body of uniform thickness to have the same tenacity throughout, must have a transverse section $F = \frac{P}{T - L\gamma}$. If $L\gamma$ is small as compared with T, $\frac{L\gamma}{T}$ is a small fraction, so that we may put:

$$F_n = \frac{P}{T}\left[1 + \frac{L\gamma}{T} + \frac{1}{2}\left(\frac{L\gamma}{T}\right)^2\right] \text{ and}$$

$$F = \frac{P}{T}\left[1 + \frac{L\gamma}{T} + \left(\frac{L\gamma}{T}\right)^2\right],$$

further, the weight of the first body is

$$= \frac{F_1 + F_n}{2} . L\gamma = \left[1 + \frac{1}{2}\frac{L\gamma}{T} + \frac{1}{4}\left(\frac{L\gamma}{T}\right)^2\right]\frac{P}{T} L\gamma;$$

and that of the second $= F . L\gamma$

$$= \left[1 + \frac{L\gamma}{T} + \left(\frac{L\gamma}{T}\right)\right]\frac{P}{T} L\gamma;$$

hence the prismatic body is heavier, and on that account more costly than one having at each point in its length a cross section corresponding to the load it has to bear, and which may therefore be called a body of uniform resistance, or *a body of the strongest form*.

Examples.—1. What cross section ought a wrought iron shaft 100 feet long to have, when besides its own weight it has to sustain a load $P = 75000$ lbs.? The modulus of tenacity or strain is taken at $T = \frac{1}{6} K = 10311$ lbs., and the weight of a cubic inch of wrought iron $\gamma = \frac{7,60 . 62,4}{12 . 12 . 12} = 0,27444$ lbs. The section sought is $F = \frac{P}{T - L\gamma} =$

put $x=0$, Then $F=F_1$, but $P=F_1T$ hence $F_1=\frac{P}{T}$ and $F=\frac{P}{T}$, $0=\log\frac{P}{T}+C$ ∴ $C=-\log\frac{P}{T}$ $\frac{xy}{T}=\log F-\log\frac{P}{T}$ passing to the limit $x=0$ and $x=l$ we have $\frac{ly}{T}=\log F_n-\log\frac{P}{T}$ $=\log\frac{F_n}{\frac{P}{T}}$ ∴ $\frac{F_n}{\frac{P}{T}}=e^{\frac{ly}{T}}$ hence $F_n=\frac{P}{T}e^{\frac{ly}{T}}$. Here the value of the (or) section has been determined, as F_n is a function of x, $fx=\frac{P}{T}e^{\frac{ly}{T}}$, wherefore the strongest form of beam is bounded by the logarithmic curve. To obtain the weight. $W=\int F dx\,y$. But any section $F_x=\frac{P}{T}e^{\frac{xy}{T}}$, now F is an arbitrary section hence $W=\frac{P}{T}\int e^{\frac{xy}{T}}dx\,y$. Let us now integrate the above expression: it is known that $de^x=e^x dx\,le$, $de=$ ∴ $de^{\frac{xy}{T}}=e^{\frac{xy}{T}}dx\frac{y}{T}$. hence $\int e^{\frac{xy}{T}}dx\,y=\int T\,de^{\frac{xy}{T}}$ $\int e^{\frac{xy}{T}}dx\,y=Te^{\frac{xy}{T}}+C$, To determine C, put $x=0$ then $W=0$ hence $Pe^0+C=0$ then $C=-P$, ∴ $W=Pe^{\frac{ly}{T}}-P=P(e^{\frac{xy}{T}}-1)=$ Expression for the Weight.

$\frac{75000}{10311 - 1200 . 0{,}27444} = 7.51$ square inches, and the weight of the shaft $G = F . L\gamma = 7{,}51 . 1200 . 0{,}27444 = 2473$ lbs.—2. If we were to give to this shaft the form of a body of uniform resistance, we should then obtain for the least section $F = \frac{P}{T} = \frac{75000}{10311} = 7{,}28$ square inches; for the greatest section $F_n = 7{,}28 . e^{0{,}27444 . 0{,}116} = 7{,}28 . e^{0{,}03183} = 7.513$ square inches, and the weight $= \left(\frac{7{,}28 + 7{,}513}{2}\right) . 329{,}3 = 2435{,}5$ lbs. (approximately).

§ 186. *Numerical Values.*—In the following table are given the mean values of the different moduli, of elasticity, tenacity, and strength of the materials most commonly occurring in construction.

TABLE I.

THE MODULI OF ELASTICITY AND STRENGTH.

NAMES OF THE SUBSTANCES.	Extension at the limits of the elasticity. $\frac{\lambda}{l}$.	Modulus of elasticity. E.	Modulus of working load. T.	Modulus of strength. K.	Modulus of safety. K_1.
Box, oak, fir, firm Scotch fir -	$\frac{1}{600}$	1856005	3094	12373	1237
Iron in wires - - - - - -	$\frac{1}{1250}$	26808964	21650	87645	14436
Iron in bars - - - - - -	$\frac{1}{1520}$	29902306	20622	59805	10311
Iron in plates - - - - - -		26808969		56712	9280
Cast iron - - - - - - -	$\frac{1}{1200}$	17528938	14436	19592	3094
Steel - - - - - - - - -	$\frac{1}{835}$	30933420	37120	123700	20622
Hard cast steel - - - - -	$\frac{1}{4500}$	45369016	98987	150543	24740
Copper - - - - - - - -				38151	6187
Copper wire - - - - - -				75271	12370
Brass - - - - - - - - -	$\frac{1}{1320}$	97955830	7218	18560	3093
Brass wire - - - - - -	$\frac{1}{742}$	149511530	20622	75271	12370
Bell metal - - - - - - -	$\frac{1}{1590}$	48462358	3093	35058	5774
Lead - - - - - - - - -	$\frac{1}{477}$	721779	1547	928	329
Leaden wire - - - - - -	$\frac{1}{1500}$	1031114	722	2062	351
Marble - - - - - - - -		2680896		2062	206
Ropes under 1 inch - - -				9280	3093
" 1 — 3 inches - - -				7218	2371
" above 3 " - - - -				5156	1753
Straps - - - - - - - -					299

The values contained in the second vertical column of this table, of the relative extension $\left(\frac{\lambda}{l}\right)$ at the limits of elasticity, give likewise the relation $\frac{T}{E}$ of the values of the fourth and third columns. The sixth column is derived from the fifth, if we divide the woods by 10, the metals by 6, and the cords by 3. The strength of wires is always greater than that of rods, because the enveloping crust of wires is stronger than their nucleus.

§ 187. *Flexure of Bodies.*—A prismatic body *ABCD*, Fig. 200, is fixed at one extremity, for instance, imbedded in a wall, and at the other extremity acted upon by a force *P*; strains then take place in this body, in consequence of which, one part is extended, and the other compressed, and the whole becomes deflected. If we imagine the whole body to be decomposed into thin laminæ by planes parallel to the axis, and at right angles to the direction of force, we may then assume that there is a certain mean lamina *KLM*, which is called the *neutral surface* or the *neutral axis* of the laminæ, which is not strained by this flexure, and remains unaltered in length, while the laminæ on the convex side undergo an extension, and those on the concave side a compression. Let $A\,B\,C_1\,D_1$, Fig. 201, be the longitudinal section of the body, *KL* its neutral axis, NO_1 an extended and UV_1 a shortened or compressed lamina. If the flexure had taken place without any change of volume, *KL* would be = $AD = NO$, &c.; *i. e.* the length of all the lamina would be one and the same; the body also would have the form *ABCD*, but because the body has sustained extensions and compressions, certain laminæ, such as *AD*, *NO*, &c., have undergone the elongations DD_1, OO_1, &c., and others, as *BC* and *UV*, the compressions CC_1, VV_1, &c., and the form of the body has changed to that of ABC_1D_1. In every case the elongations DD_1, OO_1, and the compressions CC_1, VV_1, &c., are proportional to the distances *LD*, *LO*, *LC*, *LV*, &c., from the neutral axis. But the strains in the direction of the laminæ are in the ratio of the elongations and compressions effected by them; we must, therefore, assume that these strains are proportional to the distances from

Fig. 200.

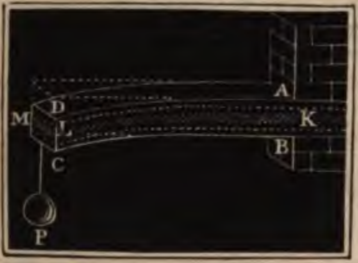
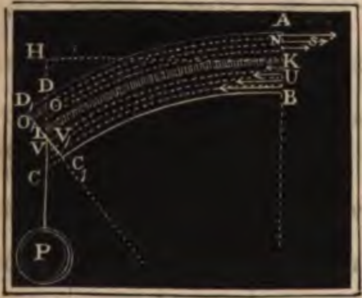


Fig. 201.

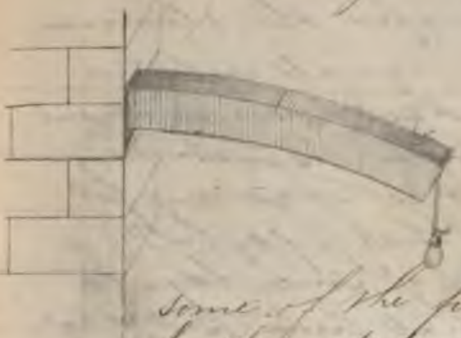


---

* See Appendix.

Flexure of bodies

If a body be supported at one end, for instance built into a wall, and deflected by a weight attached at the other extremity, then



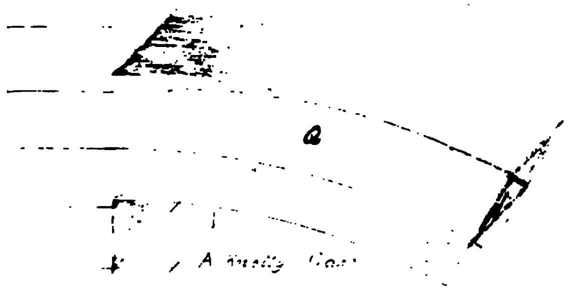
some of the fibres of the body will be elongated and some will be compressed. Therefore it is evident that there will be a lamina where these two strains meet and whose fibres will be neither elongated nor compressed. This is called the neutral plane, or lamina of the body. Let S be the strain on fibre or collection of fibres of unit section and at unit distance from the neutral plane. Let l = length of beam and neutral axis. Let x = distance of a section from the ~~neutral axis~~ plane of suspension, let y be the distance in that plane of any element from the neutral axis. Let z = breadth of section. The strains are proportional to their distances from the neutral axis. Therefore $Sydydz$ = strain on element, Sy^2dydz = moment of strain on eleme

$\int S y^2 dy\, dz$ = whole strain = $\int S y^2 dy\, b$ because the breadth is constant $z = b$ therefore the elementary portion equals $dy\,b$ instead of $dy\,dz$. $P(l-x)$ = moment of the force P with respect to the section. It is necessary for equilibrium that $\int S y^2 dy\, b = P(l-x)$ But $\int S y^2 dy\, b = S\int dA\, y^2$ since $b\,dy = dA$ = the differential area. But $\int dA\, y^2 = I$ = moment of Inertia. Then $P(l-x) = \underline{M}$ = moment of ~~Inertia~~ applied force therefore $S\int y^2 dy\, b = S\int dA\, y^2 = SI = P(l-x) = M$ Hence $SI = M$ this formula is the same as $WS = M$ of the text I is used instead of W because it shows more clearly what ~~is meant by the~~ moment of inertia a resistance. Moment of Flexure.

The general value of s is $s = \frac{\lambda}{l} E E_1$ But $E_1 = 1$ therefore $s = \frac{\lambda}{l} E$ hence $SI = \frac{\lambda}{l} EI = M$ With the For l substitute dx for dx for dx is length of the small portion under consideration.

Therefore $M = \frac{d\lambda}{dx} EI$, But $\frac{\rho}{dx} = \frac{1}{d\lambda}$ (see fig) hence $\frac{d\lambda}{dx} = \frac{1}{\rho}$ substituting the value of $\frac{\lambda}{dx}$ we shall have $M = \frac{EI}{\rho}$ is is the fundamental expression the same as $\frac{WE}{\rho}$

$\frac{1}{\rho}$



Deflection

Case of a beam built in at one end ~~support~~ deflected by a uniformly distributed weight and a force applied at the extremity. Let us take any section a distance x from the plane at which the beam is built in then the forces acting upon this section will be Pp with an arm of lever l-x, and, if we term by q the weight distributed over a unit of length of the beam, Q the total distributed ~~part~~ weight upon the part (l-x) equal to (l-x)q and acting at the centre of gravity of the part lying beyond the section or with an arm as respects our section of $\frac{l-x}{2}$

whence the moments of the two forces as respects the section are $P(l-x)$ and $\frac{q}{2}(l-x)^2$ hence the moment of flexure $E\frac{d^2y}{dx^2} = P(l-x)+\frac{q}{2}(l^2-2lx+x^2)$. Integrating $E\frac{dy}{dx} = P(lx-\frac{x^2}{2})+\frac{q}{2}(l^2x-lx^2+\frac{x^3}{3})+C$. to find the value of this constant make $x=0$ and we are at the point of building in and there since the beam is horizontal there tangent $\frac{dy}{dx}$ made by the beam with the horizontal $=0$ and $\therefore C=0$, Integrating again $Ey = P(\frac{lx^2}{2}-\frac{x^3}{6})+\frac{q}{2}(\frac{l^2x^2}{2}-\frac{lx^3}{3}+\frac{x^4}{12})+B$. To find the value of this constant make $x=0$ then $y=0$ $\therefore B=0$ and reducing and passing to the limit by making $x=l$ we have $Ey = \frac{Pl^3}{3}+\frac{ql^4}{8}$ but $ql = Q$ therefore on substituting $Ey = \frac{Pl^3}{3}+\frac{Ql^3}{8}$.

We come now to discuss the case of a beam ~~beam~~ supported at both extremities and deflected by a force P applied at the middle; each of the points of support will under these circumstances sustain one half the force P and will give the force of reactions $\frac{1}{2}P$ at each point

The beam may now be regarded as acted upon by the three forces $P, \frac{1}{2}P$, & $\frac{1}{2}P$. Let us now revolve this beam so that the convex side shall be uppermost it now becomes the case of a beam built in at the middle point and deflected by the force $\frac{1}{2}P$ acting with the arm $\frac{1}{2}l$, taking then any section at a distance x from the middle we have acting upon it the force $\frac{P}{2}$ with an arm $\frac{l}{2}-x$ so that our equation for moment of flexure is $E\frac{d^2y}{dx^2}=\frac{1}{2}P(\frac{l}{2}-x)$. Integrating $E\frac{dy}{dx}=\frac{1}{2}P(\frac{lx}{2}-\frac{x^2}{2})+C$ to find the constant when $x=0$ we are at the lowest point the tangent is horizontal, and $C=0$. Integrating again $Ey=\frac{1}{2}P(\frac{lx^2}{4}-\frac{x^3}{6})+B$. When $x=0$ then $y=0$ ∴ $B=0$. Reducing and passing to the limit by making $x=\frac{l}{2}$ we obtain $Ey=\frac{1}{2}P\frac{l^3}{24}=\frac{Pl^3}{48}$

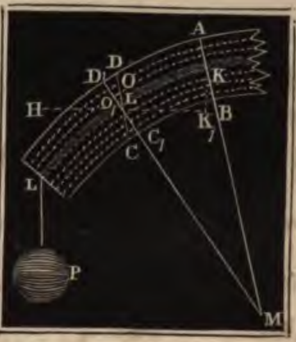
the neutral axis. If, then, we put the strain on a fibre, or layer of fibres, of a transverse section equal to unity (a square inch), and at a unit of distance (one inch) from the neutral axis $= S$; the strain for the distance $KN = z$ is Sz, and for the section F, it is FSz. If now the experimental number S represents both the extension and compression, we know the sum of all the strains $= (F_1z_1 + F_2z_2 + \ldots)\, S$, where F_1, F_2, &c., are the sections and z_1, z_2, &c., the distances from the neutral axis. In order that the tensions may produce no pressure, and therefore no alteration in the length, at the extremity K of the neutral axis, which we may regard as the fulcrum of a lever, the sum of the tensions $(F_1z_1 + F_2z_2 + \ldots)\, S$, and therefore also $F_1z_1 + F_2z_2 + \ldots$ must be $= 0$; *i. e. the neutral axis or the neutral lamina must pass through the centre of gravity of the cross section of the body.*

We may now compare the condition of the body with the equilibrium of a bent lever. The force P acts at the arm $KH = l$, the moment is, therefore, $M = Pl$, and balances the collective forces of extension and compression, whose moments are $z_1 \,.\, F_1Sz$, $z_2 \,.\, F_2Sz_2$, &c., or $F_1z_1^2 \,.\, S$, $F_2z_2^2 \,.\, S$, &c.; consequently we must put

$$M = Pl = (F_1z_1^2 + F_2z_2^2 + \ldots) \,.\, S.$$

This formula holds good for each cross section of the body, only for l we must substitute its distance each time from the point of application L of the force P. The factor $F_1z_1^2 + F_2z_2^2 + \ldots$ is dependent only on the cross section of the deflected body, and may be represented by the letter W. Hence we may put $M = Pl = WS$, and assert that the tension or strain of a transverse section is proportional to its distance l from the point of application of the force.

Fig. 202.



§ 188. From the modulus of elasticity E, the length of a fibre l at a unit of distance (an inch) from the neutral axis, and the elongation λ which it undergoes, the corresponding tension $S = \frac{\lambda}{l} E$ is known. If now ABC_1D_1, Fig. 202, is a short portion of the deflected body, $KL = l$ its length, and $MK = ML = \rho$ its radius of curvature, we have then $DD_1 : KL = LD : ML$, and also $OO_1 : KL = LO : ML$; *i. e.* $OO_1 : l = LO : \rho$. If we now assume $LO = 1$ and $OO_1 = \lambda$, we obtain $\lambda : l = 1 : \rho$, and hence $S = \frac{\lambda}{l} E = \frac{E}{\rho}$. If, finally, we substitute this value of S in the formula $M = WS$, we have the moment $M = \frac{WE}{\rho}$, and inversely, $WE = M\rho$.

The product WE is called the moment of flexure, *and hence the*

The moment of flexure is a measure for all the elastic forces developed in the system.

product of the moment M and the radius of curvature ρ *is equivalent to the moment of flexure for all cross sections.*

If we divide the neutral axis KL, Fig. 204, into n equal parts, as

Fig. 203.

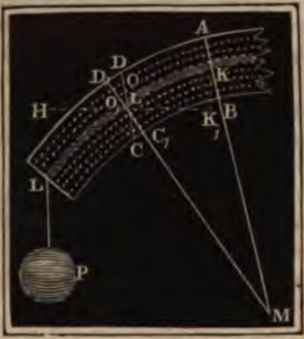
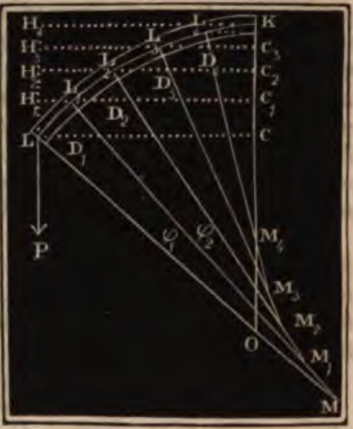


Fig. 204.



LL_1, L_1L_2, L_2L_3, &c. $= \frac{l}{n}$, and determine the radii of curvature $\overline{ML_1} = \rho_1$, $\overline{M_1L_2} = \rho_2$, &c., corresponding to these parts, the angle of curvature $\overline{LML_1} = \phi_1{}^0$, $\overline{L_1M_1L_2} = \phi_2{}^0$, &c., which every two radii of curvature include, are known, viz. $LL_1 = \frac{l}{n} = \rho_1\,\phi_1$, $L_1L_2 = \frac{l}{n} = \rho_2\phi_2$, &c., and therefore $\phi_1 = \frac{l}{n\rho_1}$, $\phi_2 = \frac{l}{n\rho_2}$, &c. If, further, we substitute $\rho_1 = \frac{WE}{M_1}$, $\phi_2 = \frac{WE}{M_2}$, &c., we then obtain $\phi_1 = \frac{M_1 l}{nWE}$, $\phi_2 = \frac{M_2 l}{nWE}$, &c.; and by the summation of all these angles we find the angle $LOK = \alpha^0$, by which a greater portion, or the whole neutral axis, is deflected.

§ 189. *Elastic Curve.*—If we suppose a small flexure, we may take the projection $CL = KH_4$, parallel to the initial direction of the undeflected beam, and equal to the length of the beam itself, and likewise the projections LD_1, L_1D_2, &c., equal to the parts LL_1, L_1L_2, &c., of the neutral axis, *i. e.* $= \frac{l}{n}$, and we obtain the moments

$M_1 = \frac{Pl}{n}$, $M_2 = \frac{2\,Pl}{n}$, $M_3 = \frac{3\,Pl}{n}$, &c. If we substitute these values in the formulæ for ϕ_1, ϕ_2, &c., then the measures of the angles of curvature are given:

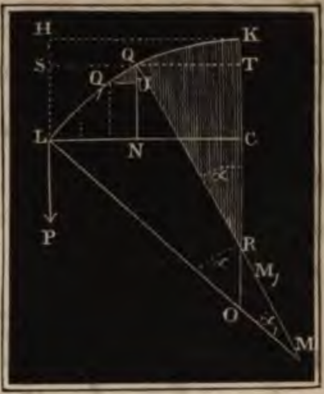
$$\phi_1 = \frac{Pl^2}{n^2\,WE},\ \phi_2 = \frac{2\,Pl^2}{n^2\,WE},\ \phi_3 = \frac{3\,Pl^2}{n^2\,WE},\ \text{\&c.};$$

hence $dy = \frac{P}{2WE}(l^2 - x^2)\,dx$ and $y = \frac{P}{WE} \times (l^2x - \frac{x^3}{3}) + C$, when $x = 0$ $y = 0$ ∴ $C = 0$ and therefore the constant disappears as its value $= 0$ when we make $x = l$ $y = \frac{P}{WE}(l^3 - \frac{l^3}{3}) = \frac{Pl^3}{3WE}$. The equation of the curve can also be deduced as follows $E\frac{d^2y}{dx^2} = P(l - x)$ Integrating $E\frac{dy}{dx} = P(lx - \frac{x^2}{2}) + C$. To find the value of the constant making $x = 0$ we conclude the point where the curve is horizontal and ∴ $\frac{dy}{dx} = 0$ ~~[illegible]~~ ~~of~~ and C disappears; again ~~differentiating~~ Integrating $Ey = P(\frac{lx^2}{2} - \frac{x^3}{3}) + C_{\prime\prime}$. To find the value of other constant we proceed as before, $x = 0$ and then $y = 0$ ∴ $C = 0$ and it disappears. Making $x = l$ we obtain $Ey = P(\frac{l^3}{2} - \frac{l^3}{6}) = \frac{Pl^3}{3}$ therefore $y = \frac{Pl^3}{3E}$ = Equation of the elastic curve.

and by addition, the measure of the whole angle of curvature KOL $= \alpha$ of the neutral axis:

$$\alpha = \frac{Pl^2}{n^2\,WE}\,(1 + 2 + 3 + \ldots + n) = \frac{Pl^2}{n^2\,WE} \cdot \frac{n^2}{2} = \frac{Pl^2}{2\,WE}.$$

Fig. 205.



With the assistance of the last formula, we may now find the equation to the curve formed by the neutral axis, KL, Fig. 205. Let us divide the absciss $LN = x$, commencing at the point L, into m equal parts, and find the parts of the ordinate $NQ = y$ corresponding to them. Since the radius of curvature QR is perpendicular to the part of the arc QQ_1, the angle $QQ_1U = QRK = \alpha_2$, and therefore the part QU of the ordinate $y_1 = Q_1U \,.\, tang.\ \alpha_2$,

or Q_1U being put $= \frac{x}{m}$ and *tang.* α_2 $= \alpha_2$, $QU\,\frac{x\alpha^2}{m}$. Now $\alpha_2 = LOK -$ $LMQ = \alpha - \alpha_1 = \frac{Pl^2}{2WE} - \frac{Px^2}{2WE} = \frac{P}{2WE}\,(l^2 - x^2)$; it follows, therefore, that $QU = \frac{x}{m} \cdot \frac{P}{2WE}\,(l^2 - x^2)$. If for x^2 we substitute successively $\left(\frac{x}{m}\right)^2, \left(\frac{2x}{m}\right)^2, \left(\frac{3x}{m}\right)^2$, &c., we then obtain by the last formula all the parts of y, and by the addition of these, the whole ordinate:

$$NQ = y = \frac{x}{m} \cdot \frac{P}{2WE}\left[l^2 - \left(\frac{x}{m}\right)^2 + l^2 - \left(\frac{2x}{m}\right)^2 + l^2 - \left(\frac{3x}{m}\right)^2 + \ldots\right] = \frac{x}{m} \cdot \frac{P}{2WE}\left[ml^2 - \left(\frac{x}{m}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + m^2)\right], \text{i. e. } y = \frac{Px}{2WE}\left(l^2 - \frac{x^2}{3}\right).$$

By this formula we may calculate for every absciss x the corresponding ordinate y, and likewise for the whole length $CL = l$, the height of the arc $CK = a$. This last is:

$$a = \frac{Pl}{2WE}\left(l^2 - \frac{l^2}{3}\right) = \frac{Pl^3}{3WE}.$$

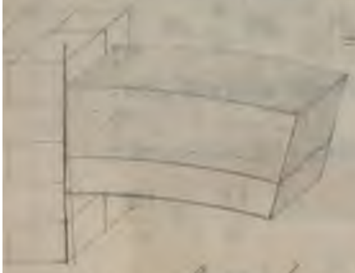
Therefore, *the height of the arc increases as the force and the cube of the length.*

If we have a by measurement, we may find from this formula the modulus of elasticity, $E = \frac{Pl^3}{3Wa}$.

§ 190. If the whole load is uniformly distributed over the beam, and if each unit of length sustains a portion $= q$, therefore, for

Uniformly Distributed Weight.

In this case the whole weight is uniform-



=ly distributed over the beam. Let g = weight on a unit of beam. Let x = distance of section from plane of suspension it is required to find the moment of the part of the beam lying beyond the section the moment of this part of the beam is entirely independent of the rest of the beam it is just the same as if the beam was built in up to the section. Let now another section be taken z distant from the first

And we suppose the deflection is small that the [illegible] respect to the sections [illegible] the axis [illegible] from

[illegible]

then $gz\,dz$ = moment of weight distributed over beam up to second section

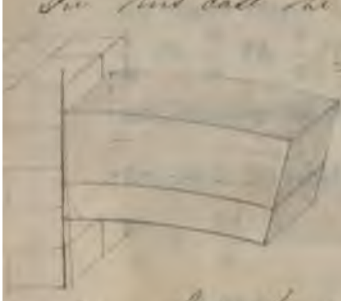
$\int_{0}^{l-x} gz\,dz = \frac{gz^2}{2} = g\frac{(l-x)^2}{2}$ = moment of that part of the beam lying beyond the sections. It is known that since $E\frac{d^2y}{dx^2}$ the moment of flexure is equal to the moment) (See appendix)

$E\frac{d^2y}{dx^2} = g\frac{(l-x)^2}{2} = g\left(\frac{l^2}{2} - lx + \frac{x^2}{2}\right)$ ~~Integrating $E\frac{dy}{dx} = g\frac{l^2x}{2}$ $\frac{lx^2}{2}$~~ $\frac{x^2}{2}$). Integrating $E\frac{dy}{dx} = g\left(\frac{l^2x}{2} - \frac{lx^2}{2} + \frac{x^3}{6}\right) + C$. To find the value of C make $x=0$ then $\frac{dy}{dx} = 0$ & $C=0$; $\frac{dy}{dx} = 0$ since the curve being horizontal at the origin the tangent of the angle made by the tangent to the curve with the horizontal at that point $= 0$. Integrating again.

equation $E\frac{dy}{dx} = \frac{P+Q}{2}(lx - \frac{x^2}{2}) + \frac{q}{2}(l^2x - lx^2 + \frac{x^3}{3})$ + C. To find the value of this const. pass to the fixed point by making $x = 0$ & then, the curve being horizontal $\frac{dy}{dx} = 0$ and $C = 0$. Integrating again $Ey = \frac{P+Q}{2}(\frac{lx^2}{2} - \frac{x^3}{6}) - \frac{q}{2}(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}) + B$. To find the value of this constant let us pass to the fixed point by writing $x = 0$ & then $y = 0$ and ∴ $B = 0$. Collecting in the above equation $Ey = \frac{P+Q}{2}\cdot\frac{l^3}{3} - \frac{q}{2}(\frac{l^4}{2} - \frac{l^4}{3} + \frac{l^4}{12} = \frac{l^4}{4})$ $= Ey = \frac{P+Q}{2}\cdot\frac{l^3}{3} - \frac{ql^4}{8}$. And $2q\frac{l}{2} = Q$ (since our l only $= \frac{1}{2}$ the beam l and the whole weight Q is $q \times$ whole length), $Ey = \frac{P+Q}{2}\cdot\frac{l^3}{3} + \frac{Ql^3}{16} = (P/6 + \frac{Q}{6}$ $+ \frac{Q}{16})l^3 = (\frac{P}{6} + \frac{5}{48}Q)l^3$. In substituting now for l its true value $\frac{l}{2}$ we have $Ey = (\frac{P}{6} + \frac{5}{48}Q)\frac{l^3}{8} = \frac{P}{48}l^3 + \frac{5Q}{8.48}l^3 = (P + \frac{5}{8}Q)\frac{l^3}{48}$.

Uniformly Distributed Weight.

In this case the whole weight is uniform-



-ly distributed over the beam. Let g = weight on a unit of beam. Let x = distance of section from plane of suspension it is required to find the moment of the part of the beam lying beyond the section the moment of this part of the beam is entirely independent of the rest of the beam it is just the same as if the beam was built in up to the section.

and we suppose the deflections so small that the [illegible] to the sections [illegible] from

Let now another section be taken z distance from the first

weight distributed over beam up to

then $gz\,dz$ = moment of second section

$$\int_x^l gz\,dz = \frac{gz^2}{2}\Big|_x^l = g\frac{(l-x)^2}{2}$$ = moment of that part of the beam lying beyond the sections. It is known that since $E\frac{d^2y}{dx^2}$ is equal to the moment

the moment of flexure

$$E\frac{d^2y}{dx^2} = g\frac{(l-x)^2}{2} = g\left(\frac{l^2}{2} - lx + \frac{x^2}{2}\right)$$ ~~Integrating $E\frac{dy}{dx} = g\frac{l^2x}{2}$ $\frac{lx^2}{2} + \frac{x^2}{2}$~~). Integrating $E\frac{dy}{dx} = g\left(\frac{l^2x}{2} - \frac{lx^2}{2} + \frac{x^3}{6}\right) + C$ To find the value of C make $x = 0$ then $\frac{dy}{dx} = 0$ & $C = 0$; $\frac{dy}{dx} = 0$ since the curve

horizontal at the origin the tan

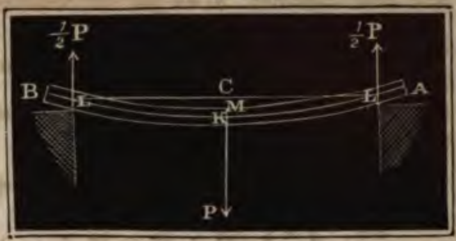
the angle made by the tangent with the horizontal at that point = 0. Integra

[illegible]

Beam [illegible]

[illegible] extremities and deflected by means a uniformly distributed weight and a weight applied at its centre. In this case let P represent the weight applied at the centre and Q the total weight distributed over the beam. Then each of the points of support sustains a pressure $\frac{P+Q}{2}$. We have only need to consider one half the [illegible] [illegible] of the beam, for we may regard the [illegible] beam [illegible] the middle section and thence deflected by the reaction or force $\frac{P+Q}{2}$. In taking such a consideration of the subject we have [illegible]iting a section at a distance x from the centre the moment of flexure with respect to this section will be the force $\frac{P+Q}{2}$ into its arm $(\frac{l}{2}-x)$ minus the whole weight uniformly distributed over this same distance acting with an arm $\frac{(\frac{l}{2}-x)}{2}$ since this latter force acting downwards is in direct opposition to the force $\frac{P+Q}{2}$ tending to deflect it upwards. [illegible] $\frac{l}{2}$ [illegible] the [illegible] $\frac{l}{2}$; [illegible] moment of flexure [illegible] $E\frac{d^2y}{dx^2} = \frac{P+Q}{2}(\frac{l}{2}-x) - \frac{q}{2}(\frac{l}{2}-x)^2$ [illegible] form [illegible] weight [illegible] of [illegible] weight [illegible] x and [illegible] And [illegible] with [illegible] arm [illegible] must [illegible] position [illegible] is in [illegible] integrating once [illegible] equation $E\frac{dy}{dx} = \frac{P+Q}{2}\left(lx - \frac{x^2}{2}\right) + \frac{q}{2}\left(l^2x - lx^2 +\right.$ + C. To find the value of this const pass to the lowest point by making $x = 0$ when, the curve being horizontal $\frac{dy}{dx} = 0$ and $C = 0$. Integrating again $Ey = \frac{P+Q}{2}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) - \frac{q}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right) + B$. To find the value of this constant let us pass to the limit by writing $x = 0$ when $y = 0$ and $\therefore B = 0$. Collecting in a line equation $Ey = \frac{P+Q}{2}\cdot\frac{l^3}{3} - \frac{q}{2}\left(\frac{l^4}{2} - \frac{l^4}{3} + \frac{l^4}{12} = \frac{l^4}{4}\right)$ $= Ey = \frac{P+Q}{2}\cdot\frac{l^3}{3} - \frac{ql^4}{8}$. but $2q\frac{l}{8} = Q$ (since our l only $= \frac{1}{2}$ the true l and the whole weight Q is $q \times$ whole length, $Ey = \frac{P+Q}{2}\cdot\frac{l^3}{3} + \frac{Ql^3}{16} = (P/6 +$ $+ \frac{Q}{16})l^3 = \left(\frac{P}{6} + \frac{15}{48}Q\right)l^3$. Substituting now for l its true value $\frac{l}{2}$ we have $Ey = \left(\frac{P}{6} + \frac{15}{48}Q\right)\frac{l^3}{8} = \frac{P}{48}l^3 + \frac{15\cdot Q}{8\cdot 48}l^3 = \left(P + \frac{5}{8}Q\right)\frac{l^3}{48}$.

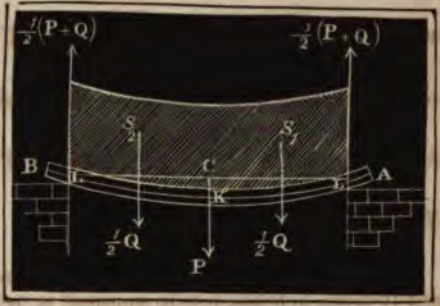
Fig. 206.



$a = \frac{\frac{1}{2}P \cdot (\frac{1}{2}l)^3}{3\,WE} = \frac{1}{16} \cdot \frac{Pl^3}{3\,WE}$ = a sixteenth of the height of the arc of the beam, which is loaded at its extremity.

If, lastly, the load $Q = ql$ is uniformly distributed over the body AB, Fig. 207, supported at both extremities, we must put in the formula $a = \left(\frac{P}{3} + \frac{Q}{8}\right) \frac{l^3}{WE}$ in place of l, $\frac{l}{2}$, in place of P, $\frac{P + Q}{2}$ and for Q, $-\frac{Q}{2}$, because with respect to K, the weight $\frac{Q}{2}$ at the arm $\frac{l}{4}$ is opposed to the reaction $\frac{P + Q}{2}$ at the arm $\frac{l}{2}$. Consequently

Fig. 207.



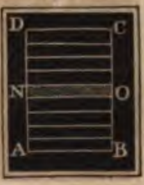
$$a = \left(\frac{P + Q}{6} - \frac{Q}{16}\right) \frac{l^3}{8\,WE} = \left(P + \frac{5}{8}\,Q\right) \frac{l^3}{48\,WE}.$$

For $P = 0$, $a = \frac{5}{8} \cdot \frac{Ql^3}{48\,WE}$; the load is, therefore, uniformly distributed over the whole arc, and the height of the arc is $\frac{5}{8}$ times as great as if the weight acted at the middle of the beam.

§ 191. *Rectangular Beams.*—In order to give the relations of flexure of a beam or other prismatic body, and the elastic curve formed by its neutral axis, the transverse section of the body must be known, and the moment of flexure WE, calculated from it.

If the section of the beam be a rectangle $ABCD$, Fig. 208, of the width $AB = CD = b$, the height $AD = BC = h$, the moment of flexure $WE = (F_1 z_1^2 + F_1 z_2^2 + \ldots) E$ will be known if we decompose this cross section by lines parallel to the neutral axis NO into $2n$ equal laminæ, each having the area $b \cdot \frac{h}{2n} = \frac{bh}{2n}$; and determine the moments of these laminæ, and

Fig. 208.



add them together. If we put successively $\frac{1}{n} \cdot \frac{h}{2}, \frac{2}{n} \cdot \frac{h}{2}, \frac{3}{n} \cdot \frac{h}{2}$ for z in $\frac{bh}{2n} \cdot z^2 E$, we shall then obtain the moments of the laminæ on one side of the neutral axis; but if we double their sum, we have the complete moment of flexure

$$WE = 2 \, . \, \frac{bh}{2n} \left[\left(\frac{h^2}{2n}\right) + \left(\frac{2h}{2n}\right)^2 + \left(\frac{3h}{2n}\right)^2 + \ldots \right] E$$
$$= \frac{bh}{n} \cdot \left(\frac{h}{2n}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + n^2) E = \frac{bh^3}{4.3} E = \frac{bh^3}{12} \cdot E.$$

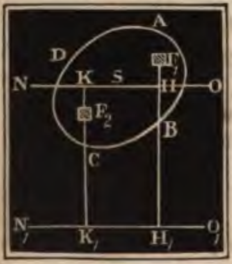
The moment of flexure, therefore, of a rectangular beam *increases as the width and the cube of the depth of the beam.*

If we put this value of WE into the formula $a = \frac{Pl^3}{3\,WE}$ of § 189, we shall obtain $a = 4 \, . \, \frac{P\,l^3}{b\,h^3\,E}$, but if into the formula $a = \frac{1}{48} \frac{P\,l^3}{WE}$ of § 190, then $a = \frac{P\,l^3}{4\,b\,h^3 E}$. Inversely, the modulus of elasticity follows from the height of the arc a $E = \frac{4\,P\,l^3}{a\,b\,h^3}$ for the one, and $E = \frac{P\,l^3}{4\,a\,b\,h^3}$, for the other case.

Example.—1. A wooden beam, 10 feet = 120 inches in length, 8 inches in width, and 10 inches in height, is to be supported at both its ends, and bear a uniform load $Q = 10000$ lbs., what flexure will it undergo? The height of the arc is $a = \frac{5}{8} \frac{Q\,l^3}{4\,b\,h^3\,E} = \frac{5}{32} \cdot \frac{10000 \, . \, 120^3}{8 \, . \, 10^3 \, . \, E} = \frac{50000 \, . \, 12^3}{32 \, . \, 8\,E} = \frac{1350000}{4 \, . \, E}$. Now E being put = 1800000 lbs. it follows that $a = \frac{135}{4 \, . \, 180} = 0{,}1875$ inches.—2. If a rectangular cast iron bar, 2 inches wide and ½ inch thick, has been deflected ¼ inch by a weight $P = 18$ lbs. lying in the middle of it, whilst the distance of the supports amounts to 5 feet, the modulus of elasticity of cast iron will be $E = \frac{P\,l^3}{4\,a\,b\,h^3} = \frac{18 \, . \, 60^3}{4 \, . \, \frac{1}{4} \, . \, 2 \, . \, (\frac{1}{2})^3} = \frac{18 \, . \, 60^3}{\frac{1}{4}} = 72 \, . \, 216000 =$ 15552000 lbs.

§ 192. *Reduction of the Moment of Flexure.*—If we know the moment of flexure of a body, *ABCD*, Fig. 209, about an axis N_1O_1, lying without the centre of gravity, the moment about another axis NO, passing through the centre of gravity S, and running parallel with the former, may be found. If the distance $HH_1 = KK_1$ of both axes $= d$, and the distances of the elementary surfaces F_1 F_2, &c., from the neutral axis $NO = z_1, z_2$, &c., we shall have the distances from the axis $N_1O_1, = d + z_1, d + z_2$, &c., and the moment of flexure will be $W_1E = [F_1\,(d + z_1)^2 + F_2\,(d + z_2)^2 + \ldots] E = [F_1\,(d^2 + 2dz_1 + z^2_1) + F_2\,(d^2 + 2dz_2 + z^2_2) + \ldots] E = [d^2(F_1 + F_2 + \ldots) + 2d\,(F_1\,z_1 +$

Fig. 209.



$Ey = \frac{Q}{l}\left(\frac{l^2x^2}{4} - \frac{lx^3}{6} + \frac{x^4}{12}\right) + C$ [illegible] find the value of

[illegible] of a [illegible] supported at both extremities and loaded with a [illegible] uniformly distributed over it. Let l = length of [illegible], Q = [illegible] distributed [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] at the middle.

For taking a [illegible] at distance [illegible] [illegible] $\left(E\frac{d^2y}{dx^2} = \frac{Q}{l}\left(\frac{l}{2} - x\right) - \frac{Q}{2l}(l-x)^2\right)$ [illegible] [illegible]

[illegible]

$$E\frac{dy}{dx} = \frac{Q}{2}\left(\frac{lx}{2} - \frac{x^2}{2}\right) - \frac{Q}{2l}\left(l^2x - lx^2 + \frac{x^3}{3}\right) + C$$ [illegible] constant [illegible] disappears

$$Ey = \frac{Q}{2}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) - \frac{Q}{2l}\left(\frac{l^2x^2}{2} - \frac{x^3l}{3} + \frac{x^4}{12}\right) + C$$ this constant also disappearing [illegible] [illegible]

$\therefore Ey = \frac{Q}{2}\cdot\frac{l^3}{3} - \frac{Q}{2l}\left(\frac{l^4}{2} - \frac{l^4}{3} + \frac{l^4}{12}\right) + \frac{3}{12}$ [illegible] $\therefore Ey = \frac{6l^3 \; 30l^3}{6 \; 2 \cdot 24}$ [illegible]

[illegible] value [illegible] $Ey = \frac{Ql^3}{24} \cdot \frac{3}{8}$ [illegible] $\frac{5}{384}Ql^3$.

$\int ds\, z^2 + \int ds\, 2z\, \delta z + \int ds$ [illegible]

$Ey = g\left(\frac{l^2x^2}{4} - \frac{lx^3}{6} + \frac{x^4}{24}\right) + C$ To find the value of this constant make $x = 0$ then $y = 0$ & $C = 0$ Passing to the limits $x = 0$ & $x = l$ we have $Ey = gl^4\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{24}\right) = \frac{gl^4}{8} = \frac{Ql^3}{8}$ (since $Q = lg$) Therefore $y = \frac{Ql^3}{8}$. Moment of ~~Flexure~~

Case of rectangular beam.

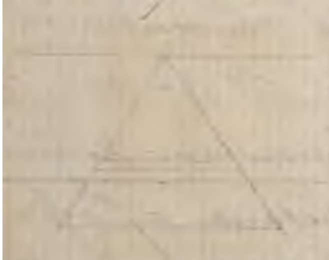
ydx = elementary section, b the breadth of the rectangle being constant and $= b$. $E'bx^2dx$ = elementary moment of ~~flexure~~ flexure $E'b\int_{-\frac{1}{2}h}^{\frac{1}{2}h} x^2dx$ = moment of ~~flexure~~ flexure of whole beam $= \frac{E'b\left(\frac{h^3}{8} + \frac{h^3}{8}\right)}{3} = \frac{bh^3}{12}E'$

Reduction of the moment of ~~flexure~~ [illegible]

Let ABC be any body XX be any axis; Let GG be an axis passing through the centre of Gravity Let A = Area of surface then dA = Elementary portion of surface; Let Z = distance of elementary portion of surface from GG Therefore $\delta + Z$ equals distance from XX, Let I_x = measure of the moment of ~~flexure~~ Inertia with respect to XX Let I_g = measure of the moment of ~~inertia~~ flexure with respect to GG. $I_x = \int dA(Z+\delta)^2 = \int dA(Z^2 + 2Z\delta + \delta^2)$ $= \int dAZ^2 + \int dA\,2Z\delta + \int dA\delta^2 = I_g + 0 + A\delta^2$ for

$\int dA z^2 = I_g =$ moment of inertia Inertia ~~flexure~~ with respect to G G, Also $\int z\,dz\,dA = 0$ with respect to GG which for equilibrium must be $= 0$ because it is the statical mo- ~~because is is the moment of~~ -ment of the body with respect to the axis ~~the body~~ hence $I_g = I_x - A\delta^2$. That is the measure of the moment of flexure about the axis passing through the centre of gravity (or the neutral axis), is equal to the measure about any parallel axis less the product of the area into the square of the distance between the two axes. Moment of Flexure Case of Triangular beams ~~Page~~ Art 193

through the. Let XX be the axis vertex, GG the axis through the centre of Gravity, h the height and b the base of the triangle



$x =$ distance of elementary portion of ~~ba~~ bt from the vertex of the triangle $y\,dx =$ elementary portion, $\frac{y}{x} = \frac{b}{h}$ ∴ $y = \frac{bx}{h}$ ∴ $y\,dx = \frac{b}{h}x\,dx$, $\frac{b}{h}x^3dx$ mom. flex. element portion

$\int_0^h \frac{b}{h}x^3dx = \frac{bh^4}{4h} = \frac{bh^3}{4} = I_x$. But $I_g = I_x - A\delta^2$, $A = \frac{bh}{2}$, $\delta = \frac{2h}{3}$

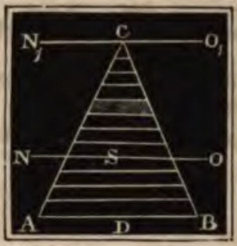
∴ $I_g = \frac{bh^3}{4} - \frac{bh}{2}\cdot\frac{4}{9}h^2 = \frac{bh^3}{36}$. $I_gE = \frac{1}{3}\frac{bh^3}{12}E =$ mom. flex. tri

$F_2 z_2 + \ldots) + (F_1 z_1^2 + F_2\, z_2^2 + \ldots)]$. But $F_1 + F_2 + \ldots$ as the sum of all the elements $=$ the transverse section F of the whole body; further, $F_1 z_1 + F_2 z_2 + \ldots$ as the sum of the moments about an axis passing through the centre of gravity of the body $= 0$, and $F_1 z_1^2 + F_2\, z_2^2 + \ldots$ is the moment of flexure WE about the neutral axis NO; it follows, therefore, that $W_1 E = (F\, d^2 + W)\, E$, or $W_1 = F\, d^2 + W$; and inversely, $W = W_1 - F\, d^2$.

The measure W of the moment of flexure about the neutral axis is equal to the measure W_1 of the moment of flexure about a second parallel axis, less the product of the transverse section F and the square (d^2) of the distance of both axes. Hence it follows, that of all the moments of flexure, that about the neutral axis is the least.

The moments of flexure of many bodies about any axis may be easily found; we may therefore avail ourselves of these to determine, by means of the formulæ found, the moments about the neutral axis.

Fig. 210.



§ 193. To find the moment of flexure of a prism having a triangular transverse section ABC, we must decompose this section by lines parallel to the base AB into n thin laminæ, and determine the moments of these about the axis $N_1 O_1$ passing through the point C parallel to AB. If h is the height CD, and b the breadth AB of the triangular section ABC, we have the height of these laminæ $= \frac{h}{n}$, their lengths $= \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}$, &c., to $\frac{nb}{n}$, and their distances from $N_1 O_1 = \frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c., to $\frac{nh}{n}$. From these the areas of the laminæ are $F_1 = \frac{b\,h}{n^2}$, $F_2 = \frac{2\,b\,h}{n^2}$, $F_3 = \frac{3\,b\,h}{n^2}$, and their moments $F_1 z_1^2 = \frac{b\,h^3}{n^4}$, $F_2 z_2^2 = 2^3 \cdot \frac{b\,h^3}{n^4}$, $F_3 z_3^2 = 3^3 \cdot \frac{b\,h^3}{n^4}$, &c., and the moment of flexure about the axis $N_1 O_1$:

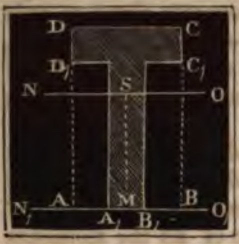
$$W_1 = \frac{bh^3}{n^4}(1^3 + 2^3 + 3^3 + \ldots + n^3) = \frac{bh^3}{n^4} \cdot \frac{n^4}{4} = \frac{bh^3}{4}.$$

The distance of the centre of gravity S from the point C is $d = \frac{2}{3}\, h$, and the area of the whole triangle $F = \frac{bh}{2}$; therefore $Fd^2 = \frac{bh}{2} \cdot \frac{4}{9}\, h^2 = \frac{2\,b\,h^3}{9}$, and the moment of flexure about the neutral axis NO sought is:

$$WE = (W_1 - Fd^2)\, E = \left(\frac{bh^3}{4} - \frac{2\,b\,h^3}{9}\right) E = \frac{1}{3} \cdot \frac{bh^3}{12}\, E =$$

a third of the moment of flexure of the rectangular beam, which has the same depth and width as the triangular one. But since this beam has double the volume, it then follows, that under otherwise similar circumstances, the triangular beam has $\frac{2}{3}$ of the moment of flexure of the rectangular.

Fig. 211.



We may find in the same manner the moments of flexure of many other bodies used in construction. For the transverse section of a T-shaped body A_1B_1CD, Fig. 211, whose dimensions are $AB = b$, $AB - A_1B_1 = AA_1 + BB_1 = b_1$, $AD = BC = h$ and $AD_1 = BC_1 = BC - CC_1 = h_1$, the moment of flexure about the lower edge $A_1B_1 =$ the moment of the rectangular figure $ABCD$, less the moments of the rectangles A_1D_1 and B_1C_1, *i. e.* $W_1 = \frac{1}{2} \cdot \frac{b(2h)^3}{12} - \frac{1}{2} \cdot \frac{b_1(2h_1)^3}{12} = \frac{bh^3 - b_1h_1^3}{3}$, as follows, if we consider each of these rectangles as the half of rectangles having double the height with the neutral axis N_1O_1. Now the area $A_1C_1D = bh - b_1h_1$, and its moment $Fd = bh \cdot \frac{h}{2} - b_1h_1 \cdot \frac{h_1}{2} = \frac{1}{2}(bh^2 - b_1h_1^2)$; hence it follows that the arm $MS = d = \frac{bh^2 - b_1h_1^2}{2(bh - b_1h_1)}$, the moment $Fd^2 = \frac{1}{4}(bh^2 - b_1h_1^2)^2 \div (bh - b_1h_1)$, and the moment of flexure about the neutral axis passing through the centre of gravity S:

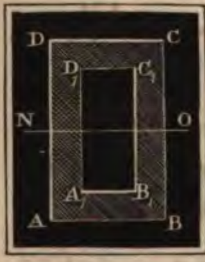
$$W = W_1 - Fd^2 = \frac{bh^3 - b_1h_1^3}{3} - \frac{1}{4}(bh^2 - b_1h_1^2)^2 \div (bh - b_1h_1)$$

$$= \frac{4(bh^3 - b_1h_1^3)(bh - b_1h_1) - 3(bh^2 - b_1h_1^2)^2}{12(bh - b_1h_1)}$$

$$= \frac{(bh^2 - b_1h_1^2)^2 - 4\,bh\,b_1h_1(h - h_1)^2}{12(bh - b_1h_1)}.$$

§ 194. *Hollow Beams.*—The moment of flexure of a hollow rectangular beam $ABCD$, Fig. 212, is determined, if we deduct from the moment of the complete beam that of the hollow part. $AB = b$ is the external breadth, and $BC = h$ the height, and $A_1B_1 = b_1$ the internal breadth, and $B_1C_1 = h_1$ the height, we then have the moments of flexure of both $= \frac{bh^3}{12}$ and $\frac{b_1h_1^3}{12}$, and by subtraction we get the moment of flexure of the hollow beam $W = \frac{bh^3 - b_1h_1^3}{12}$.

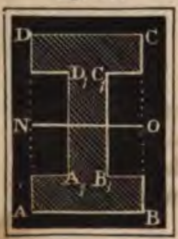
Fig. 212.

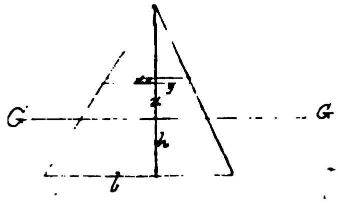


We may find in an exactly similar manner the moment of flexure of a body $ABCD$, Fig. 213, hollowed out at the sides. $AB = b$ is the outer breadth and $BC = h$ the height; and if $AB - A_1B_1 = b_1$, and $B_1C_1 = h_1$, the sum of the breadth and the heights of both hollows, by subtraction we have again:

$$W = \frac{bh^3 - b_1h_1^3}{12}.$$

Fig. 213.



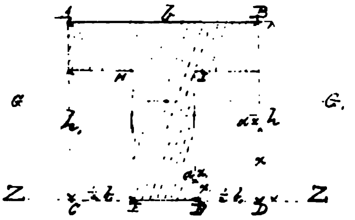


To find the moment of [illegible] of a [illegible] [illegible] [illegible] [illegible] axis [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] and [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] $u = \frac{2}{3}b - \frac{b}{h}x$ [illegible] for [illegible] [illegible] [illegible] [illegible]

[illegible]

$$= \frac{bh^3}{36}$$

[illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] therefore [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]



[illegible] of

T[illegible]

[illegible]

will be equal to [illegible] CB. [illegible] [illegible] CH and DI = $\int$ [illegible]

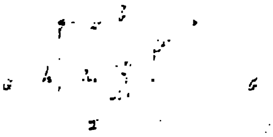
[illegible] $I_G = I_{gz} - A\bar{x}^2$ [illegible]

[illegible]

$$-\frac{3b^2h^4 + 6bb_1h^2h_1^2 - 3b_1^2h_1^4}{[illegible]} = [illegible]$$

$$= [illegible]$$

$$= [illegible]$$



H[illegible]

[illegible] $I_G =$ [illegible]

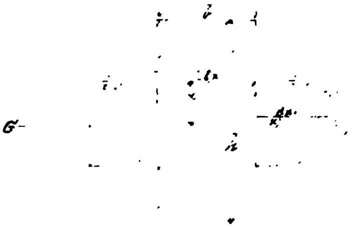
$I_G = \frac{[illegible]}{12}$ [illegible]

$= \frac{bh^3 - b_1h_1^3}{12}$ identical with the above, [illegible]

[illegible] GG may be taken at the centre of [illegible]

[illegible] that the [illegible]

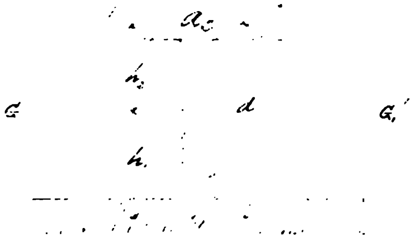
[illegible] the [illegible]



Cross [illegible] [illegible]

I_g [illegible]

[illegible]



Flanged Girder [illegible] unequal flanges. [illegible] a_1 and a_2 [illegible] and h_1 & h_2 [illegible] unknown. [illegible]

$I_g = a_1h_1^2 + a_2h_2^2$. [illegible]

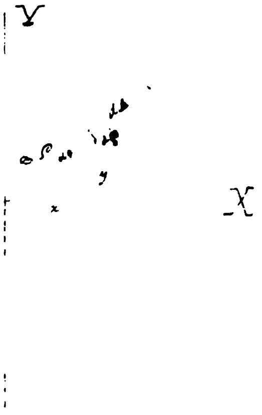
[illegible] and [illegible] d [illegible] $= a_2h_2$ [illegible] $d = h_1 + h_2$ [illegible] $h_2 = \frac{a_1d}{A}$ and $h_1 = \frac{a_2d}{A}$ [illegible] equation $I_g = \frac{a_1a_2^2d^2}{A^2} + \frac{a_2a_1^2d^2}{A^2} = \frac{d^2a_1a_2}{A}$ [illegible]

[illegible] upper flange equal to lower i.e. $a_1 = a_2$ and [illegible] vertical [illegible] flange [illegible]

[illegible] Flange = [illegible] $I_g =$ [illegible]

[illegible]

$I =$ [illegible]



[illegible]

$I_x =$ [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

= [illegible]

[illegible] $I_y =$ [illegible]

[illegible]

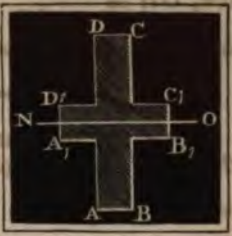
cos² = 1 + cos 2φ − cos² [illegible] cos² = 1 + cos 2φ / 2 [illegible]

$I_y = \int_0 \int \rho^3 d\varphi \, d\rho \, \frac{1 + \cos 2\varphi}{2}$ [illegible]

will disappear) = [illegible]

The moment of flexure of a body $ABCD$, Fig. 214, of a cross-shaped section, may be obtained in the same manner. Here $AB = b$ the width, and $BC = h$ the height of the middle piece, and if $A_1B_1 - AB = b_1$ and $A_1D_1 = h_1$ are the sum of the breadths and the height of the side ribs; by addition we have the moment of flexure: $W = \frac{bh^3 + b_1h_1^3}{12}$.

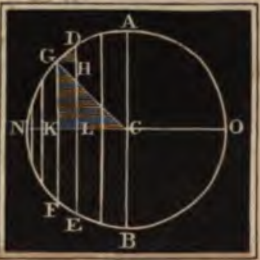
Fig. 214.



It is besides easy to see, that deep, hollow, and ribbed or flanged sections of the same area have a greater moment of flexure than square sections. Because this moment increases with the transverse section F and the square (z^2) of the distance from the neutral axis, one and the same fibre affords, therefore, a greater resistance to flexure, the further it is distant from the neutral axis. If, for example, the height h of a massive rectangular beam be equal to double its breadth b, its moment of flexure will be either $W = \frac{b \,.\, (2\,b)^3}{12} = \frac{2}{3}\,b^4$ or $= \frac{2\,b \,.\, b^3}{12} = \frac{1}{6}\,b^4$, according as we put up the beam with the lesser breadth b, or the greater $2\,b$; in the first case, therefore, the moment of flexure is four times greater than in the second. If we replace the massive beam of the cross section bh by a hollow one, whose hollow bh is equal to the massive part of the section $b_1h_1 - bh$, if, therefore, $b_1h_1 - bh = bh$, *i. e.* $b_1h_1 = 2bh$, or $b_1 = \sqrt{2}$ and $h_1 = h\,\sqrt{2}$, we shall obtain the moment of flexure of the last $\frac{b_1h_1^3 - bh^3}{12} = \frac{b\,\sqrt{2}\,(h\,\sqrt{2})^3 - bh^3}{12} = \frac{3}{12}\,bh^3$, *i. e.* three times as great as for the first.

§ 195. *Cylinders.*—The moment of flexure of a cylinder is determined in the following manner. Let $AOBN$, Fig. 215, be the circular transverse section, and NO the neutral axis of the cylinder. The diameter AB, divides this section into two equal parts, having equal moments of flexure, and the moment of flexure of the whole may be found by doubling the moment of the half ANB. The half may be divided by sections DE, FG, &c., parallel to AB, and at right angles to NO into thin lamina, which may be considered as rectangular. The moment of flexure of such a portion $DEFG$, $= \frac{KL \,.\, \overline{DE}^3}{12}$. Now $CA = CN = r$ the radius of the circular section, a quadrant AN has, therefore, the area $\frac{\pi\, r}{2}$, and if we divide this into n equal parts, any such part $DG = \frac{1}{n} \,.\, \frac{\pi\, r}{2} = \frac{\pi\, r}{2\, n}$. The

Fig. 215.



projection parallel to CN, $GH = KL$ corresponds to this part, and may be determined by putting, $GH : GD = GK : CG$, and, therefore, $GH = \frac{GD \cdot GK}{CG} = \frac{\pi}{2n} \cdot GK$. Hence we have for the moment of flexure of the part

$$DEFG = \frac{\pi}{2n} \cdot GK \cdot \frac{(2\,GK)^3}{12} = \frac{\pi}{3n}\,(GK)^4.$$

If we put the variable angle corresponding to the section GF, $ACG = \phi^\circ$, we shall obtain the ordinate $GK = r \cos.\,\phi$, and for the last moment of flexure $= \frac{\pi}{3n}\, r^4 \cos.\,\phi^4 = \frac{\pi}{3n}\, r^4 \cdot \frac{3 + 4 \cos.\,2\,\phi + \cos.\,4\,\phi}{8}$. The moment of flexure of the half cylinder will be now found, if for ϕ we successively put the values $\frac{1}{n} \cdot \frac{\pi}{2}, \frac{2}{n} \cdot \frac{\pi}{2}, \frac{3}{n} \cdot \frac{\pi}{2}$, &c., to $\frac{n}{n} \cdot \frac{\pi}{2}$, and add the results. But $\frac{\pi\, r^4}{3\,n} \cdot \frac{1}{8} = \frac{\pi\, r^4}{24 n}$ is a common factor; we have, therefore, only to consider the sum of such values, as $3 + 4 \cos.\,2\,\phi + \cos.\,4\,\phi$. The number 3 added n times gives $3\,n$; the sum of all values of the $\cos.\,2\,\phi$ which present themselves, when ϕ is made to increase from 0 successively to $\frac{\pi}{2}$, and, therefore, $2\,\phi$ from 0 to π, equal to 0, because the cosines in the second quadrant are equal and opposite to the cosines in the first; lastly, the sum of all the cosines of all angles from 0 to $2\,\pi = 0$, hence the sum of all values of $3 + 4 \cos.\,2\phi + \cos.\,4\,\phi$ taken between the limits $\phi = 0$ and $\phi = \frac{\pi}{2}$ is $= 3\,n$, and the measure of the moment of flexure of the half cylinder $= \frac{\pi\, r^4}{24 n} \cdot 3\,n = \frac{\pi r^4}{8}$, and, lastly, that of the whole cylinder:

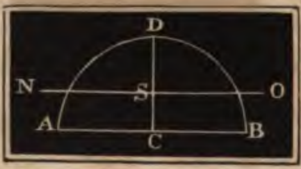
$$W = \frac{\pi}{4}\, r^4 = 0{,}7854\, r^4.$$

For a tube or hollow cylinder with the outer radius r_1 and the inner r_2

$$W = \frac{\pi}{4}\,(r_1^4 - r_2^4).$$

To find the moment of flexure of a body having a semi-circular transverse section ADB, Fig. 216, we may make use of the rule found in § 192, from which the moment about the axis NO passing through the centre of gravity S is equivalent to the moment about the diameter AB, considered as a second axis, less the transverse section F $(= \frac{1}{2}\,\pi r^2)$ times the square of the distance CS of both axes.

Fig. 216.



From this we obtain the moment sought $= \frac{1}{2} \cdot \frac{\pi}{4} r^4 - \frac{1}{2} \pi r^2 \cdot \overline{CS}^2$ $= \frac{\pi r^4}{8} - \frac{1}{2} \pi r^2 \cdot \left(\frac{4r}{3\pi}\right)^2$ (§ 108) $= \pi r^4 \left(\frac{1}{8} - \frac{8}{9\pi^2}\right) = 0{,}110 \cdot r^4$.

§ 196.—*Relative Strength.*—When we know the moment of flexure of a prismatic body, we may determine from it by simple multiplication the *working load* and the *absolute* strength of the body. If a single fibre, or layer of fibres, is extended or compressed to the limits of elasticity, the body has then attained the limits of its tenacity. If we again represent by T the modulus of tenacity and the distance of the furthermost fibre from the neutral axis by e, we shall have $T = \frac{\lambda}{l} E$, and $\frac{\lambda}{l}$, or the relative elongation, $= \frac{e}{\rho}$, hence $\frac{E}{\rho} = \frac{T}{e}$. *If we substitute* $\frac{T}{e}$ *for* $\frac{E}{\rho}$ *in the formula for the moment of flexure, it will then give the statical moment of the tenacity.* We have $Px = SW = \frac{EW}{\rho}$, therefore, also, $Px = \frac{TW}{e}$. It is evident that this moment is a maximum when $x = l$, or when the arm $= l$; from this we may conclude, that at the extremity where the beam is fixed, the greatest flexure ensues, and the limit of elasticity is first attained. Accordingly, the *working load of a beam* is determined by the formula

$$P = \frac{TW}{e\,l}.$$

In like manner, the *strength*, or the resistance to rupture of the beam, may be determined. If a fibre is strained to the point of rupture, the breaking of the whole beam takes place, because the beam has now a section smaller by the section of these fibres, and therefore a greater deflexion ensues, and thus a rupture of the succeeding fibres or layer of fibres follows. If we put the modulus of strength $= K$, we have $\frac{E}{\rho} = \frac{K}{e}$, and, therefore, the force for the rupture of the beam:

$$P = \frac{KW}{e\,l},$$

In a uniform rectangular beam, the distance of the outermost lamina of fibres from the neutral axis $= \frac{h}{2}$, hence the formula $Pl = \frac{E}{\rho} \cdot \frac{bh^3}{12}$ (§ 191) gives the resistance to rupture

$$P = \frac{2K}{h} \cdot \frac{bh^3}{12\,l} = \frac{bh^2}{6\,l} \cdot K.$$

If the beam is hollow, as in Fig. 212, we have $P = \frac{bh^3 - b_1h_1^3}{6\,hl} \cdot K$, so that the formula also holds good for a body, as in Fig. 213, hollowed out at the sides.

In a prismatic body of a triangular cross section, as in Fig. 210, $e = \frac{2}{3}h$, hence $P = \frac{K}{\frac{2}{3}h} \cdot \frac{bh^3}{36\,l} = \frac{bh^2}{24\,l} \cdot K$. According to this, rectangular beams for a similar section have twice the tenacity of triangular beams.

For a cylinder of radius r, $e = r$, therefore,

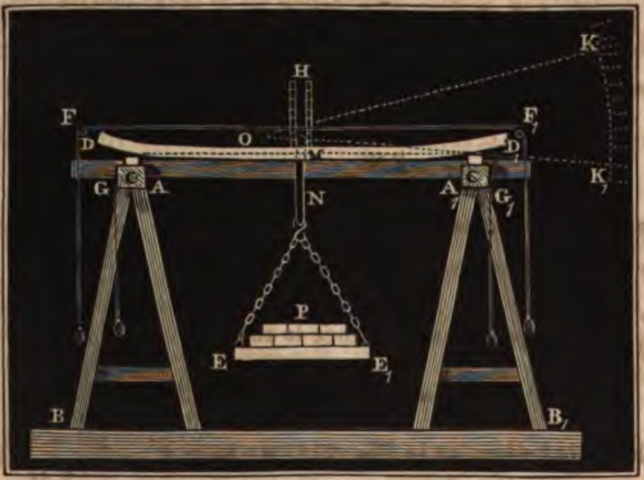
$$Pl = \frac{K}{r} \cdot \frac{\pi}{4} r^4 = \frac{\pi}{4} r^3\, K.$$

If the cylinder is hollow, we have $Pl = \frac{\pi}{4}\left(\frac{r_1^4 - r_2^4}{r}\right) K.$

If we substitute the modulus of the working load T for that of the strength, or for K, an aliquot part, *i. e.* $\frac{1}{10}$th, the working load is given by the formula already found.

§ 197. *Experiments.**—To find the deflexion and tenacity of beams, we may make use of the experimental values for E and T in § 186; but as concerns the strength of beams, it is safer to replace the modulus of strength there given and derived from experiments on tensile strain, by those values of K which have been found from experiments on compression. A perfect accordance cannot exist between the moduli found by these two methods, because in rupture, not only an extension, but also a compression takes place, and both of these not only in the direction of the axis, but also in the transverse section, though here not to the same amount. Besides, many other circumstances affect the elasticity, tenacity and strength of bodies, on which account, considerable variations in the results always present themselves. Timber, for example, is stronger at the core and at the root than at the sap and the top. Timber will also bear a greater strain when the force acts perpendicular to the annual rings,

Fig. 217.



---

* See Appendix.

than when parallel to them. Lastly, the soil and the situation where it has grown, temperature, dryness, age, &c., affect the resistance of woods. Besides, the deflexion of a body after it has been loaded for a long time, is always somewhat greater than on the immediate application of the load.

Experiments upon elasticity and strain were made by *Eytelwein* and *Gerstner*, with the apparatus represented in Fig. 217. AB A_1B_1 are two tressels, C and C_1 two iron supports. DD_1 the rectangular beam for experiment resting upon them. The load P for the flexure of the body lies upon a scale-pan EE_1 suspended to a stirrup MN, whose upper and rounded extremity lies in the middle M of the beam. In order to find the deflexion corresponding to a load P, *Eytelwein* applied two fine horizontal threads FF_1 and GG_1 and likewise a scale M resting upon the middle of the beam; *von Gerstner*, on the other hand, availed himself of a long one-armed delicate lever OK, whose fulcrum was at M, and whose extremity, like the hand of a watch, indicated upon a vertical scale KK_1 the deflexion of M to fifteen times its amount.

Remark. Experiments on elasticity, &c., have been made by Banks, Barlow, Buffon, Burg, Ebbels, Eytelwein, Finchan, von Gerstner, Gauthey, Muschenbroek, Rennie, Rondelet, Tredgold, &c. An ample summary of these, and besides a theory somewhat different from the above, is given by Burg in the 19th and 20th vols. of the "Jahrbücher des polytechnischen Instituts in Wien." The experiments of Eytelwein and von Gerstner are described in Eytelwein's "Handbuch der Statik fester Körper," vols. ii., and in von Gerstner's "Handbuch der Mechanik," vol. i. The Treatise printed from the transactions of the Association of Prussian Industry, "Elementare Berechnung des Widerstandes prismatischer Körper gegen Biegung," by Brix, has been used for the preparation of the foregoing article.

§ 198. *Modulus of Relative Strength.*—The following table contains the mean values of the modulus of rupture for several bodies met with in the arts. To find, with the assistance of these, the pressures which bodies can sustain with safety for a long duration, we must put for wood the tenth, for metals and stones, from the third to the fourth of K.*

TABLE II.

THE MODULUS OF FRACTURE OR MODULUS OF STRENGTH FOR THE FLEXURE OF BODIES.

Names of Substances.	Modulus of Fracture K.	Names of Substances.	Modulus of Fracture K.
Box	10000 to 24000	Elm	6000 to 12000
Oak	8000 " 24000	Cast Iron	24000 " 56000
Pine	8000 " 13000	Limestone	700 " 1700
Scotch Fir	7000 " 17000	Sandstone	600 " 800
Deal	7000 " 14000	Brick	180 " 340

According to this, we may assume for wood as a mean $K = 12000$ and for cast-iron $K = 40000$ pounds, and we shall then obtain for a rectangular beam imbedded in a wall at one extremity and loaded at the other:

---

* See Appendix.

1. $Pl = 200 \,.\, bh^2$, if it consist of wood, and tenfold security be allowed.
2. $Pl = 1000 \,.\, bh^2$, if the beam be of cast-iron, and fourfold security be given.

If the body be cylindrical, we then have for wood

3. $Pl = 950\, r^3$, and for cast-iron
4. $Pl = 4700\, r^3$.

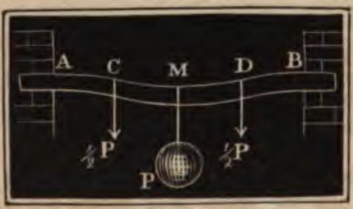
P, l, b, h, r, have the denominations hitherto used.

For wrought iron K is taken 20 per cent. less, because this bends more than cast iron; here therefore we must put

$$Pl = 800\, bh^2 = 3600\, r^3.$$

If the load Q be uniformly distributed over the beam, the beam will bear as much again, wherefore the above co-efficients must be *doubled.* If the beam rest at its extremities on points of support, whose distance is l, and if the load P act in the middle between these points, then for P we must put $\frac{P}{2}$ and for l, $\frac{l}{2}$, wherefore Pl becomes $\frac{Pl}{4}$, and the the tenacity quadrupled. But if the load between the points be uniformly distributed over the beam, we then shall have for the pressure $\frac{Q}{2}$, which acts from below upwards at a point of support, the moment $\frac{Q}{2} \,.\, \frac{l}{2}$; and for the opposite pressure $-\frac{Q}{2}$ as the half of the load pulling downwards at the centre of gravity, the moment $-\frac{Q}{2} \cdot \frac{1}{2} \cdot \frac{l}{2} = -\frac{Ql}{8}$; hence there will remain as the pressure for rupture at the middle, the moment $\frac{Ql}{4} - \frac{Ql}{8} = \frac{Ql}{8}$, and therefore $Ql = 8 \,.\, \frac{bh^2}{6} K$, also $= 8 \,.\, \frac{\pi}{4} br^3 K$, therefore the strength or tenacity is twice as great as if the load acted at the middle, and eight times as great as if it pulled downwards at one extremity whilst the other remained fixed.

Fig. 218.



If a beam, Fig. 218,[*] is imbedded in a wall at both extremities, or if its extremities are fixed, then the beam sustains as much again as if it rested freely at its extremities; for in this case the greatest flexure is not only in the middle, but likewise at the extremities; the beam, therefore, breaks at the same time in the middle and at the extremi-

---

[*] See Appendix.

Let us now consider the case of a beam fixed at each extremity and loaded in the middle. We may obtain the [illegible] of [illegible] of [illegible] upon the beam by two forces P, and P_1 one acting at the point of loading and another at a distance from the first end (…) distance is undetermined [illegible] [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

found [illegible] [illegible] [illegible] [illegible] integrating

we come to the point where the

being tangential $\frac{dy}{dx} = 0$ ∴ C = 0 ∴ C =

∴ Integrating again we will get

3) $Ey = P(\frac{lx^2}{4} - \frac{x^3}{6}) - P_1(\frac{lx^2}{2} - \frac{x^3}{6}) + P(\frac{l_1x^2}{2} - \frac{x^3}{6}) + B$. for the

constant make $x = 0$ then $y = 0$ ∴ $B =$

Having now found this expression

for the flexure of the first curve we

now pass to the consideration of the

second. Let us now take a

section at a distance from the end

of the first half the second being

unaffected by the forces P_1 and P_2

equal to x_1 then the moment of

flexure or expression will be equal

1) $E\frac{d^2y_1}{dx_1^2} = -P_1(l - x_1) + P_2(l_1 - x_1)$. Integrating

2) $E\frac{dy_1}{dx_1} = -P_1(lx_1 - \frac{x_1^2}{2}) + P_2(l_1x_1 - \frac{x_1^2}{2}) + C$. To find

the value of this constant we

find at the point where the two

curves have a common tangent,

this will be the point where x

$= x_1$ and $\frac{dy}{dx} = \frac{dy_1}{dx_1}$ hence the x_1 in [illegible] equation is the same as the x in the new equation of our last curve and equating them since $\frac{dy_1}{dx_1} = \frac{dy}{dx}$ we have $-P_1(lx-\frac{x^2}{2})+P_2(l_1x-\frac{x^2}{2})+C = P(\frac{lx}{2}-\frac{x^2}{2})-P_1(lx-\frac{x^2}{2})+P_2(l_1x-\frac{x^2}{2})$. The P_2 and P_1 cancel out leaving $C = P(\frac{lx}{2}-\frac{x^2}{2})$, but x for the point where these curves have a common tangent $=\frac{l}{2}$ since it is at this point that the curves have a point in common, i.e. substituting this value for x in the value for our constant it becomes $C = P(\frac{l^2}{4}-\frac{l^2}{8}) = \frac{Pl^2}{8}$, putting in this value of the constant and placing it first we have integrating again $Ey = \frac{Pl^2x_1}{8} - P_1(\frac{lx_1^2}{2}-\frac{x_1^3}{6})+P_2(\frac{l_1x_1^2}{2}-\frac{x_1^3}{6})+C_1$. To find the value of this constant make $x = \frac{l}{2}$ then we are at the point of application of the force P for which point $y = y_1$ and therefore our last equation and the equation (3) of our previous curve are equal, $x = x_1$ ∴ $\frac{Pl^2x}{8} - P_1(\frac{lx^2}{2}-\frac{x^3}{6}) + P_2(\frac{l_1x^2}{2}-\frac{x^3}{6}) + C_1 = P(\frac{lx^2}{4}-\frac{x^3}{6}) + P_1(\frac{lx^2}{2}-\frac{x^3}{6}) + P_2(\frac{l_1x^2}{2}-\frac{x^3}{6})$ ∴ $C_1 = P(\frac{lx^2}{4}-\frac{x^3}{6}-\frac{l^2x}{8})$ putting in x its value $\frac{l}{2}$ we have $C_1 = P(\frac{l^3}{16}-\frac{l^3}{48}-\frac{l^3}{16}) = -P\frac{l^3}{48}$.

Putting [illegible] we have $Ey = -\frac{Pl^3}{48} + \frac{Pl^2x}{8} - P_1\left(\frac{lx^2}{4} - \frac{x^3}{6}\right) + P_2\left(\frac{lx^2}{4} - \frac{x^3}{6}\right)$ (3) [illegible] quantities P_1 and P_2, also l_1, [illegible] values of these quantities [illegible] P_1 [illegible] beam [illegible] $y = y_1$ [illegible] $= 0$, as [illegible] at $\frac{dy}{dx} = 0$ [illegible] and [illegible] called

$$\frac{Pl^3}{8} - \frac{P_1l^2}{2} + P_1\left(ll_1 - \frac{l_1^2}{2}\right) = 0 \quad -\frac{Pl^3}{48} + \frac{Pl^3}{8} - \frac{P_1l^3}{5} + P_1\left(\frac{l_1l^2}{2} - \frac{l^3}{6}\right) = 0$$

[illegible] to eliminate P_1 from each of these equations [illegible] the coefficients of P_1 equal hence multiplying (1) by $\frac{2}{3}l$, it gives $\frac{Pl^3}{12} - \frac{P_1 l^3}{3} + P_2(\frac{2l_1l^2}{3} - \frac{l^3}{3}) = 0$, $\frac{5}{48}Pl^3 - \frac{P_1l^3}{8} + P_2(\frac{l_1l^2}{2} - \frac{l^3}{6}) = 0$ and subtracting $\frac{Pl^3}{48} + P_2(-\frac{l_1l^2}{6} + \frac{l^3}{6}) = 0$ ∴ we obtain for the value of P_2, $P_2 = \frac{\frac{Pl^3}{48}}{\frac{l_1l^2-l^3}{6}} = \frac{Pl^3}{8(l_1l^2-l^3)} = \frac{Pl}{8(l_1-l)}$. Substituting this value in (2) we have $P\frac{l^3}{12} - P_1\frac{l^3}{3} + P\frac{l}{8(l_1-l)}(\frac{2l_1l^2}{3} - \frac{l^3}{3})$ multiplying the last two it reduces to $P\frac{l^3}{12} - P_1\frac{l^3}{12} + P(\frac{2l_1l^3}{24(l_1-l)} - \frac{l^4}{24(l_1-l)})$ ∴ by transposition we obtain the value of P_1 another of the unknown quantities $P_1\frac{l^3}{3} = P(\frac{l^3}{12} + \frac{2l_1l^3}{24(l_1-l)} - \frac{l^4}{24(l_1-l)}) = P(\frac{4l_1l^3 - 2l^4 + 2l_1l^3 - l^4}{24(l_1-l)})$ $= P\frac{l^3}{24}(\frac{4l_1-3l}{l_1-l})$ ∴ $P_1 = P\frac{l^3}{24} \times \frac{3}{l^3}(\frac{4l_1-3l}{l_1-l}) = \frac{P}{8}\,\frac{4l_1-3l}{l_1-l}$

We have however this value of P_1 in terms of another unknown quantity l_1, but we will see in the course of further deduction that this "takes care of itself" for l_1 disappears. It is but an equality of moments that we make in this discussion & when we substitute these values of P_1 and P_2 either equation (3) or (4) we will eliminate them both. l_1 disappears as we have said in the reduction giving an equation [illegible] in the flexure curve [illegible] P and l. Let us [illegible] substitute in equation 3 first curve [illegible].

$$Ey = P\left(\frac{lx^2}{4} - \frac{x^3}{6}\right) - P/8 \cdot \frac{4l_1 - 3l}{l_1 - l}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + \frac{Pl}{8(l_1 - l)}\left(\frac{l_1x^2}{2} - \frac{x^3}{6}\right)$$

$$Ey = P\left(\frac{lx^2}{4} - \frac{x^3}{6} - \left(\frac{4l_1 - 3l}{8(l_1 - l)}\right)\frac{lx^2}{2} \underset{+}{+} \frac{4l_1 - 3l}{8(l_1 - l)} \cdot \frac{x^3}{6} + \frac{l}{8(l_1 - l)} \cdot \frac{l_1x^2}{2} - \frac{l}{8(l_1 - l)} \cdot \frac{x^3}{6}\right)$$

$$= P\left(\frac{lx^2}{4} - \frac{x^3}{6} - \frac{lx^2(4l_1 - 3l)}{16(l_1 - l)} + \frac{x^3(4l_1 - 3l)}{48(l_1 - l)} + \frac{ll_1x^2}{16(l_1 - l)} - \frac{lx^3}{48(l_1 - l)}\right)$$

$$= P\left(\frac{lx^2}{4} - \frac{x^3}{6} - \frac{4ll_1x^2 - 3l^2x^2}{16(l_1 - l)} + \frac{4l_1x^3 - 3lx^3}{48(l_1 - l)} + \frac{l_1lx^2}{16(l_1 - l)} - \frac{lx^3}{48(l_1 - l)}\right)$$

$$= P\left(\frac{lx^2(12(l_1 - l)) - 8x^3(l_1 - l) - 12ll_1x^2 + 9l^2x^2 + 4l_1x^3 - 3lx^3 + 3ll_1x^2 - lx^3}{48(l_1 - l)}\right)$$

$$= P\left(\frac{-12l^2x^2 + 9l^2x^2 - 8l_1x^3 + 4l_1x^3 + 8lx^3 - 3lx^3 - lx^3 + 3ll_1x^2}{48(l_1 - l)}\right)$$

$$= P\left(\frac{-3l^2x^2 - 4l_1x^3 + 4lx^3 + 3ll_1x^2}{48(l_1 - l)}\right)$$

$$= P\left(\frac{3x^2l(l_1 - l) - 4x^3(l_1 - l)}{48(l_1 - l)}\right) = P\left(\frac{(3x^2l - 4x^3)(l_1 - l)}{48\,(l_1 - l)}\right)$$

$$Ey = P \cdot \frac{3x^2l - 4x^3}{48} = P\,\frac{x^3(3l - 4x)}{48}$$

and the maximum deflection is at the point of ap-plication of the weight for which $x = \frac{l}{2}$

$$\therefore Ey = P \cdot \frac{\frac{3}{4}l^2 l - \frac{4l^3}{8}}{48} = P\,\frac{\frac{3}{4}l^3 - \frac{2}{4}l^3}{48} = P\,\frac{l^3}{192}$$

There is another point about this case of a beam built in at both ends and that is to find the points of contrary flexure, or the point at which the curve formed by the beam changes from convex to concave on the upper side and Vice Versa.

[illegible]

[illegible]

[illegible]

[illegible] that [illegible]

[illegible] corresponded [illegible]

[illegible]

[illegible]

[illegible] singularity [illegible]

[illegible]

[illegible]

[illegible]

[illegible] and [illegible]

[illegible] singularity [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

ties; whilst at the intermediate points C and D, where the convexity passes into concavity, no flexure at all ensues. Consequently, for a portion AC, the pressure $= \frac{P}{2}$, the arm $= \frac{l}{4}$, and the moment $= \frac{P}{2} \cdot \frac{l}{4} = \frac{Pl}{8}$. If, finally, in this last case the load Q is uniformly distributed over the beam, the moment presents itself $= \frac{Ql}{16}$, because we may suppose, that the one half of Q is immediately sustained by the points of support, and that the other half acts in the middle of Q.

The weight G of a beam acts exactly as if the load Q were distributed uniformly over the beam; for a beam fixed at one extremity, therefore, the moment $= Pl + \frac{1}{2} Gl$; but for a beam resting on both extremities and loaded in the middle, it is $= \frac{P}{2} \cdot \frac{l}{2} + \frac{G}{2} \cdot \frac{l}{2} - \frac{G}{2} \cdot \frac{l}{4} = (P + \frac{1}{2} G) \frac{l}{4}$, &c.

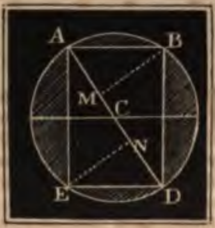
Example.—1. A rectangular beam of fir, 7 inches thick and 9 inches in depth, is to rest on both its extremities, so that the distance of the points of support may amount to 20 feet; what load, suspended from the middle, will it sustain? $b = 7$, $h = 9$, $l = 20$ feet $= 240$ inches; hence $240 \,.\, P = 4 \,.\, 200 \,.\, 7 \,.\, 9^2$; consequently this load $P = 70 \,.\, 27 = 1890$ lbs.—2. A round wooden water-wheel, and its axle, 10 feet long, is to sustain at the wheel, together with its own weight, a uniformly distributed load $Q = 10000$ lbs.; what diameter must the wheel have? $Ql = 10000 \,.\, 120 = 1200000, = 8 \,.\, 950 \,.\, r^3$, or $r^3 = \frac{1200000}{8 \,.\, 950} = 157{,}9$; hence the radius sought $r = \sqrt[3]{157{,}9} = 5{,}4$ inches, and the diameter of the axle $2r = 10{,}8$ inches, for which we may assume one foot.—3. To what height may the corn in a granary be heaped up if the bottom rest upon beams of 25 feet in length, 10 inches in breadth, and 12 in depth, the distance between the axes of any two beams $= 3$ feet, and one cubic foot of corn weighs 48,5 lbs.? If we apply the formula $Ql = 16 \,.\, 200 \,.\, bh^2$, we must put $b = 10$, $h = 12$, $l = 25 \,.\, 12 = 300$; consequently $Q = \frac{16 \,.\, 200 \,.\, 10 \,.\, 144}{300} = 15360$ lbs. A parallelopiped, 25 feet long, 3 feet broad, x feet deep, weighs $= 25 \,.\, 3 \,.\, x \,.\, 48{,}5$ lbs.; hence, if we put this value $= Q$, it follows that $x = \frac{15360}{75 \,.\, 48{,}5} = 4{,}22$ feet, the requisite height to which the grain may be heaped up.

§ 199. *Strongest Beams.*—Bodies of equal section very often possess different relative strengths; the formula $Pl = \frac{K}{6} \,.\, bh^2$ shows that the *strength increases, as the breadth, as the square of the depth, and inversely as the length of the beam.* The depth has consequently a greater influence upon the tenacity than the breadth; a beam of double the breadth bears twice as much, *i. e.* as much as two single beams; on the other hand, a beam of double the depth, four times that of a beam of the same depth. For this reason beams are made, namely, when they are of cast iron, much deeper than broad; they are hollowed out near the middle, and what is taken away replaced by parts at a greater distance from the neutral axis; but this rule must be particularly attended to, viz., always to lay the beam on the least side, or rather so to lay it, that the pressure may act in the direction of the greater side.

The strength of a round trunk, or of any other cylindrical body, is $P = \frac{\pi}{4} \cdot \frac{r^3}{l} K$, that of a square with equal breadths and depths $2r$, $= P_1 = \frac{2r \cdot (2r)^2}{l} \cdot \frac{K}{6} = \frac{4}{3} \cdot \frac{r^3}{l} K$; if we compare both pressures with each other, $\frac{P}{P_1} = \frac{\pi}{4} \cdot \frac{3}{4} = 0,588$; the cylindrical body has, therefore, only about 59 per cent. the strength of a beam having a square transverse section. Wooden beams are hewn or cut from round trunks of trees, and thereby are much weakened. But the question now is, which is the strongest form of beam that can be cut from a cylindrical trunk?

Let *ABDE*, Fig. 219, be the section of the trunk, $AD = d$ its diameter, further $AB = DE = b$ the breadth, and $AE = BD = h$ the depth of the beam. Then $b^2 + h^2 = d^2$, or $h^2 = d^2 -$
moment of ruptu

Fig. 219.



$(b \pm x) d^2 - (b$
provided we neglect
$x + 3bx^2$. That the
than the last, the di
positive, whether we t ut this is only
possible if $d^2 - 3b^2 =$ ce then $= 3bx^2$, therefore
positive, whereas, if is a real positive or negative value, $3bx^2$ may be neglected, and the difference may be put $= \mp (d^2 - 3b^2) x$, which if x has the same sign, is at one time positive, at another negative. But if we put $d^2 - 3b^2 = 0$, we obtain the breadth sought $b = d\sqrt{\frac{1}{3}}$, and the corresponding depth $h = \sqrt{d^2 - b^2} = d\sqrt{\frac{2}{3}}$; therefore, the ratio of the depth to the breadth: $\frac{h}{b} = \frac{\sqrt{2}}{\sqrt{1}} = 1,414$ or about $\frac{7}{5}$. The trunk must be so fashioned that it shall produce a beam whose depth to its breadth is as 7 to 5. To find the section corresponding to greatest strength, let us divide the diameter *AD* into three equal parts, raise at the points of division *M* and *N* perpendiculars *MB* and *NE*, and finally connect the points of intersection *B* and *E* by the circle with the extremities *A* and *D* of the straight line *AD*. *ABDE* is the section of greatest resistance; for since $AM : AB = AB : AD$ and $AN : AE = AE : AD$, $AB = b = \sqrt{AM \cdot AD} =$

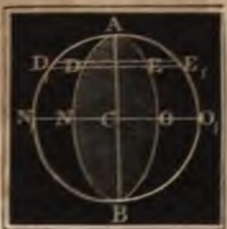
$\sqrt{\frac{1}{3} d \,.\, d} = d\sqrt{\frac{1}{3}}$ and $AE = h = \sqrt{AN \,.\, AD} = \sqrt{\frac{2}{3} d \,.\, d} = \sqrt{\frac{2}{3}}$, therefore $\frac{h}{b} = \frac{\sqrt{2}}{1}$, which is actually requisite.

Remark. The trunk has the moment of rupture $Pl = \frac{\pi K}{4} . r^3$, but the beam of greatest resistance formed from it $Pl = \frac{K}{6} . d\sqrt{\frac{1}{3}} . \frac{2}{3} d^2 = \frac{K}{\sqrt{243}} . d^3 = \frac{8\,K}{\sqrt{243}} r^3$; the trunk, therefore, loses by squaring about $1 - \frac{8}{\sqrt{243}} . \frac{4}{\pi} = 1 - 0{,}65 = 0{,}35$, *i. e.* 35 per cent. of its strength. To spare this loss, the trunk is often hewed not quite square, but the corners rounded off. A beam with a square section formed from the same trunk, has the moment $Pl = \frac{K}{6} . d\sqrt{\frac{1}{2}} . \frac{d^2}{2}$, because here the breadth = the depth $= d\sqrt{\frac{1}{2}} = 0{,}707\, d$, hence the loss here $= 1 - \frac{8}{6 \,.\, 2\sqrt{2}} . \frac{4}{\pi} = 1 - \frac{8}{3\pi\sqrt{2}} = 1 - 0{,}60 = 0{,}40$ *i. e.* 40 per cent.

§ 200. *Hollow and Elliptical Beams.*—Very frequently bodies are hollowed at the inside or outside, and provided with ribs or flanges, either with a view to save material, or what comes to the same thing, to gain in strength. For a hollow rectangular beam of iron $P = 1000 \,.\, \frac{bh^3 - b_1 h_1^3}{lh}$, the hollow may be of the depth h_1 and breadth b_1, made within or without at the sides. For a hollow cylindrical body $P = 4700 \,.\, \frac{r_1^4 - r_2^4}{l r_1}$. In such cases the thickness of the solid part $r_1 - r_2$ is commonly made $= \frac{2}{5}$ of the outer radius r_1; whence it follows: $P = 4700 \,.\, \frac{r_1^4 - (0{,}6 r_1)^4}{l r_1} = 4700 \,.\, \frac{0{,}8704\, r_1^3}{l} = 4090\, \frac{r_1^3}{l}$. An equally heavy solid cylinder has the radius $r = \sqrt{r_1^2 - r_2^2} = \sqrt{r_1^2 - 0{,}36\, r_1^2} = 0{,}8\, r_1$; hence its moment of resistance $= 4700 \,.\, (0{,}8\, r)^3 = 2406\, r^3$, namely, about 41 per cent. less than that of the hollow cylinder.

We gain also in strength, when, instead of a cylinder, we apply a prismatic body with an elliptical section, and place its greater axis upright or parallel to the direction of the pressure. If we suppose a circle AO_1BN_1 whose radius $CA = CB = a$ the semi-axis major, described about this elliptical section $AOBN$, Fig. 220, the strength of resistance of the body having an elliptical section may be calculated simply from that having a circular section. The length of any element DE of the elliptic elements parallel to its minor axis $NO = 2\,b$ is always $\frac{b}{a}$ of the length of the circular element D_1E_1; but now the elasticity and strength are proportional to these dimensions singly; therefore, also the strength of the elliptic element to that of the circular element, is as b to a, and, finally, the

Fig. 220.



strength for the whole ellipse $= \frac{b}{a}$ times the strength of the whole circle, *i. e.*

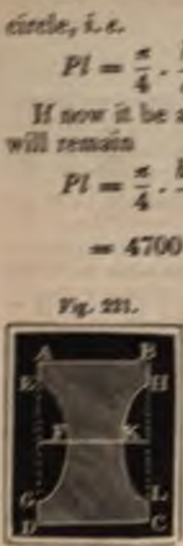
$$Pl = \frac{\pi}{4} \cdot \frac{b}{a} \cdot a^3 K = \frac{\pi}{4} b\, a^2 K, \text{ for cast iron } = 4700\, a^2 b.$$

If now it be an elliptical hollowing whose axes are a_1 and b_1, there will remain

$$Pl = \frac{\pi}{4} \cdot \frac{ba^3 - b_1 a_1^3}{a} K, \text{ for cast iron,}$$

$$= 4700 \cdot \frac{a^3 b - a_1^3 b_1}{a}.$$

Fig. 221.



If, lastly, a body having a rectangular section $ABCD = bh$, Fig. 221, be hollowed at the flanks by the semi-ellipses EFG, HKL, and if the semi-axes of these are $= a_1$ and b_1, we shall have then

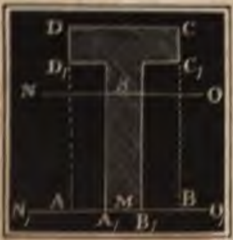
$$Pl = bh_1 \frac{K}{6} - \frac{\pi b_1 a_1}{4} K = \frac{2\, b\, h^3 - 3\, \pi\, b_1 a_1^3}{12\, h} \cdot K;$$

for cast iron

$$Pl = 200 \cdot \frac{bh^3 - 4{,}712\, a_1^3 b_1}{h}.$$

Examples.—1. A transverse beam of oak, 9 inches broad and 11 inches deep, of known sufficient tenacity, is to be replaced by a hollow cast iron beam, of 5 inches in outer breadth and 10 in depth; of what thickness of metal must it be cast? Let this thickness $= x$, we have then for the breadth of the hollowing $= 5 - x$, and its depth $= 10 - x$; consequently, for the hollow beam $b_1 h_1^3 - b_2 h_2^3 = 5 \cdot 10^3 - (5 - x)(10 - x)^3 = 2500\, x - 450\, x^2 + 35\, x^3 - x^4$. Since the moment of resistance of the wooden beam $= 200 \cdot 9 \cdot 11^2 = 217800$, we shall have to put: $\frac{1000}{10}(2500\, x - 450\, x^2 + 35\, x^3 - x^4) = 217800$, or $2500\, x - 450\, x^2 + 35\, x^3 - x^4 = 2187$. As a first approximation $x = \frac{2178}{2500} = 0{,}9$ inches. But this value gives $450 \cdot x^2 = 450 \cdot 0{,}81 = 364{,}5$; $35\, x^3 = 25{,}5$, $x^4 = 0{,}7$; we may therefore put: $x = \frac{2178 + 364{,}5 - 25{,}5 + 0{,}7}{2500} = \frac{2517{,}7}{2500} = 1{,}01$ inch for the requisite thickness of iron.—2. If in a T-shaped girder of cast iron, the breadth $AB = CD = b$ is equal to the depth h and the thickness $A_1 B_1 = CC_1 = \frac{1}{5} b$, therefore $b_1 = \frac{4}{5} b$, and $h_1 = \frac{4}{5} b$; we shall then have for the moment of resistance (§ 193):

Fig. 222.



$$Pl = \frac{K}{12} \cdot \frac{(bh^2 - b_1 h_1^2)^2 - 4\, bhb_1 h_1\, (h - h_1)^2}{(bh - b_1 h_1)\, e},$$

or by substituting $e = \frac{1}{2} \cdot \frac{bh^2 - b_1 h_1^2}{bh - b_1 h_1}$,

$$Pl = \frac{K}{6} \cdot \frac{(bh^2 - b_1 h_1^2)^2 - 4\, bb_1\, hh_1\, (h - h_1)^2}{bh^2 - b_1 h_1^2}$$

$$= 1000 \cdot \frac{(b^3 - 0{,}512\, b^3)^2 - 4 \cdot 0{,}64\, b^4 \cdot (b - 0{,}8\, b)^2}{b^3 - 0{,}512\, b^3}$$

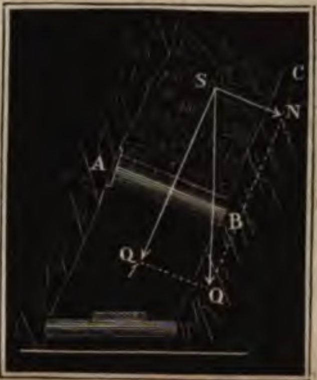
$$= 1000 \cdot \frac{0{,}2381 - 2{,}56 \cdot 0{,}04}{0{,}488} \cdot b^3 = \frac{135{,}6}{0{,}488} \cdot b^3 = 278 \cdot b^3.$$

If, now, such a girder, 4 feet in length, rest on both its extremities, and is to bear a load in its middle of 7400 lbs., Pl would then $= 7400 \cdot 4 \cdot 12 = 355200$, and therefore, $4 \cdot 278\, b^3 = 355200$; whence we should have the extreme depth and breadth $b = h = \sqrt[3]{\frac{355200}{1112}} = 6{,}84$ in. and the thickness of iron $\frac{1}{5} b = 1{,}35$ inches.

This is an [illegible] used in the [illegible] with accompanying fig. [illegible] evidently [illegible] [illegible] the fact that the distance in the expression for the moment of inertia is not the same as distance of the farther fibre. The moment of inertia of the parallelogram is $\frac{ah^3}{72}$ and that of the ellipse is $\frac{\pi}{4}a^3$ and from the symmetry of the fig. the moment of inertia of the beam equals the difference of these two moments, hence our formula for rupture is $R = \frac{Kr}{[illegible]}$ [illegible] and in this case [illegible] $R = [illegible] = \frac{[illegible]}{12h}K(ah^2 - 3\pi a a^3)$. Referring back to our table of moduli of rupture we find that of [illegible], cast iron, after making proper allowance for security [illegible] [illegible] [illegible] is [illegible] [illegible]

§ 201. *Oblique Pressure.*—If the pressure P act obliquely to the axis of a beam, which for example is inclined to the horizon whilst the pressure acts vertically, we have then only to take into account its components directed at right angles to the axis. If, for example, the inclined stretcher AB, Fig. 223, supports an accumulated load Q, this may be decomposed into the components Q_1 and N, and for an inclination α to the horizon of the stretcher, the pressure Q_1, counteracted by the stretcher $= Q \,.\, \cos.\, \alpha$, and the pressure N counteracted by the lateral wall $BC = Q \sin.\, \alpha$. Taking the friction into account $= Q_1 = Q \,.\, (\cos.\, \alpha - f \sin.\, \alpha)$ and hence for a round stretcher: $Q\,(\cos. \quad - f \sin.\, \alpha) = 8 \,.\, \frac{950\, r^3}{l}$, r being the radius and l the length of the stretcher.

Fig. 223.

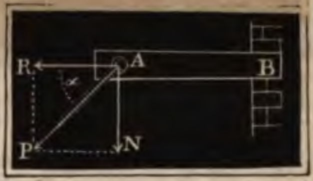


If the pressure P be applied directly to the beam AB, Fig. 224, deviating from the axis by the angle $PAR = \alpha$, two components present themselves, $N = P \sin.\, \alpha$ and $R = P \cos.\, \alpha$, of which the one brings into play the relative, and the other the absolute elasticity of the beam. If F be the cross section of the beam, every unit of it is stretched by the force $\frac{P \cos.\, \alpha}{F}$, and, therefore, the modulus of elasticity K must be taken at $\frac{P \cos.\, \alpha}{F}$ less; therefore, we must substitute for K, $K - \frac{P \cos.\, \alpha}{F}$, whence it follows that:

$$P \sin.\, \alpha = \left(K - \frac{P \cos.\, \alpha}{F}\right) \frac{W}{el};$$

therefore, for a rectangular beam $P \sin.\, \alpha = \left(K - \frac{P \cos.\, \alpha}{F}\right) \frac{bh^2}{6\, l}$, and the pressure for rupture: $P = \frac{K}{\frac{6\, l \sin.\, \alpha}{bh^2} + \frac{\cos.\, \alpha}{F}}$. For $\alpha^\circ = 90^\circ$, $\sin.\, \alpha = 1$, $\cos.\, \alpha = 0$, hence $P = \frac{K}{6} \,.\, \frac{bh^2}{l}$, for $\alpha^\circ = 0$, $\sin.\, \alpha = 0$, $\cos.\, \alpha = 1$, hence $P = KF$, as it should be, for we have here only to consider the absolute strength.

Fig. 224.



Example. What distance from each other must the 10-inch stretchers of ASB, Fig. 222, be laid, if it be 4½ feet wide, and run for 60 feet up a vein having a slope or inclination of 70°; the weight of a cubic foot of the ground to be supported being 65 lbs., the co-efficient of friction upon the supports is taken at $\frac{1}{3}$? Let x feet be the distance of two rafters, the weight sustained by one rafter $= 4{,}5 \,.\, 60 \,.\, 65\, x = 17550 \,.\, x$ lbs., and from the theory of the inclined plane, this rafter will have only to sustain the pressure $Q_1 = (\sin. 70^\circ - \frac{1}{3} \cos. 70) \,.\, 17550\, x = (0{,}9397 - 0{,}1140) \,.\, 17550\, x = 0{,}8257 \,.\, 17550\, x = 14492\, x$ lbs. But the rafter sustains $8 \,.\, 950 \,.\, \frac{r^3}{l} = 8 \,.\, \frac{950 \,.\, 5^3}{54} = 17592$; we must, therefore, put: $14492\, x = 17592$, and $x = \frac{17592}{14492} = 1{,}214$ feet $= 14{,}6$ inches. We must, therefore, only leave an interval between any two rafters of 14,6 inches.

§ 202. *Loading beyond the middle.*—If a pressure P acts upon a beam, supported at both ends, not at the middle, but at a point D at distances $D\mathfrak{A} = l_1$ and $DB = l_2$ from the points of support, the beam then can bear a greater load. According to the equality of certain statical moments, the point of support $\mathfrak{A}$ sustains the pressure $P_1 = \frac{l_2}{l_1 + l_2} P$, and the point B the pressure $P_2 = \frac{l_1}{l_1 + l_2} P$, hence the moment of rupture at the point of application $D = D\mathfrak{A} \,.\, P_1 = DB \,.\, P_2 = \frac{l_1 l_2 P}{l_1 + l_2}$. For any other point E this moment $EB \,.\, P_1$ is less, because the arm EB is less than the arm $DB = l_2$; the greatest deflexion also takes place at D, and fracture first occurs at this point. Accordingly we must put $\frac{P l_1 l_2}{l_1 + l_2} = \frac{KW}{e}$ or the whole length $l_1 + l_2$ being represented by l, $\frac{P l_1 l_2}{l} = \frac{K}{6} \,.\, bh^2$, if the beam is rectangular. The pressure $P = \frac{K}{6} \,.\, \frac{l}{l_1 l_2} \,.\, bh^2$ is moreover $= \infty$, when l_1 or l_2 very nearly $= 0$, and is infinitely less, the more l_1 and l_2 approach to equality. If, lastly, $l_1 = l_2$, *i. e.* if the pressure P acts in the middle of the beam, P becomes a minimum, because, if we put $l_1 = \frac{l}{2} + x$ and $l_2 = \frac{l}{2} - x$, the product forming the denominator $l_1\, l_2 = \frac{l^2}{4} - x^2$ is always less than $\frac{l_2}{4}$, whether $\frac{l}{2}$ be made somewhat (x) greater or less. A beam, therefore, supported at its extremities, sustains least when the load is applied at its middle, and one so much the greater the nearer the load approaches one of the points of support.

Fig. 225.

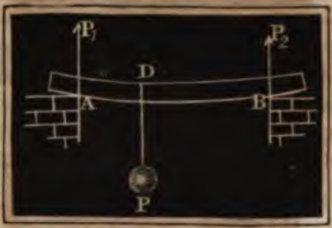
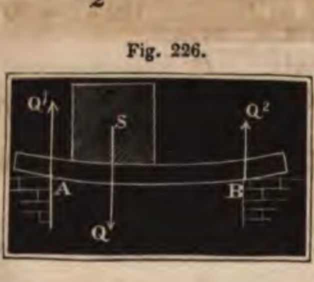


Fig. 226.



If a load Q be uniformly distributed over the length c, the centre of which is l_1 and l_2 distant from the points of support $\mathfrak{A}$ and B, Fig. 226, we shall then have to take the difference $\frac{Ql_1 l_2}{l} - \frac{Q}{2} \cdot \frac{c}{4}$ for the moment of rupture, because the pressure $Q_1 = \frac{Ql_2}{l}$ at the arm l_1, and half the weight $\frac{Q}{2}$ acting at the arm $\frac{c}{4}$ is opposed to it. Therefore

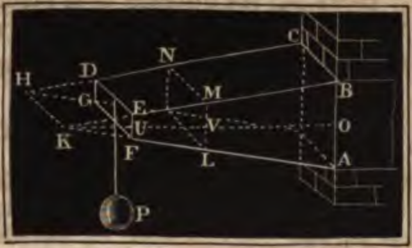
$$Q\left(\frac{l_1 l_2}{l} - \frac{c}{8}\right) = \frac{K}{6} \cdot bh^2.$$

Example. What load does a hollow cast iron beam sustain, if its outer depth and breadth amount to 8 inches and 4 inches, and inner breadth and depth 6 inches and 2 inches; and if further, the middle of the load, uniformly distributed over 3 feet in length, is distant from one point of support 4, and from the other 2 feet? It is $\frac{b_1 h_1^3 - b_2 h_2^3}{h_1} = \frac{4 \cdot 512 - 2 \cdot 216}{8} = 202$; further, $\frac{l_1 l_2}{l} - \frac{c}{8} = \left(\frac{4 \cdot 2}{6} - \frac{3}{8}\right) 12 = \frac{23}{2}$ inches; hence, $\frac{23}{2} Q = 1000 \cdot 202$; and consequently, $Q = 17565$ lbs.

§ 203. *Plane of Rupture.*—If the beams are not prismatic, if they have different transverse sections at different places, the plane of rupture, *i. e.* the plane in which rupture will ensue, will no longer be the same as for prismatic bodies, because this place is not only dependent on the arm x, but also on the transverse section. If we suppose a rectangular section of variable breadth w, and height z, and assume the beam to be fixed at one extremity, and at the other acted upon by a pressure P, and the distance of the transverse section wz from the extremity where the pressure acts $= x$, we must then put $P = \frac{K}{6} \cdot \frac{wz^2}{x}$, and find the minimum value of $\frac{wz^2}{x}$ in order to determine the weakest part, or plane of rupture of the beam.

Here many cases present themselves; let us consider only the following. Let the body $ABEG$, Fig. 227, be a truncated wedge, or have the form of a prism with a trapezoidal base, let the breadth $DE = FG$ at the extremity $= b$, the depth $EF = DG = h$, and the distance UK of the edge cut off from the terminating surface EG, $= c$. Let us now assume that the plane of rupture NL is distant $UV = x$ from the terminating surface, we shall then obtain for it the depth $ML = z = h + \frac{x}{c} h = h\left(1 + \frac{x}{c}\right)$, whilst the uniform breadth is $MN = w = b$.

Fig. 227.



The value $\frac{wz^2}{x} = \frac{bh^2}{x}\left(1 + \frac{x}{c}\right)^2 = bh^2\left(\frac{1}{x} + \frac{2}{c} + \frac{x}{c^2}\right)$ increases and mum, when this latter term is of the last value. But if in place of x, we put $c + u$, where u is a small number, we shall then obtain for it:

$$\frac{1}{c+u} + \frac{c+u}{c^2} = \frac{1}{c\left(1+\frac{u}{c}\right)} + \frac{1}{c} + \frac{u}{c^2}$$

$= \frac{1}{c}\left(2 - \frac{u}{c} + \frac{u^2}{c^2} - \ldots\right) + \frac{u}{c^2} = \frac{2}{c} + \frac{u^2}{c^3}$. As now in this last expression u appears only as a square, it follows that every other value, which is obtained when the distance x is assumed greater or less than c, gives a greater value than for $x = c$, that consequently for $x = c$, $\frac{1}{x} + \frac{x}{c^2}$, and, therefore, also $\frac{w\,z^2}{x} = bh^2 \left(\frac{1}{c} + \frac{2}{c} + \frac{1}{c}\right) = \frac{4\,bh^2}{c}$ is a minimum. From hence it follows, that the magnitude of the surface of rupture $= b\,.\,2\,h = 2\,bh$, and is distant from the terminating surface $EG = bh$ as much again as the edge HK of the portion cut off.

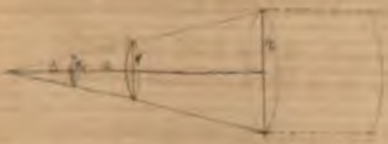
In a similar manner, the distance of the plane of rupture from the terminating surface of a truncated pyramid or truncated cone is equal to half the height of the supplementary pyramid or supplementary cone.

§ 204. *Beams of the Strongest Form.*—A beam, which opposes an equal resistance to rupture throughout all its sections, of which, therefore, each may be considered as a plane of rupture, is called a *beam of the strongest form.* Of all beams of equal strength, the body of equal resistance at each point of its length has the least quantity of material, and is, therefore, the most suitable, and that which should be selected for architectural construction, and for machines, not only out of regard to economy, but also, that the weight may not be increased unnecessarily.

If we put the distance of a plane of rupture from the further extremity $= x$, and the measure of the moment of flexure for that section $= W$, we then have the pressure requisite for rupture $P = \frac{WK}{e\,x}$. As K is a constant factor, a beam of the strongest form $\frac{W}{e\,x}$ must be constant also, *i. e.* it must be of the same value for every possible section. If for a beam of a rectangular section the variable breadth $= u$, and the depth $= v$; but the breadth at the origin, or end supposed fixed $= b$, and the depth there $= h$, we have generally $W = \frac{uv^3}{12}$, and $e = \frac{v}{2}$, hence $P = \frac{uv^2}{x} \cdot \frac{K}{6}$, and for the origin, for which x has become l, $P = \frac{bh^2}{l} \cdot \frac{K}{6}$.

If we make these two values of P equal, we obtain the equation $\frac{uv^2}{x} = \frac{bh^2}{l}$ for the beam of the strongest form. In a beam of equal

In order to show the proper import-
ance of always getting the proper
limits of integration let us tak



Case of Cone

$c : h :: c+x : r \therefore r = h\frac{(c+x)}{c}$

$\pi x = \frac{\pi}{4}\times d^2$ but $d^2 = \frac{h^2}{c^2}(c+x)^2 \therefore \pi x = \frac{\pi}{4}\times\frac{h^2}{c^2}(c+x)^2 \therefore z = \frac{\pi}{4}\times\frac{h^2}{c^2}\frac{(c+x)^3}{x}$

$u = \frac{(c+x)^3}{x} \therefore du = \frac{x\,d(c+x)^3 - (c+x)^3dx}{x^2} = \frac{3x\,dx(c+x)^2 - (c+x)^3dx}{x^2} \therefore \frac{du}{dx} = \frac{3x(c+x)^2 - (c+x)^3}{x^2}$

$3x(c+x)^2 - (c+x)^3 = 0 \quad 3x - (c+x) = 0 \quad$ or $x = \frac{c}{2}$

∴ substituting, volume $= \frac{hl}{b^4}\int y^4 dy$ integrating
this between the limits $y = 0$ and $y = b$
$V = \frac{hl}{5}\frac{b^5}{b^4} = \frac{1}{5}hlb$ Integrating now in a dif-
ferent method $\iiint dx\,dy\,dz = \int yz\,dx$ but
$y = b\sqrt[3]{\frac{x}{l}} \therefore z = h\sqrt[3]{\frac{x}{l}}$ substituting these va
$dV = bh\int\sqrt[3]{\left(\frac{x}{l}\right)^2}dx \therefore V = \frac{3}{5}\frac{bh}{l^{\frac{2}{3}}}x^{\frac{5}{3}} = \frac{3}{5}bhl$. This last answ
is the correct one. We may integrate in
yet different ways and obtain dif-
ferent in some cases and again
in others identical results.

[illegible]

$\iint dx\,dy\,dz$ [illegible]

$x = \frac{y^3 l}{b^3}$ $y = \sqrt[3]{\frac{x}{l}}\cdot b$, $x = \frac{z^3 l}{h^3}$ $z = h\sqrt[3]{\frac{x}{l}}$, $y = \frac{zb}{h}$

(1) $\iint x\,dy\,dz$

$\frac{b}{l^3}\iint y^3\,dy\,dz$

$\frac{b}{l^3}\int \frac{y^4}{4}\,dz$

$\frac{l^4 b}{4\cdot b^3 h^4}\int z^4\,dz$

$\frac{l b^4 z^5}{4\cdot5\cdot b^3 h^4} = \frac{lhb}{20}$

$\iint x\,dy\,dz$

$\frac{b}{h^3}\iint z^3\,dz\,dy$

$\frac{b}{h^3}\int \frac{z^4\,dy}{4}$

$\frac{h^4 b}{4\cdot h^3 l^4}\int y^4\,dy$

$\frac{b h^4 y^5}{4\cdot5\cdot h^3 l^4} = \frac{lhb}{20}$

(1) $\iint x\,dy\,dz$

$\frac{b}{l^3}\iint y^3\,dy\,dz$

$\frac{b}{l^3}\int z y^3\,dy$

$\frac{hb}{l^4}\int y^4\,dy$

$\frac{h b y^5}{5\,l^4} = \frac{hbl}{5}$

(2) $\iint y\,dx\,dz$

$b\iint \sqrt[3]{\frac{x}{l}}\,dx\,dz$

$b\int z\sqrt[3]{\frac{x}{l}}\,dx$

$\frac{bh}{l^{\frac{1}{3}}}\int x^{\frac{1}{3}}\,dx$

$\frac{3}{5}\cdot\frac{bh}{l^{\frac{1}{3}}}\,x^{\frac{5}{3}} = \frac{3}{5}$ [illegible]

$\iint$ [illegible] $dx\,dz$ (3)

$h\iint \sqrt[3]{\frac{x}{l}}\,dy\,dx$

$\frac{h}{l^{\frac{1}{3}}}\int x^{\frac{1}{3}}\,dx$ [illegible]

$\frac{hb}{l^{\frac{1}{3}}}\int x^{\frac{1}{3}}\,dx$

$\frac{3}{5}\,\frac{hb}{l^{\frac{1}{3}}}\,x^{\frac{5}{3}} = \frac{3}{5}\,bhl$

$\iint y\,dx\,dz$ (4)

$b\iint \sqrt[3]{\frac{x}{l}}\,dx\,dz$

$\frac{3}{4}\,\frac{bl^{\frac{4}{3}}}{h^4}\int z^4\,dz$

$\frac{3}{4}\,\frac{bl^{\frac{4}{3}}}{h^4}\,\frac{z^5}{5}$

$= \frac{3}{20}\,bhl^{\frac{4}{3}}$

$\int z\,y\,dx$ (3)

$h\iint \left(\frac{x}{l}\right)^{\frac{1}{3}} y\,dx$

$hb\int \left(\frac{x}{l}\right)^{\frac{1}{3}}\,dx$

$\frac{3\,hb\,x^{\frac{5}{3}}}{5\,l^{\frac{1}{3}}}$

$= \frac{3}{5}\,bhl$

$\iint x\,dy\,dz$ (4)

$\frac{l}{h^3}\iint z^3\,dz\,dy$

$\frac{l}{h^3}\int y z^3\,dz$

$\frac{bl}{h^4}\int z^4\,dz$

$5\cdot\frac{blh^5}{h^4}\cdot 5\,blh$

breadth $u = b$ is $\frac{v^2}{x} = \frac{h^2}{l}$, therefore, $\frac{v^2}{h^2} = \frac{x}{l}$, which is the equation to the parabola (§ 35, *Remark*), and points out that the longitudinal section *ABE*, Fig. 228, must have the form of a parabola whose vertex is the extremity or point of suspension *E* of the load. If the beam *AB*, Fig. 229, rests upon its extremities, and sustains a load in its middle *C*, or if a beam *AB*, Fig. 230, is supported in its middle *C*, and two pressures, balancing each other, are applied at the extremities *A* and *B*, then the longitudinal profile has the form of two parabolas meeting in the middle. The last case occurs in balances, which, as they are weakened by the holes at the points *A*, *C*, *B*, are provided with ribs, or have a middle piece *AB* given to them. If the depth $v = h$ is constant $\frac{u}{x} = \frac{b}{l}$ or $\frac{u}{b} = \frac{x}{l}$, for the breadth u is proportional to its distance from the extremity, the horizontal projection, therefore, of the beam *ABD*, Fig. 231, forms a triangle *BCD*, and the whole beam a wedge with a vertical edge coinciding with the direction of force. If the body *ABD*, Fig. 232, is to have similar transverse sections, we shall then have $\frac{v}{h} = \frac{u}{b}$, hence $\frac{u \cdot u^2h^2}{b^2x} = \frac{bh^2}{l}$, *i. e.* $\frac{u^3}{b^3} = \frac{x}{l}$, therefore, the breadth and the depth in-

Fig. 228.

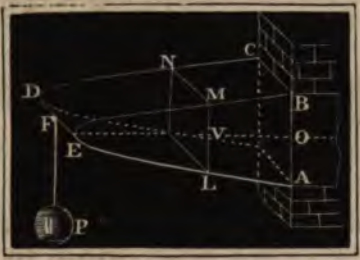


Fig. 229.

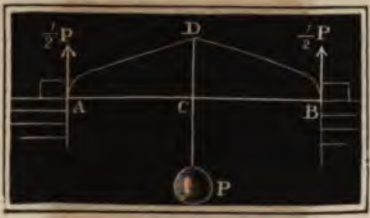


Fig. 230.

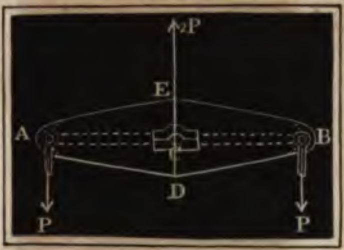


Fig. 231.

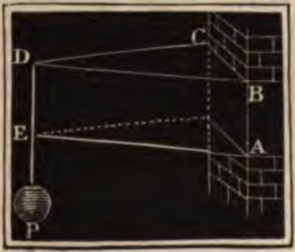
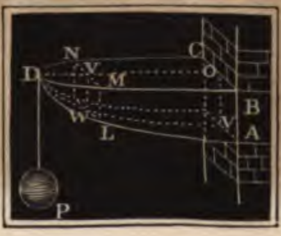


Fig. 232.



crease as the cube roots of the corresponding arms. For example, a section eight times further from the outer end than a given section, would only have double the height and breadth of that of the given section.

If a beam be uniformly loaded, we have the variable load $Q = qx$, and its arm $= \frac{x}{2}$, hence, instead of Px, we must put $xq \cdot \frac{x}{2} = \frac{x^2q}{2}$, whence $\frac{x^2q}{2} = uv^2 \cdot \frac{K}{6}$ and also $\frac{l^2q}{2}$ must be taken $= bh^2 \frac{K}{6}$, and consequently $\frac{uv^2}{bh^2} = \frac{x^2}{l^2}$. Were the breadth invariable, that is $u = b$, we should have $\frac{v^2}{h^2} = \frac{x^2}{l^2}$, therefore, also $\frac{v}{h} = \frac{x}{l}$, and, therefore, a triangle *ABE* for the longitudinal section, and a wedge *ABED*, Fig. 233, for the body of the strongest form. If we take a uniform depth $v = h$, we then obtain $\frac{u}{b} = \frac{x^2}{l_2}$, and, therefore, for the plane a surface *BDC*, bounded by a parabolic arc, as in Fig. 234. If we again make similar transverse sections, then $\frac{u^3}{b^3} = \frac{x^2}{l^2}$ so that we have both in the vertical as in the horizontal profile, a cubic parabola, in which the cubes of the ordinates increase, as the squares of the abscisses.

Fig. 233.

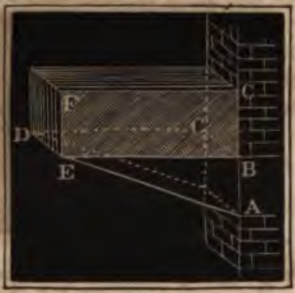
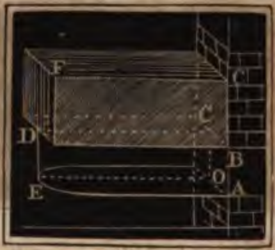


Fig. 234.



If a body *AB* supported at both extremities, Fig. 235, is uniformly loaded over its whole length, we have for the moment of rupture at a distance from a point of support $AO = x$:

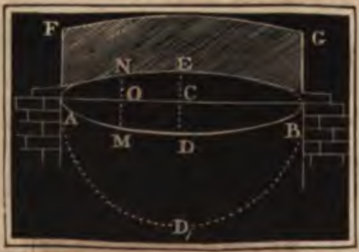
$$\frac{Q}{2} \cdot x - qx \cdot \frac{x}{2} = \frac{q}{2}(lx - x^2),$$

on the other hand for the middle point:

$$\frac{Q}{2} \cdot \frac{l}{2} - \frac{Q}{2} \cdot \frac{l}{4} = \frac{Ql}{8} = \frac{ql^2}{8}.$$

If we suppose the body to be of

Fig. 235.



uniform breadth, we have to put $\frac{q}{2}(lx - x^2) = bv^2 \cdot \frac{K}{6}$ and $\frac{ql^2}{8} = bh^2 \cdot \frac{K}{6}$, and by division $\frac{v^2}{h^2} = \frac{lx - x^2}{\frac{1}{4}l^2}$, or $v^2 = \left(\frac{h}{\frac{1}{2}l}\right)^2 (lx - x^2)$. Were $h = \frac{1}{2}l$, v^2 would be $= lx - x^2$, and therefore the longitudinal section would be a circle AD_1B described with l as a radius, but because $lx - x^2$ must still be multiplied by $\left(\frac{h}{\frac{1}{2}l}\right)^2$ in order to obtain the square v^2 of every ordinate OM, this circle passes into an ellipse ADB, whose semi-axes are $CA = \alpha = \frac{1}{2}l$ and $CD = \beta = h$.

The same relations exist for bodies with circular sections as for those with similar rectangular sections. In the case of a beam imbedded in a wall at one extremity, and loaded at the other $\frac{u^3}{r^3} = \frac{x}{l}$, *i. e.* the radii increase as the cubes of the distances from the point of application.

§ 205. *The Thickness of Axles.*—In the parts of machines, as the shafts, axles, &c., flexures may prejudicially affect the working of machines, by giving rise to vibrations and shocks; and it is here, therefore, often more desirable to determine the sections, not according to their strength, but according to their degree of flexure. *Gerstner* and *Tredgold* maintain that a beam of wood, supported at both extremities and loaded in the middle, may suffer a deflexion $\alpha = \frac{1}{288} \cdot l$ without disadvantage, and that such a beam of cast or wrought iron can only undergo a deflexion or height of arc $\alpha = \frac{1}{480} \cdot l$. But now from § 190: $a = \frac{P l^3}{48\, WE}$, and from § 191: $W = \frac{bh^3}{12}$, hence follows the height of arc: $a = \frac{P l^3}{4\, bh^3\, E}$ and $\frac{a}{l} = \frac{P l^2}{4\, bh^3\, E}$. If now we put $\frac{a}{l} = \frac{1}{288}$, and $E = 1800000$, we obtain for wooden beams the tenacity or strength at the middle:

$$P = \frac{a}{l} \cdot \frac{4\, bh^3\, E}{l^2} = \frac{1}{288} \cdot \frac{4\, bh^3 \,.\, 1800000}{l^2} = 25000 \cdot \frac{bh^3}{l^2}.$$

For cast iron $\frac{a}{l} = \frac{1}{480}$, and $E = 17000000$, hence:

$$P = \frac{1}{480} \cdot \frac{4\, bh^3}{l^2} \,.\, 17000000 = 142000 \,.\, \frac{bh^3}{l^2}.$$

If further we take for cast iron $\frac{a}{l} = \frac{1}{480}$, and $E = 29000000$ lbs., we obtain for a rectangular beam of this material:

$$P = 242000 \,.\, \frac{bh^3}{l^2}.$$

The co-efficients 25000, 142000, 242000 must be multiplied by

$3\,\pi = 9{,}42$, and h and b be replaced by r, for cylindrical beams as round axles, &c. The following table gives the dimensions of the transverse sections, l being expressed in feet, b, h, r in inches, and P in pounds.

Substances.	Rectangular section.	Circular section.
Wood - - - - -	$bh^3 = \frac{P\,l^2}{170}$	$r^4 = \frac{P\,l^2}{1600}$
Cast iron - - - -	$bh^3 = \frac{P\,l^2}{980}$	$r^4 = \frac{P\,l^2}{9250}$
Wrought iron - -	$bh^3 = \frac{P\,l^2}{1680}$	$r^4 = \frac{P\,l^2}{15800}$

If the load Q be uniformly distributed over the beam, P must be replaced by $\frac{5}{8}\,Q$, § 190, and if the weight of the beam be taken into account, by $P + \frac{5}{8}\,G$. If it be the case of a beam which is fixed at one extremity and loaded at the other, P and l must then be doubled, therefore, $P\,l^2$ to be multiplied by eight; if, lastly, the beam fixed at one extremity sustains a load Q uniformly distributed, for $P\,l^2$, we must substitute $\frac{3}{8}\,.\,8\;Ql^2 = 3\;Ql^2$ for $P\,l^2$.

Examples.—1. What load will a wooden beam, 20 feet long, 7 inches thick and 9 deep, reposing on both its extremities, sustain for a length of time? This load is $P = 170\,.\,\frac{b\,h^3}{l^2} = 170\,.\,\frac{7\,.\,9^3}{20^2} = 1190\,.\,\frac{729}{400} = 2170$ lbs.

In § 198 P was found $= 1890$ lbs.—2. What thickness must an iron axle, 12 feet long, be cast, if the same has to sustain a uniformly distributed load $Q = 40000$ lbs., without any detrimental flexure? $r^4 = \frac{\frac{5}{8}\,Q\,l^2}{15800}$, therefore here $r^4 = \frac{5}{8}\,.\,\frac{40000\,.\,12^2}{15800} = 228$, and $r = \sqrt[4]{228} = 3{,}89$ inches; consequently, the thickness of the axle $2\,r = 7{,}78$, or about $7\frac{3}{4}$ inches. By the formula for strength, if the modulus of tenacity of wrought iron be taken at $\frac{14}{9}$ times that of cast iron: $r^3 = \frac{Ql}{8\,.\,\frac{14}{9}\,.\,4700} = \frac{40000\,.\,144\,.\,9}{8.14\,.\,4700} = 98{,}5$; hence, $r \sqrt[3]{98{,}5} = 4{,}62$ inches, and $2\,r = 9{,}24$ inches.

§ 206. *Rupture by Compression.*—If prismatic bodies are so strongly compressed in the direction of their axes, as to amount to rupture, their resistance to compression has to be overcome. This rupture may take place in two ways. If the body be short, if it approximates to a cube, it will fall to pieces without undergoing flexure, but if the body is longer than it is broad and thick, flexure similar to that which takes place will precede the rupture. The one kind of rupture consists in a crushing, bruising, transverse strain, or splitting asunder of the body or its parts; the other, in a fracture or destruction of a section of the body. Hence *a distinction is made between the crushing strength and strength of rupture under compression.*

The resistance to crushing is, for similar sections, proportional to their areas; for regular sections, however, somewhat greater than for irregular, and greatest of all for circular sections. It is besides independent for the most part of the length of the body. Short wooden prisms split asunder in the direction of their length, or form a bulge; stones break into several pieces or separate along an inclined plane. Ten times the absolute strength is given to wood and stones; to iron, only one of five times; and to walls of rough stones, twenty times. If K_1 be the modulus of resistance to crushing, and F the transverse section of the bodies, the working load will be

$P = FK_1$ and $F = \frac{P}{K_1}$, where for K_1, $\frac{1}{5}K$ to $\frac{1}{20}K$ must be substituted.

TABLE

OF THE MODULUS OF RESISTANCE TO CRUSHING.

Names of substances.	Modulus K.	Names of substances.	Modulus K.
Basalt - - -	27000	Brick - - -	550 to 2200
Gneiss - - -	5100	Oak - - -	2800 " 6800
Granite - - -	6000 to 11000	Pine - - -	6800 " 8000
Limestone - -	1500 " 6000	Fir - - -	2000
Marble - - -	3800 " 12000	Cast iron - -	146000
Mortar - - -	450 " 900	Wrought iron -	72000
Sandstone - -	1400 " 13000	Copper - - -	60000

The values of K contained in the preceding table are not unfrequently, especially for wooden columns, applicable even when the bodies are very long, only it has been found necessary to diminish these values by one, two, or three-sixths, when the columns are twelve, twenty-four or forty-eight times as long as they are thick. Accordingly, for a column of oak, one foot thick and twenty-four long, K must be taken at from $2800\,(1 - \frac{2}{6}) = 1900$ lbs. to $6800 \,.\, \frac{2}{3} = 4500$ lbs. The formulæ developed in § 185 for the transverse section of bodies of considerable weight, and of bodies of the strongest form, here find their application.*

Examples.—1. What load can a round column of pine, 12 feet long and 11 inches diameter, sustain? $F = \frac{\pi \,.\, 11^2}{4} = 95$ square inches; if we now take for K a mean value $= \frac{6800 + 8000}{2} = 7400$, and diminish the value one-sixth, because the length is 13 times that of the thickness, and therefore put $K = 7400 \,.\, \frac{5}{6} = 6200$ lbs.; and give a ten-times security, we shall then have $P = \frac{6200 \,.\, F}{10} = 620 \,.\, 95 = 58900$ lbs.—2. How thick must be the foundation walls of a massive building of 20000000 lbs. weight, 60 feet outer length, and 40 feet breadth, if for this purpose we use well finished blocks of gneiss? Let x be the requisite thickness, $60 - x$ is the mean length, and $40 - x$ the breadth; therefore the mean perimeter $2\,(60 - x + 40 - x) = 200 - 4\,x$; if we multiply this by x, we obtain the base of the walls $(200 - 4\,x)\,x$ square feet $= 144\,(200 - 4\,x)\,x = 576\,(50 - x)\,x$ square inches. For a twenty-fold security, a square inch of gneiss sustains a pressure $= \frac{5100}{20} = 255$ lbs.; hence we have to put $255 \,.\, 576\,(50 - x)\,x = 20000000$, or $50\,x - x^2 = \frac{20000000}{146880} = 136$. Now $x = \frac{136 + x^2}{50}$, or about $x = \frac{136}{50}$

---

* See Appendix.

$= 2{,}7$ feet. Now x^2 being put $= 2{,}7^2 = 7$, more accurately $x = \frac{136 + 7}{50} = \frac{143}{50}$ $= 2{,}86$ feet, for which we may take $2{,}9$ feet, $= 35$ inches.

Fig. 236.

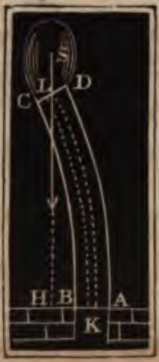
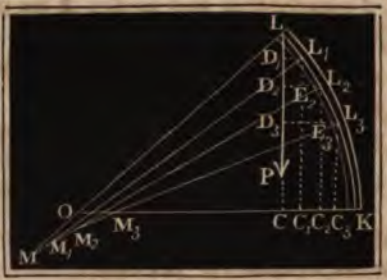


Fig. 237.



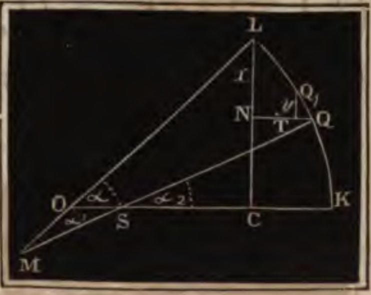
§ 207. *Rupture under Compression.*—If a prismatic body $ABCD$, Fig. 236, be fixed at one extremity, and at the other be acted on by a pressure P, which acts in the direction of the axis of the body, the relations of deflexion will come out otherwise than when the pressure acts perpendicular to the axis. The neutral axis KL assumes another form, because the arms of the pressure P are not formed by the abscisses, but by the ordinates, as HK. From § 188, we have for the angles of curvature LML_1, $L_1M_1L_2$, &c., of the neutral axis KL, Fig. 237, $\phi_1 = \frac{M_1 \, . \, \overline{LL_1}}{WE}$, $\phi_2 = \frac{M_2 \, . \, \overline{L_1L_2}}{WE}$, &c., but here the moments are $M_1 = P \, . \, \overline{D_1L_1}$, $M_2 = P \, . \, \overline{D_2L_2}$, &c., hence we have the measures of the angles: $\phi_1 = \frac{P \, . \, \overline{D_1L_1} \, . \, \overline{LL_1}}{WE}$, $\phi_2 = \frac{P \, . \, \overline{D_2L_2} \, . \, \overline{L_1L_2}}{WE}$, &c. If we introduce the tangential angles $L_1LD_1 = KOL = \alpha$, $L_2L_1E_2, = \alpha_1 = \alpha - \phi_1$, $L_3L_2E_3 = \alpha_2 = \alpha_1 - \phi_2 = \alpha - \phi_1 - \phi_2$, &c., and if we suppose only a small curvature, we may then write: $LL_1 = \frac{D_1L_1}{\alpha}$, $L_1L_2 = \frac{E_2L_2}{\alpha_1} = \frac{E_2L_2}{\alpha - \phi_1}$, $L_2L_3 = \frac{E_3L_3}{\alpha_2}$, &c.; if further we divide the entire height of the arc $CK = a$ into n equal parts, we may then put: $D_1L_1 = CC_1 = E_2L_2 = C_1C_2$, &c. $= \frac{a}{n}$, but $D_2L_2 = \frac{2a}{n}$, $D_3L_3 = \frac{3a}{n}$, &c., and by the substitution of these values it follows that:

$$\phi_1 = \frac{P \, . \frac{a}{n} \, . \frac{a}{n\alpha}}{WE} = \frac{Pa^2}{WE\, n^2\alpha}, \phi^2 = \frac{P \, . \frac{2a}{n} \, . \, \frac{a}{n\,\alpha_1}}{WE} = \frac{2\,P\,a^2}{WE\, n^2\alpha_1},$$

$\phi_3 = \frac{3\,Pa^2}{WE\,n^2 a_2}$, &c., or $\phi_1\ a = \frac{Pa_2}{WE\,n^2}, \phi_2\,a_1 = \frac{2\,Pa^2}{WE\,n^2}$, $\phi_3\,a_2 = \frac{3\,Pa^2}{WE\,n^2}$, &c.

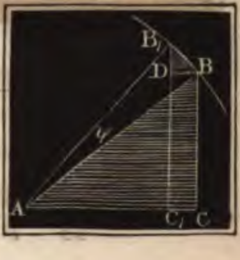
The sum $\phi_1 a + \phi_2\,a_1 + \phi_3\,a_2 + \ldots = \frac{Pa^2}{WE\,n^2}(1 + 2 + 3 + \ldots + n) = \frac{Pa^2}{WE . n^2} \cdot \frac{n^2}{2} = \frac{Pa^2}{2\,WE}$, and may be also found, if a be divided into m equal parts, and any such part $\frac{a}{m}$, be put $= \phi_1 = \phi_2 = \phi_3$, &c. We shall then obtain $\phi_1\,a + \phi_2\,a_1 + \phi_3\,a_2 + \ldots = \frac{a}{m} \cdot a + \frac{a}{m}\left(a - \frac{a}{m}\right) + \frac{a}{m}\left(a - \frac{2a}{m}\right) + \ldots + \frac{a}{m} \cdot \frac{a}{m}$ by taking out the common factor $\left(\frac{a}{m}\right)^2$, and writing it in an inverse order $= \left(\frac{a}{m}\right)^2 (1 + 2 + \ldots + m) = \frac{a^2}{m^2} \cdot \frac{m^2}{2} = \frac{a^2}{2}$, and by making these two sums equal to each other $\alpha^2 = \frac{Pa^2}{WE}$, an equation between the angle of curvature $LOK = \alpha°$, and the height of the arc $CK = a$.

Fig. 238.



For the equation of the elastic line LK, Fig. 238, let us take $LN = x$ and $NQ = y$ as co-ordinates, and put the corresponding angle of curvature $LMQ = \alpha_1$. In the last equation, if we put a for y, and y for a, we then have to replace the sum $\phi_1\,a + \phi_2\,a_1 + \phi_3\,a_2 + \ldots$ by $\frac{\alpha^2 - (\alpha - \alpha_1)^2}{2}$; hence, if we represent the supplementary angle $QSK = \alpha - \alpha_1$ by α_2, we afterwards obtain $\alpha^2 - \alpha_2^{\,2} = \frac{Py^2}{WE}$, or $\alpha_2^{\,2} = \alpha^2 - \frac{Py^2}{WE}$, and so $\alpha^2 = \sqrt{\frac{P}{WE}} \cdot \sqrt{a^2 - y^2}$. But since $\alpha_2° = \angle\ QQ_1T$, and *tang.* $QQ_1T = \frac{TQ}{TQ_1} = \frac{\text{element } \delta \text{ of the ordinate } y}{\text{element } \varepsilon \text{ of the absciss } x}$, we have $\frac{\delta}{\varepsilon} = \sqrt{\frac{P}{WE}} \cdot \sqrt{a^2 - y^2}$.

Fig. 239.



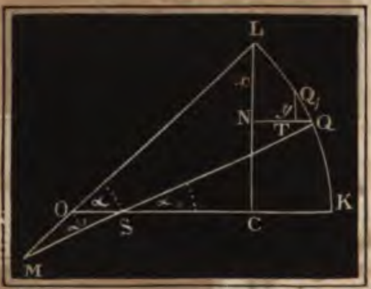
If in a rectangular triangle ABC, Fig. 239, with the hypothenuse $AB = a$, the angle CAB increases by a small amount $BAB_1 = \psi°$, the perpendicular $BC = y$ increases by the amount

$DB_1 = \delta$, and the base AC decreases by the amount $DB = C_1C$, then $\frac{DB_1}{BB_1} = \frac{AC}{AB}$, *i. e.* $\frac{\delta}{a\psi} = \frac{\sqrt{a^2 - y^2}}{a}$; and hence $\frac{\delta}{\psi} = \sqrt{a^2 - y^2}$. By comparing the two expressions $\frac{\delta}{\varepsilon} = \sqrt{\frac{P}{WE}} \cdot \sqrt{a^2 - y^2}$ and $\frac{\delta}{\psi} = \sqrt{a^2 - y^2}$, we obtain the equation $\psi = \varepsilon \sqrt{\frac{P}{WE}}$, or $\frac{\varepsilon}{\psi} = \sqrt{\frac{WE}{P}}$. Therefore the ratio of the element ε of the absciss to the element of the arc ψ is invariable, and $= \sqrt{\frac{WE}{P}}$, and hence, also, the ratio of the absciss x to the whole arc $A = \sqrt{\frac{WE}{P}}$, *i. e.* $\frac{x}{A} = \sqrt{\frac{WE}{P}}$, and $A = x\sqrt{\frac{P}{WE}}$. If, finally, we substitute this value of A in the equation $BC = AB\ sin.\ A$; *i. e.* $y = a\ sin.\ A$, we obtain the equation sought:

$$y = a\ sin. \left(x\sqrt{\frac{P}{WE}}\right).$$

With the assistance of this last formula we may find the ordinate $NQ = y$ corresponding to any absciss $LN = x$, Fig. 240. If in this we put $x = l$ and $y = a$, we then obtain $a = a\ sin. \left(l\sqrt{\frac{P}{WE}}\right)$, *i. e.* $1 = sin. \left(l\sqrt{\frac{P}{WE}}\right)$, whence it follows that $l\sqrt{\frac{P}{WE}} = \frac{\pi}{2}$ and $P = \left(\frac{\pi}{2l}\right)^2 WE$.

Fig. 240.



As this formula does not contain the height of the arc a, it follows that the force P is capable of maintaining equilibrium for every deflexion of the body. This remarkable circumstance is explained by the fact that an increase of the arm or of the statical moment is combined with an increase of the deflexion. Hence, therefore, the force for rupture is:

$$P = \left(\frac{\pi}{2l}\right)^2 WE.$$

§ 208. *Columns.*—If we put in the formula $P = \left(\frac{\pi}{2l}\right)^2 \cdot WE$ for $W = \frac{bh^3}{12}$, we then obtain in P the resisting strength of a rectangular column $P = \frac{\pi^2}{48} \cdot \frac{bh^3}{l^2} E$.

The strength, therefore, of a parallelopiped increases as the breadth or greater dimension, and the cube of the thickness or less dimension of the transverse section, and inversely as the square of the length.

If, on the other hand, we put $W = \frac{\pi}{4} r^4$, then for a cylindrical column we have $P = \frac{\pi^3}{16} \cdot \frac{r^4 E}{l^2}$.

The strength of a cylinder increases, therefore, as the fourth power of the diameter, and inversely as the square of the length.

For a hollow column with the radii r_1, and r_2

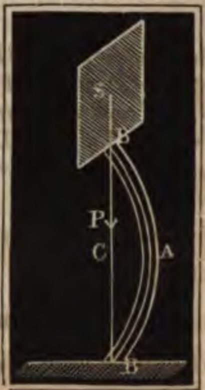
$$P = \frac{\pi^3}{16} \cdot \frac{(r_1^4 - r_2^4) E}{l^2}.$$

If the column be not fixed at the lower extremity, it will assume a curvature BAB_1, Fig. 241, by which the lower half will be as strongly deflected as the upper, and the greatest curvature take place in the middle. Therefore this beam must be regarded as the double of one imbedded in a wall, and for l, $\frac{l}{2}$ must be substituted, so that for the rectangular and for the cylindrical columns,

$$P = \frac{\pi^2}{12} \cdot \frac{b h^3}{l^2} E, \; P = \frac{\pi^3}{4} \cdot \frac{r^4}{l^2} E;$$

in both cases, however, there is a fourfold tenacity. These formulæ, when the columns are not very long, give generally a greater tenacity than the formula for the crushing strength, wherefore the ratios of the sections are often determined from the last. It is at least advisable only to make use of the formula for rupture under compression when the length is at least twenty times that of the thickness, and then, further, to allow a twenty-fold security.*

Fig. 241.



Examples.—1. For a column of fir, 12 feet long and 11 inches thick, the tenacity is $P = \frac{\pi^3}{16} \cdot \frac{r^4}{l^2} \cdot \frac{E}{20} = 31 \cdot \left(\frac{11}{24}\right)^4 \cdot \frac{1800000}{20} = 31 \,.\, 0{,}044 \,.\, 90000 = 123000$ lbs.; in Example No. 1, of § 206, 58900 lbs. only, therefore about $\frac{1}{2}$ of the above, was found.—2. How thick must a column of oak, 30 feet high, be in order to be able to bear a load of 60000 lbs.?

Here $r = \sqrt[4]{\frac{4 P l^2}{\pi^3 E}} = \sqrt[4]{\frac{4 \,.\, 60000 \,.\, (30 \,.\, 12)^2}{31 \,.\, \frac{1800000}{20}}} = \sqrt[4]{\frac{8 \,.\, 6 \,.\, 360^2}{31 \,.\, 18}} = 10{,}3$ inches; consequently, the thickness is about 21 inches. The strength of crushing requires, if K be put $= \frac{1}{10} \cdot \frac{4}{6} \cdot \frac{2800 + 6800}{2} = 320$ lbs., the transverse section $F = \frac{60000}{320} = 188$ square inches; whence, $r = \sqrt{\frac{188}{\pi}} = 13{,}7 \,.\, 0{,}564 = 7{,}7$ inches, and the thickness should be $15\frac{1}{2}$ inches. For this case the first value must be taken.

§ 209. *Torsion.*—When a body ABC, Fig. 242, fixed at one extremity, is acted upon by a force whose direction lies in the plane normal to the axis, and, therefore, endeavors to turn the body about the axis, or when two forces of revolution P and Q act in different normal planes upon a body AB, fixed by its axis, Fig. 243, the fibres running parallel to the axis undergo a wrenching or torsion, the amount of which we wish to determine. Let AB, Fig. 242, be a fibre before, and AD the same fibre during the torsion, and, therefore, let the extremity of the fibre B be advanced by the force of torsion to D. If now l be the initial length AB, and λ its extension, therefore $l + \lambda$ the length AD during the torsion, and if s be the corresponding torsion BD, we have after the Pythagorean law to put

---

* See Appendix.

Fig. 242.

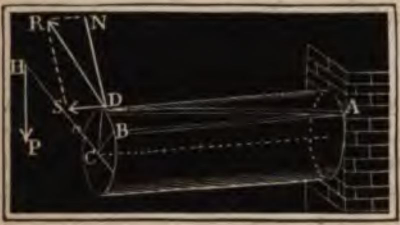
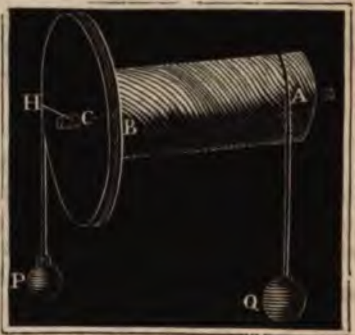


Fig. 243.

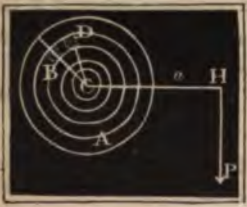


$$\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2$$

$(l + \lambda)^2 = l^2 + s^2$, or $l^2 + 2\,l\lambda + \lambda^2 = l^2 + s^2$, may be put approximately $= \lambda = \frac{s^2}{2l}$. If further F be the section of such a fibre, we then have for the force required to produce this extension in the direction of the fibre $S = \frac{s^2}{2l^2} \cdot F \cdot E$. But this force or tension (S) of a fibre is only a component of the force of torsion R, which produces besides a further pressure N, normal to the fibres. From the similarity of the triangles RDS and BDA, it follows that $S : R = s : l$, hence $S = \frac{Rs}{l}$, and by equating both values of S:

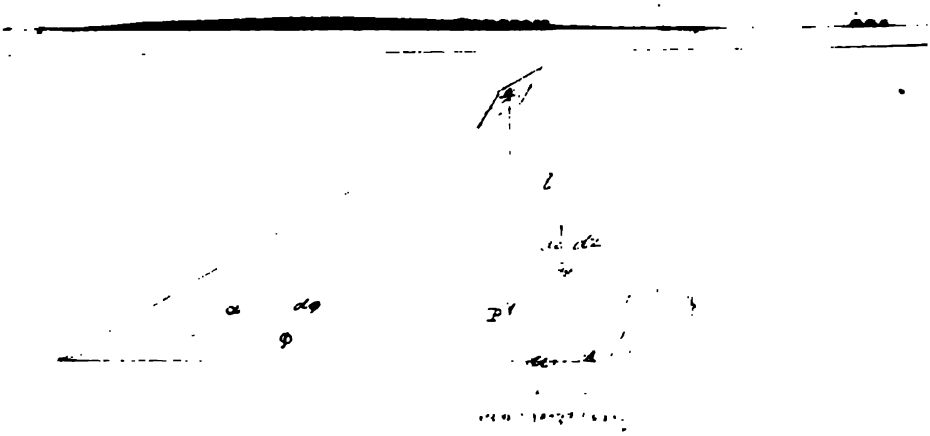
$$R = \frac{s}{2l} \cdot F \cdot E.$$

Fig. 244.



Therefore the force of torsion of a fibre increases as the torsion (s), *and the transverse section F, and inversely as the length* (l) *of the fibre.*

To find the force of torsion of a cylindrical axle CBA, Fig. 244, let us divide its radius r into n equal parts, and suppose concentric circles passing through the points of division, so that the transverse section becomes decomposed into annular elements of the


l
dz
α
dφ
φ
p'

[illegible]

[illegible] $\Delta L = Py$ [illegible]

[illegible] $\frac{Py\,dy}{\phi}$ [illegible]

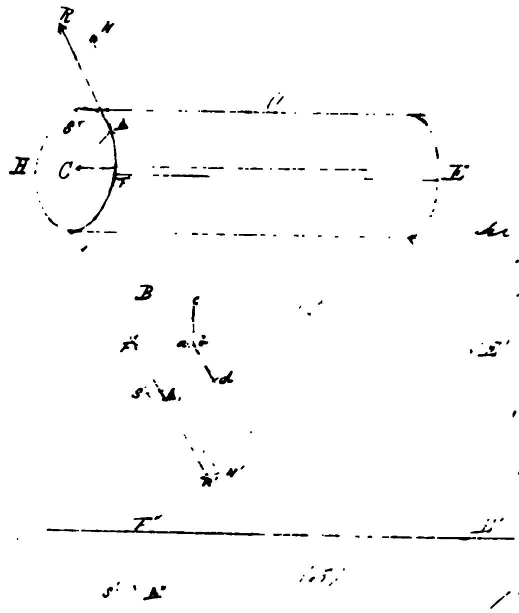
[illegible] $a = 0,\ y = a$ [illegible]

[illegible] $c\phi^2 = P(a^2 - y^2)$ [illegible]

[illegible] $x^2 = \frac{Pa^2}{c}$ [illegible]

$\phi = \sqrt{\frac{P}{c}(a^2 - y^2)}$ and finding [illegible] $\frac{dy}{dx}$ [illegible] $\frac{dy}{dx} = \sqrt{\frac{P}{c}(a^2 - y^2)}$

multiplying out by dx and dividing by $\sqrt{a^2-y^2}$, $\frac{dy}{\sqrt{a^2-y^2}} = \sqrt{\frac{p}{e}}\cdot dx$. Integrating $\sin^{-1}\frac{y}{a} = \sqrt{\frac{p}{e}}\cdot x + C$. make $y=0$ when $x=0$ and $C=0$, $\therefore \frac{y}{a} = \sin\sqrt{\frac{p}{e}}\cdot x$. When we are at the uppermost point of the curve $y=a$ and $x=e$, hence $\sin^{-1}1 = \sqrt{\frac{p}{e}}\cdot e = \frac{\pi}{2}$; solving with respect to p, $p = \left(\frac{\pi}{2e}\right)^2 e$.



[illegible] HALL [illegible] a cylinder [illegible] reflected [illegible] a plane [illegible] E [illegible] fig. 7. AB the [illegible] [illegible] the plane of the end. AH [illegible] [illegible] normal to the point A. Before [illegible] R was normal to the plane [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] position R is no longer normal to [illegible] [illegible] [illegible] [illegible] [illegible]

[illegible] given point [illegible]
[illegible] A [illegible] A [illegible] B'A' [illegible]
[illegible] joined [illegible]
[illegible] perpendicular [illegible]
[illegible] B, [illegible]
[illegible] vertical plane [illegible] perpen-
[illegible] E'B, and drawing a line ca perpendic-
-ular [illegible] the [illegible] Carry in
the horizontal plane and will be the
projection of the
[illegible] of the perpendicular [illegible] AE [illegible]
to [illegible] vertical plane of the
[illegible] parallel [illegible]
the line FA' and [illegible]
[illegible] the [illegible] N parallel [illegible]
and S [illegible] projection [illegible]
B'A', [illegible] plane
[illegible] B'B" [illegible] element
F'A' [illegible]
[illegible] F"A [illegible] A'A' [illegible]
[illegible] A"A'= A'A', [illegible]
[illegible] A"N [illegible]
A'B" [illegible]
perpendicular [illegible] S:K::S:L [illegible]
[illegible] S:K::S:L.

[illegible]

[illegible] case of torsion

[illegible] round axis.

$R = E \cdot \frac{\delta F}{l}$ [illegible] constant [illegible] and [illegible] particular cases [illegible] of the [illegible] consideration [illegible] elementary area [illegible] and [illegible] moment [illegible] general equation [illegible]

[illegible] $= E \frac{a}{l} \frac{\pi r^4}{2}$ [illegible] $a = \frac{2 P a l}{\pi r^4 E}$ from [illegible] having the [illegible] of torsion [illegible] constant [illegible] and [illegible]

For rectangular section, $dx\,dy$ = elementary section [illegible] $x^2 + y^2 = r^2$, [illegible] described [illegible] $R = \frac{E}{2l}$ $dx\,dy$ [illegible]

$= \frac{Ea}{2l} \iint r^2\,dx\,dy = \frac{Ea}{2l} \iint (x^2+y^2)\,dx\,dy =$ [illegible] $x^2\,dx\,dy + y^2\,dx\,dy$

$= \frac{E}{2l}$ [illegible] $= E$ [illegible] $= \frac{a\,b^4}{3\,l} E$

[illegible]

$Pa = E$ [illegible] u, $u = \frac{3 P a l}{[illegible](b^2 + [illegible]) E}$

thickness $\frac{r}{n}$, and radii $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n} \dots \frac{nr}{n}$. The solid contents of these elements are $F_1 = 2\pi \cdot \frac{r}{n} \cdot \frac{r}{n} = 2\pi \left(\frac{r}{n}\right)^2$, $F_2 = 2\pi \cdot \frac{2r}{n} \cdot \frac{r}{n} = 4\pi \left(\frac{r}{n}\right)^2$, $F_3 = 2\pi \cdot \frac{3r}{n} \cdot \frac{r}{n} = 6\pi \left(\frac{r}{n}\right)^2$, &c. If all the fibres are twisted by the angle $BCD = \alpha^\circ$, they have the corresponding torsions $s_1 = \frac{r}{n}\alpha$, $s_2 = \frac{2r}{n}\alpha$, $s_3 = \frac{3r}{n}\alpha$, and hence the forces of torsion are : $R_1 = \frac{r\alpha}{2\,nl} \cdot 2\pi \left(\frac{r}{n}\right)^2 E = \frac{\alpha\pi}{l} \left(\frac{r}{n}\right)^3 E$, $R_2 = \frac{4\alpha\pi}{l} \left(\frac{r}{n}\right)^3 E$, $P_3 = \frac{9\alpha\pi}{l} \left(\frac{r}{n}\right)^3 E$, &c. If further we multiply these forces by the arms $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, and add together the values so obtained, we have for the moments of torsion $Pa = \frac{\alpha\pi}{l} \left(\frac{r}{n}\right)^4 (1^3 + 2^3 + 3^3 + \dots + n^3) E$, *i. e.* $Pa = \frac{\alpha\pi}{l} \left(\frac{r}{n}\right)^4 \cdot \frac{n^4}{4} = \frac{\alpha\pi r^4}{4\,l} \cdot E$, and inversely, the measure of the angle of torsion:

$$\alpha = \frac{4\,l \,.\, Pa}{\pi r^4 E}.$$

If the axle be hollow and have radii r_1 and r_2, we have then

$$Pa = \frac{\alpha\pi}{4\,l} E\,(r_1^4 - r_2^4), \text{ therefore } \alpha = \frac{4\,Pal}{\pi E\,(r_1^4 - r_2^4)}.$$

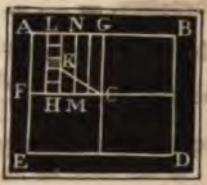
The application of hollow axles gives also with respect to torsion a saving in material, for if we put $r_2 = r$, and $r_1 = r\sqrt{2}$, we then obtain for the hollow axle, which has the same section as a solid one, the moment of torsion:

$$= \frac{\alpha\pi E}{4\,l}(4r^4 - r^4) = 3 \,.\, \frac{\alpha\pi r^4 E}{4\,l}, \textit{ i. e.}$$

thrice as great as for the solid axle.

§ 210. For a shaft or axle of a rectangular section *ABDE*, Fig. 245, the moment of torsion is found in the following manner. If we divide half the breadth $AG = b$ into n equal parts, and carry through the points of division the parallel planes HL, MN, &c., we obtain elements of equal sections, each $= \frac{b}{n} \,.\, h$, where h represents half the height $AF = GC$ of the section. If now we divide one of these elementary strips into m equal parts, we have for its area $\frac{1}{m} \cdot \frac{bh}{n} = \frac{bh}{mn}$. Let the normal distance CH of the strip HL from the centre C, $= c$, and the distance KH of the element

Fig. 245.



K from the normal CH, $= e$, then the distance of the element from the axis is $CK = \sqrt{c^2+e^2}$, accordingly the arc of torsion $= a\sqrt{c^2+e^2}$, and the moment of torsion

$$= \frac{a\sqrt{c^2+e^2}}{2\,l} \cdot \frac{bh}{mn}\sqrt{c^2+e^2} \,.\, E = \frac{abh}{2mnl}(c^2+e^2)\,E.$$

If now we successively put $e = \frac{1}{m}h, \frac{2}{m}h, \frac{3}{m}h$, &c., and sum the results, we have the moment of the strip:

$$HL = \frac{abh}{2mnl}\left(c^2 + \frac{h^2}{m^2} + c^2 + \frac{4h^2}{m^2} + c^2 + \frac{9h^2}{m^2} + \ldots\right)E$$

$$= \frac{abh}{2mnl}\left[mc^2 + \left(\frac{h^2}{m^2}\right)(1+4+9+\ldots+m^2)\right]E.$$

But $1+4+9+\ldots+m^2 = \frac{m^3}{3}$, hence the moment of the strip $= \frac{abh}{2nl}\left(c^2 + \frac{h^2}{3}\right)E$. To obtain the moments of all the strips, let us again put $c = \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}$, &c., and again sum the results, we shall then have:

$$\frac{abh}{2nl}\left[\left(\frac{b}{n}\right)^2 + 4\left(\frac{b}{n}\right)^2 + 9\left(\frac{b}{n}\right)^2 + \ldots + \frac{nh^2}{3}\right]E =$$

$$\frac{abh}{2nl}\left[\left(\frac{b}{n}\right)^2(1+4+\ldots n^2) + \frac{nh^2}{3}\right]E = \frac{abh}{2nl}\left(\frac{b^2}{n^2}\cdot\frac{n^3}{3} + \frac{nh^2}{3}\right)E$$

$$= \frac{abh}{2\,l}\left(\frac{b^2+h^2}{3}\right)E.$$

Generally the sections are square, and therefore $b = h$. As we have only considered a fourth part of the shaft, it follows for the whole shaft that:

$$Pa = \frac{4\,a\,b^4}{3\,l}\,E.$$

For a cylindrical axle $P_1a_1 = \frac{a\pi\,r^4}{4\,l}E$; if we put $b = r$ we then obtain $Pa = \frac{4}{3}\cdot\frac{4}{\pi}\cdot\frac{a\pi\,r^4}{4\,l}E = \frac{16}{3\pi}P_1a_1$, the moment of the square is, therefore, $= \frac{16}{3\pi} =$ 1,756 times as great as that of the round axle. But if we make $4b^2 = \pi r^2$, and, therefore, both sections equal, we then obtain $Pa = \frac{a\,.\,\pi^2r^4}{4\,.\,3l}E = \frac{\pi^2}{4.3}\cdot\frac{4}{\pi}\,.\,P_1a_1 = \frac{\pi}{3}\,P_1a_1$, therefore the square shaft is only a little stronger than the cylindrical axle.

If the axle be hollow, and the outer and inner radii be $2b_1$ and $2b_2$, we then have:

$$Pa = \frac{4aE}{3\,l}(b_1^4 - b_2^4).$$

§ 211. *Breaking Twist.*—When the torsion exceeds a certain limit, the fibres are torn asunder, and the cylindrical axle is *twisted asunder*. For the moment of rupture of the fibre furthest from the axis $\frac{\lambda}{l} = \frac{K}{E}$, but $\frac{\lambda}{l}$ is also $= \frac{s^2}{2\,l^2} = \frac{\alpha^2 r^2}{2\,l^2}$, hence it follows that $\frac{\alpha\,r}{l} = \sqrt{\frac{2\,K}{E}}$. The statical moment of twisting for the round axle is:

$$Pa = \sqrt{\frac{2\,K}{E}} \cdot \frac{\pi\,r^3}{4}\,E = \frac{\pi\,r^3}{2}\sqrt{\frac{KE}{2}};$$

but for the square shaft, where the greatest distance of a fibre is half the diagonal $b\,\sqrt{2}$, it follows that

$$\frac{K}{E} = \frac{2\,\alpha^2\,b^2}{2\,l^2} \text{ since } \frac{\alpha\,b}{l} = \sqrt{\frac{K}{E}} \text{ and } Pa = \frac{4\,b^3}{3}\sqrt{KE}.$$

Since the fibres are not only extended by torsion, but also compressed, and as we have only had regard to extension in our development, so it may be expected that the formulæ found do not in their quantitative relations quite correspond with experiment and, therefore, it is necessary to take the constants E and $\sqrt{KE}$ from experiments made especially to determine them.

If α be given in degrees, such observations admit of our putting for the torsion:

Substances.	Circular section.	Square section.
Wood	$Pa = 3500 \cdot \frac{\alpha^\circ r^4}{l}$.	$Pa = 5800\ \frac{\alpha^\circ b^4}{l}$.
Cast iron	$Pa = 160000\,\frac{\alpha^\circ r^4}{l}$.	$Pa = 280000\,\frac{\alpha^\circ b^4}{l}$.
Steel and wrought iron .	$Pa = 280000\,\frac{\alpha^\circ r^4}{l}$.	$Pa = 470000\,\frac{\alpha^\circ b^4}{l}$.

In what relates to the strength of torsion, numerous experiments made upon cast iron have given $\sqrt{\frac{KE}{2}} = 30000$ to 66000 lbs., if therefore, a five-fold security be taken, then is $\frac{\pi}{2}\sqrt{\frac{KE}{2}} = 12600$ lbs. therefore, for the round cast iron axle $Pa = 12600\,r^3$, and for the square $= 15000\,b^3$.

The same formulæ hold good for axles of wrought iron, but for wooden ones we may put $Pa = 1260\,r^3$ and $= 1500\,b_3$, *i. e.* the modulus of strength $= \frac{1}{10}$ that of iron axles. The modulus of strength for steel $\sqrt{\frac{KE}{2}}$ must be taken at twice that of iron, and gun metal at one half.*

---

* See Appendix.

Examples.—1. The iron upright axle of a turbine exerts at the circumference of a toothed wheel of 15 inches radius reposing upon it, a force of 2500 lbs.; what thickness must be given to it? $Pa = 2500 \,.\, 15 = 37500$, and if we put $r^3 = \frac{Pa}{12600} = \frac{37500}{12600} = \frac{375}{126}$, we shall obtain $r = \sqrt[3]{\frac{375}{126}} = 1{,}44$ inches; hence, the thickness of the axle $2r = 2{,}88$ inches, for which 3 inches may be assumed. If the distance of the toothed wheel from the water wheel is 60 inches, the torsion of the axle $= \alpha^{\circ} = \frac{Pal}{160000\, r^4} = \frac{37500 \,.\, 60}{160000 \,.\, 1{,}44^4} = \frac{375 \,.\, 6}{160 \,.\, 4{,}28} = \frac{14{,}06}{4{,}28} = 3^{\circ}\ 3'$, therefore very considerable. —2. On a square axle of fir, a force $P = 500$ lbs., acts at an arm of 20 feet, whilst the load is applied at an arm of 2 feet, the distance measured in the direction of the axis $l = 10$ feet; how thick must this axle be made, and how great is the torsion? It is $Pa = Qb = 500 \,.\, 2 \,.\, 12 = 120000$ inch lbs.; but the load $Q = \frac{a}{b} P = 5000$ lbs.; half the side b of the axle is determined by $b^3 = \frac{Pa}{1500} = \frac{120000}{1500} = 80$; hence $b = \sqrt[3]{80} = 4{,}31$ inches, and the whole side $= 8{,}62$ inches. The torsion amounts to $\alpha^{\circ} = \frac{Pal}{5800 \,.\, b^4} = \frac{120000 \,.\, 12 \,.\, 10}{5800 \,.\, 4{,}31^4} = \frac{144000}{58 \,.\, 345} = 7^{\circ}$; therefore, here very considerable. In general, less torsion is allowed, and therefore the axles are made much stronger. Generally, this angle does not amount to ½ a degree. If, however, we put $\alpha^{\circ} = \frac{1}{2}^{\circ}$, for this case we shall obtain $b^4 = \frac{144000}{58 \,.\, \frac{1}{2}} = 4965$, hence $b = \sqrt[4]{4965} = 8{,}4$ inches, and $2b = 16{,}8$ inches. According to Gerstner, the angle of torsion of an axle ought not to amount to more than $0{,}1^{\circ}$.

Remark. If an axle has to sustain relative elasticity and that of torsion, we must make the calculation for both, and apply the greater ratio of the dimensions found.

SECTION IV.

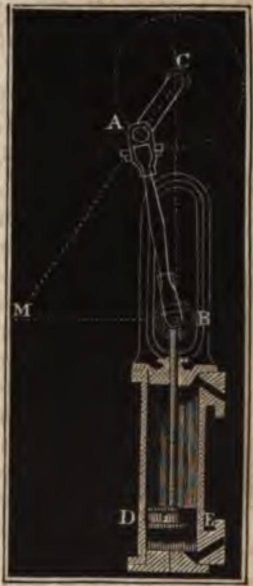
DYNAMICS OF RIGID BODIES.

CHAPTER I.

DOCTRINE OF THE MOMENT OF INERTIA.

§ 212. *Kinds of Motion.*—The motion of a rigid body is either one of *translation*, or one of *rotation*, or both. The spaces described by the particles of a body in motion of translation or progression are parallel and equal to each other; on the other hand, in motion of revolution or rotation, the particles of a body describe concentric circles about a certain straight line, which is called the *axis of revolution*. Compound motion may be regarded as a motion of revolution about a *moving axis*. The latter is either uniform or variable.

Fig. 246.



The piston *DE*, or the piston-rod *BF* of a pump or steam-engine, Fig. 246, has a motion of translation, whilst the arm or the crank *AC* has one of rotation, and the connecting rod *AB* a compound motion; for one of its extremities *B* has a progressive, the other *A* a rotatory motion. The axis of revolution, in a rotating cylinder, is invariable; that of the connecting rod *AB*, on the other hand, is variable; for it is the intersection *M* of the perpendicular to the direction of the axis of the rod *BM*, and the prolongation of the crank *CA*.

§ 213. *Rectilinear Motion.*—The laws of motion of a material point found in § 81, &c., have their direct application in rectilinear progressive motion. The particles of the mass M_1, M_2, M_3, &c., of a body, moving with the acceleration p by their inertia, offer resistance to motion with the forces M_1p, M_2p, M_3p, &c. (§ 53); and since the motions of all these take place in lines parallel to each other, the directions, therefore, of these forces are parallel; hence, the resultant of all these forces arising from the inertia is equivalent to the sum $M_1 p, + M_2 p + M_3 p, + \ldots = (M_1 + M_2 + M_3 + \ldots)\, p = Mp$, where M represents the mass of the whole body, and its point of application coincides with the centre of gravity of the body. In order, therefore, to change the motion of an otherwise freely moving body of the mass M or of the weight $G = Mg$ into one rectilinear and progressive, a force $P = Mp = \frac{Gp}{g}$, whose direction passes through the centre of gravity S of the body, is requisite. If, in virtue of the action of the force P, the velocity c acquired while the body describes the space s, increases to the velocity v, then the mechanical effect accumulated in the mass (§ 71) is:

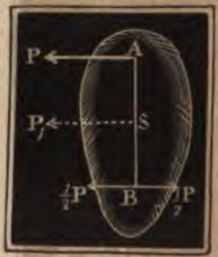
$$Ps = \left(\frac{v^2 - c^2}{2}\right) M = \left(\frac{v^2 - c^2}{2\,g}\right) G = (h - h_1)\, G.$$

Example. The piston of a pump, with its rod or that of a steam engine or blowing machine, has a variable motion; it has no velocity at its highest and lowest positions, and at its mean position its velocity is a maximum. If the weight of the piston and rod be $= G$, and its maximum velocity at the middle of its ascent and descent $= v$, the effect which it will accumulate by virtue of its inertia in the first half of its path, and will give out again in the second half $= \frac{v^2}{2g}\, G$. For $G = 800$ lbs. and $v = 5$ ft. this effect $= 0{,}0155 \,.\, 5^2 \,.\, 800 = 310$ ft. lbs.; were half the path of the piston $s = 4$ feet, we should then have the mean force which would be requisite to accelerate the motion of the piston in the first half of this path, and which it would exert in the second half by its retardation:

$$P = \frac{v^2}{2\,gs} \,.\, G = \frac{310}{4} = 77 \text{ lbs.}$$

§ 214. *Motion of Rotation.*—If the moving force P of a body AB, Fig. 247, does not pass through the centre of gravity S, the body will take up a rotation about this point, and this motion will go on as if the force were directly applied to it, as may be proved in the following manner. From the centre of gravity let a perpendicular SA be drawn to the direction of the force; let this be prolonged backwards and the prolongation SB be made equal to the perpendicular, and let two forces, balancing each other and acting parallel to P, the one $+ \frac{1}{2}\, P$, and the other $- \frac{1}{2}\, P$, be applied to the point B. The force $+ \frac{1}{2}\, P$ will give, when combined with half the force P applied at A, the force applied at the centre of gravity S, $P_1 = \frac{1}{2}\, P + \frac{1}{2}\, P = P$, for which the force $- \frac{1}{2}\, P$ will form a couple with the second half of P applied at A; there will result, therefore, from the excentrically acting force P, a force P acting through the centre of gravity, which will move forward this point, together with the whole body, and a couple $(\frac{1}{2}\, P, - \frac{1}{2}\, P)$, which will cause the body to rotate about the centre of gravity without producing a pressure on it. The statical moment of this couple will be $= SA \,.\, \frac{1}{2}\, P + SB \,.\, \frac{1}{2}\, P = SA \,.\, P$ equivalent to the statical moments of the

Fig. 247.



[illegible]

A, B, [illegible] C [illegible]

$A =$ [illegible], $B =$ [illegible]

[illegible]

$A + B$ [illegible]

[illegible]

[illegible] of [illegible] given

the axis OX; then these three [illegible]-

-[illegible] axes [illegible] principal axes

[illegible] ABC principal [illegible].

The centre of gravity and the prin-

-cipal axes have the advantage of

simplifying the equations of motion

and by making a number of their

terms disappear. When they are taken

for the origin and coordinate axes they

may also [illegible] dynamical

[illegible]. The determination of

the motion of a solid body requires

the consideration of certain definite in-

-tegrals taken for the whole extent of

the body. It may be assumed that

a [illegible] [illegible]
[illegible]

[illegible] distance from each other

[illegible] ignored any [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

framing [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

and [illegible]

[illegible]

will [illegible]

[illegible] and [illegible] AN [illegible] equal and [illegible] equation of [illegible] surface, designate by [illegible] AN [illegible]

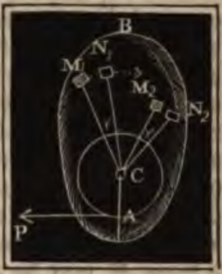
[illegible] surface of [illegible]

force applied at A; consequently the resultant rotation will be the same as if the centre of gravity S were fixed, and P acted alone.

According to this, therefore, every arbitrarily directed force communicates two motions to a body; one of translation and one of rotation; and hence it is necessary to study the laws of this latter.

If, finally, a body be constrained by a path or directrix to assume a motion of translation, an excentric force will then produce the same effect as if it were applied to the centre of gravity, because the forces of rotation will be taken up by the directrix.

Fig. 248.



§ 215. *Moment of Inertia.*—In the rotation of a body AB, Fig. 248, about a fixed axis C, all its points describe equal angles in equal times. If the body rotate in a certain time through the angle ϕ^0 or arc $\phi = \frac{\phi^0}{180^0} \cdot \pi$, the elements of the body $M_1, M_2 \ldots$ will describe at the distances $CM_1 = y_1$, $CM_2 = y_2$, &c., from the axis, the spaces ϕy_1, ϕy_2, &c. If the angular velocity, *i. e.* the velocity of those points of the body which are distant a unit of length (a foot) from the axis of revolution $= \omega$, then the simultaneous velocities of the elements of the mass at the distances y_1, y_2, &c., will be $= \omega y_1, \omega y_2$, &c.; hence their *vis viva* is $(\omega y_1)^2 M_1, (\omega y_2)^2 M_2$, &c., and the sum, or the *vis viva* of the whole body: $(\omega y_1)^2 M_1 + (\omega y_2)^2 M_2 + \ldots = \omega^2 (M_1 y_1^2 + M_1 y_2^2 + \ldots)$.

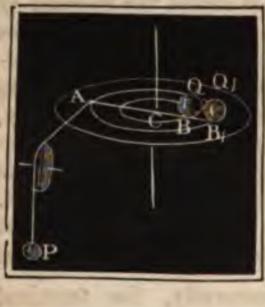
The sum of the product of the particles of the mass and the squares of their distances from the axis of revolution, is called the *moment of inertia* of the body, the *moment of rotation*, and the *moment of the mass.*? If we represent this by T, and therefore put $T = M_1 y_1^2 + M_2 y_2^2 + \ldots$, we then obtain $\omega^2 T$ for the *vis viva* of a body revolving with the angular velocity ω. Hence, to communicate to a body, already in a state of rest, an angular velocity ω, a mechanical effect $Ps = \frac{1}{2} \omega^2 T$ must be expended; and inversely, a body produces this effect when it passes from this angular velocity into a state of rest. If, generally, the initial angular velocity of a rotating body $= \varepsilon$, and the terminal angular velocity $= \omega$, we then have for the mechanical effect produced, $Ps = \left(\frac{\omega^2 - \varepsilon^2}{2}\right) T$; and inversely, the terminal velocity corresponding to an expended or accumulated mechanical effect Ps:

$$\omega = \sqrt{\varepsilon^2 + \frac{2\,Ps}{T}}.$$

Example. If a body, AB, Fig. 248, originally at rest, and capable of turning about a fixed axis C, possesses a moment of inertia of 50 ft. lbs., and by means of a cord lying over a pulley is made to rotate in a path $s = 5$ feet, the acquired angular velocity of this body will be $\omega = \sqrt{\frac{2\,Ps}{T}} = \sqrt{\frac{2\,.\,20\,.\,5}{50}} = \sqrt{4} = 2$ feet, *i.e.*, each point at the distance of one foot from the axis of revolution will describe, after the accumulation of this mechanical effect, two feet in a second. The time of a revolution $t = \frac{2\pi}{\omega} = 3{,}1416$ seconds, and the number of revolutions per minute $u = \frac{60}{t} = \frac{60}{3{,}1416} = 19{,}1$. If the angular velocity found $\omega = 2$ feet, passes into the velocity $s = \frac{3}{4}$ feet, then this mass produces the mechanical effect $P_1 s_1 = \left[2^2 - \left(\frac{3}{4}\right)^2\right] \cdot \frac{50}{2} = \left(4 - \frac{9}{16}\right) \cdot 25 = \frac{55}{16} \cdot 25$ $= 85{,}93$ ft. lbs.; for instance, a weight P_1 of 10 lbs., raised 8,593 feet high.

§ 216. *Reduction of Inert Masses.*—If the angular velocities of two masses M_1 and M_2 be equal, if, for instance, these masses belong to one and the same rotating body, their *vires vivæ* will be to each other as their moments of inertia, $T_1 = M_1 y_1^2$ and $T_2 = M_2 y_2^2$, and if these be equal to each other, the masses will possess equal *vires vivæ*. Two masses have, therefore, an equal influence on the state of motion of a revolving body, and one may be replaced by the other without causing any change in the condition of motion, if they possess equal moments of inertia, $M_1 y_1^2$ and $M_2 y_2^2$, and are inversely as the squares of their distances from the axis of revolution. By help of the formula $M_1 y_1^2 = M_2 y_2^2$, a mass may be reduced from one distance to another; *i. e.*, a mass M_2 may be found, which, at the distance y_2, has the same influence on the condition of motion of the revolving body, as a given mass M_1 at the distance y_1; namely, $M_2 = \frac{M_1 y_1^2}{y_2^2} = \frac{T_1}{y_2^2}$, *i. e., the mass reduced to the distance y_2 is the quotient of the moment of inertia of the mass and the square of that distance.*

Fig. 249.



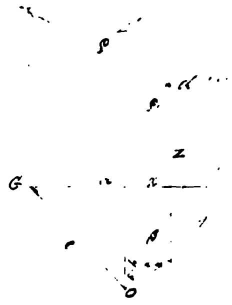
Two weights, Q and Q_1, attached to an axle, Fig. 249, with the arms $CB = b$ and $CB_1 = a$, have an equal influence, in virtue of their masses, on the motion of the wheel and axle, if $Q_1 a^2 = Qb^2$; therefore, $Q_1 = \frac{Qb^2}{a^2}$. Hence, if a force P act at the arm $CA = CB_1 = a$, and cause rotation in a mass of the weight Q at the distance $CB = b$, we shall have then to reduce the latter to the arm a of the force P; therefore, for Q substitute $Q_1 = \frac{Qb^2}{a^2}$ and the mass set into motion by P will be $= \left(P + \frac{Qb^2}{a^2}\right) \div g$; whence the accelerated motion of the weight:

$$P = p = \frac{\text{force}}{\text{mass}} = \frac{P}{P + \frac{Qb^2}{a^2}} \cdot g = \frac{Pa^2}{Pa^2 + Qb^2} \cdot g,$$

and the angular acceleration $\frac{p}{a} = \frac{Pa}{Pa^2 + Qb^2} \cdot g$.

Example. The weight of a rotating mass $Q = 360$ lbs., its distance from the axis of rotation $b = 2{,}5$ feet, the weight constituting the moving force $P = 24$ lbs., and its arm $a = 1{,}5$ feet, the inert mass put into accelerated motion by $P = \left[P + \left(\frac{2{,}5}{1{,}5}\right)^2 Q\right] \div$

a definition important to be remembered and a term much used


z
G
x
O

$g = 0{,}031 \left(24 + \frac{25}{9} \,.\, 360\right) = 0{,}031 \,.\, 1024 = 31{,}75$ lbs.; and hence the accelerated motion of the weight : $p = \frac{24}{31{,}75} = 0{,}756$ feet; on the other hand the acceleration of the motion of the mass $Q_1 = \frac{b}{a} \,.\, p = \frac{25}{15} p = \frac{5 \,.\, 0{,}756}{3} = 1{,}26$ ft., and the angular acceleration $= \frac{p}{a} = 0 \,.\, 504$ feet. After 4 seconds the acquired angular velocity will be $\omega = 0 \,.\, 504 \,.\, 4 = 2{,}016$ feet, and the corresponding distance $= \frac{2{,}016 \,.\, 4}{2} = 4{,}032$ feet; consequently the angle of revolution $\phi^\circ = \frac{4{,}032}{\pi} \,.\, 180^\circ = 1{,}28 \,.\, 180^\circ = 230^\circ\, 24'$; consequently the space described by the weight $P = \frac{pt^2}{2} = \frac{0{,}756 \,.\, 4^2}{2} = 6{,}05$ feet.

§ 217. *Reduction of the Moment of Inertia.*—When the moment of inertia of a body or system of bodies about an axis, passing through the centre of gravity S of the body, is known, the moment of inertia about any other axis running parallel to it, may be easily found. Let Fig. 250 be the first axis of revolution, passing through the centre of gravity S, D the second axis, for which the moment of inertia of the body is to be determined; further, let $SD = e$ be the distance of the two axes, and let $SN_1 = x_1$ and $N_1M_1 = y_1$ the rectangular co-ordinates of a particle of the mass M_1 of the body. Now the moment of inertia of this particle about $D = M_1 \,.\, \overline{DM_1}^2 = M_1\, (\overline{DN_1}^2 + \overline{N_1M_1}^2) = M_1\, [(e + x_1)^2 + y_1^2]$, and about $S = M_1 \,.\, \overline{SM_1}^2 = M_1 \,.\, (\overline{SN_1}^2 + \overline{N_1M_1}^2) = M_1\, (x_1^2 + y_1^2)$; hence the difference of both moments:

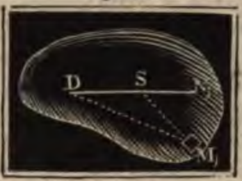
$$= M_1\, (e^2 + 2\, ex_1 + x_1^2 + y_1^2) - M_1\, (x_1^2 + y_1^2) = M_1e^2 + 2\, M_1\, ex_1.$$

For another particle of the mass M_2, it is $= M_2e^2 + 2\, M_2\, ex_2$, for a third $= M_3e^2 + 2\, M_3\, ex_3$, &c., and for all the particles together:

$$= (M_1 + M_2 + M_3 + \ldots)\, e^2 + 2e\, (M_1x_1 + M_2x_2 + M_3x_3 + \ldots).$$

But $M_1 + M_2 + \ldots$ is the sum M of all the masses, and $M_1x_1 + M_2x_2 + \ldots$ the sum Mx of their statical moments; hence the difference between the moment of inertia T_1 of the whole body about the axis D and the moment of inertia T about S: $T_1 - T = Me^2 + 2\, eMx$. But since, lastly, for every plane passing through the centre of gravity, the sum of the statical moments of the particles on the one side is as great as that of those on the other, the algebraical sum of all the particles therefore $= 0$; we have, also, $Mx = 0$, and, therefore, $T_1 - T = Me^2$; *i. e.* $T_1 = T + Me^2$.

Fig. 250.



The moment of inertia of a body about an axis not passing through the centre is equivalent to its moment of inertia about an axis running parallel to it through the centre of gravity, increased by the product of the mass of the body and the square of the distance of the two axes.

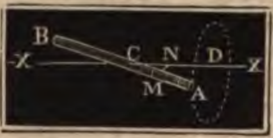
It is also seen from this that of all the moments of inertia about parallel axes, that one is the least whose axis is a line of gravity of the body.

§ 218. It is necessary to know the moments of inertia of the principal geometric bodies, because they very often come into application in mechanical investigations. If these bodies are homogeneous, as in the following we will always suppose to be the case, the particles of the mass M_1, M_2, &c., are proportional to the corresponding particles of the volume V_1, V_2, &c., and hence the measure of the moment of inertia may be replaced by the sum of the particles of the volume, and the squares of their distances from the axis of revolution. In this sense the moments of inertia of lines and surfaces may also be found.

If the whole mass of a body be supposed to be collected into one point, its distance from the axis may be determined on the supposition, that the mass so concentrated possesses the same moment of inertia as if distributed over its space. This distance is called the *radius of gyration, or of inertia.* If T be the moment of inertia, M the mass, and r the radius of gyration, we then have $Mr^2 = T$, and hence $r = \sqrt{\frac{T}{M}}$. We must bear in mind that this radius by no means gives a determinate point, but a circle only, within whose circumference the mass may be considered as arbitrarily distributed.

If into the formula $T_1 = T + Me^2$, we introduce $T = Mr^2$ and $T_1 = Mr_1^2$, we obtain $r_1^2 = r^2 + e^2$, *i. e.* the square of the radius of gyration referred to a given axis = the square of the radius of gyration referred to a parallel line of gravity, plus the square of the distance between the two axes.

Fig. 251.



§ 219. *The Rod.*—The moment of inertia of a rod AB, Fig. 251, which turns about an axis $X\overline{X}$ through its middle point C, may be determined in the following manner. The cross section of the rod = F, its half length $CA = l$, and the angle which its axis includes with the axis of rotation $ACX = \alpha$. If we divide half the length into n parts, we then obtain n portions each of the contents $\frac{Fl}{n}$; the distances of these portions from the middle C are $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c., hence their distances from the axis $X\overline{X}$ are $MN, = \frac{l}{n} \sin.\alpha\ \frac{2l}{n} \sin.\alpha\ \frac{3l}{n} \sin.\alpha$, &c., and these squares $= \left(\frac{l \sin.\alpha}{n}\right)^2, 4\left(\frac{l \sin.\alpha}{n}\right)^2, 9\left(\frac{l \sin.\alpha}{n}\right)^2$ &c. By the multiplication of these with the contents of an element $\frac{Fl}{n}$, and by the addition of all the products, we have the moment of half the rod:

$$T = \frac{Fl}{n}\left[\left(\frac{l \sin.\alpha}{n}\right)^2 + 4\left(\frac{l \sin.\alpha}{n}\right)^2 + 9\left(\frac{l \sin.\alpha}{n}\right)^2 + \ldots\right]$$

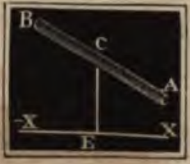
[illegible]

$= \frac{Fl^3 \sin. \alpha^2}{n^3} (1^2 + 2^2 + 3^2 + \ldots + n^2)$, or, $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n^3}{3}$, $T = \frac{Fl^3 \sin. \alpha^2}{3}$. But as the volume of half the rod Fl is to be considered as the mass M, it follows, finally, that $T = \frac{1}{3} Ml^2 \sin. \alpha^2$. The distance of the end of the rod from the axis $\overline{XX}$ is $AD = a = l \sin. \alpha$, hence it follows more simply that $T = \frac{1}{3} Ma^2$, which formula is also to be applied to the whole rod, if M be the mass of the whole. A mass M_1 at the extremity A of the rod, has the moment of inertia $M_1 a^2$, hence if we make $M_1 = \frac{1}{3} M$, it has then the same moment of inertia as the rod. Whether, therefore, the mass be uniformly distributed over the rod, or its third part be collected at the extremity A, it comes to the same thing.

If we put $T = Mr^2$, we obtain $r^2 = \frac{1}{3} a^2$, and hence the radius of gyration of the rod: $r = a \sqrt{\frac{1}{3}} = 0{,}5773 \,.\, a$.

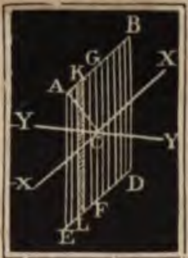
If the rod stands perpendicularly to the axis of rotation $a = l$, therefore, $T = \frac{1}{3} Ml^2$. If, lastly, the rod AB, Fig. 252, be not in the same plane with the axis of rotation, and if the shortest distance between the two axes be $CE = e$, we shall then have from § 217, the moment of inertia $T_1 = T + Me^2 = M(e^2 + \frac{1}{3} a^2)$.

Fig. 252.



§ 220. *Rectangle and Parallelopiped.*—The moment of inertia of a rectangular plate, $ABDE$, Fig. 253, which turns about an axis $\overline{XX}$ passing through its middle C, and parallel to a side, is as for a rod $= \frac{1}{3} Ml^2$, but if the axis $\overline{YY}$ stand perpendicular to the plane, the moment of inertia is then determined from the former paragraph in the following manner; the half $AEFG$ is divided by lines parallel to the side AE into strips of equal breadth, such as KL, the moments of these strips determine, and then added together. If the half length $FE = GA = l$, half the breadth $CF = CG = b$, and the number of parts $= n$, the area of a part $= \frac{l}{n} \,.\, 2\,b = \frac{2\,bl}{n}$. The distances from C of these strips are in the series $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c., therefore their squares $\left(\frac{l}{n}\right)^2, 4\left(\frac{l}{n}\right)^2, 9\left(\frac{l}{n}\right)^2$, &c., hence moments of inertia are:

Fig. 253.



$$\frac{2\,bl}{n}\left[\left(\frac{l}{n}\right)^2 + \frac{b^2}{3}\right], \frac{2\,bl}{n}\left[4\left(\frac{l}{n}\right)^2 + \frac{b^3}{3}\right], \frac{2\,bl}{n}\left[9\left(\frac{l}{n}\right)^2 + \frac{b^2}{3}\right]$$

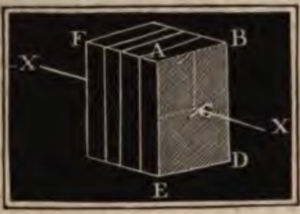
and the moment of inertia of half the plate:

$$T = \frac{2\,bl}{n}\left[\left(\frac{l}{n}\right)^2 (1 + 4 + 9 + \ldots + n^2) + n \,.\, \frac{b^2}{3}\right]$$

$$= \frac{2\,bl}{n}\left[\left(\frac{l}{n}\right)^2 \cdot \frac{n^3}{3} + \frac{n\,b^2}{3}\right] = \frac{2\,bl\,(l^2+b^2)}{3} = \tfrac{1}{3}\,M\,(l^2+b^2),$$

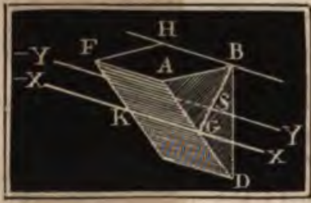
because $2\,bl$ must be considered as the mass of half the plate. As the semi-diagonal $CA = d = \sqrt{l^2+b^2}$, we may put: $T = \frac{1}{3}\,Md^2$. If M represent the whole mass, the formula holds good for the moment of inertia of the whole plate. Since, further, a parallelopiped BEF, Fig. 254, may be decomposed by parallel planes into equal rectangular plates, the above formula is also applicable to this, if the axis of rotation pass through the middle points of any two opposite surfaces. It follows, besides, from this formula, that the moment of inertia of the parallelopiped is equivalent to the moment of inertia of the third part of its mass applied at a corner A.

Fig. 254.



From the formula for the moment of inertia of a parallelopiped, that of a triangular prism may be also calculated. The diagonal plane ADF divides the parallelopiped into two equal triangular prisms with rectangular and triangular bases ABD, Fig. 255; hence, for the rotation about an axis $X\overline{X}$, passing through the middle C and K of the hypothenuse, the moment of inertia $= \frac{1}{3}\,Md^2$. If now we make use of the proposition § 217, we obtain the moment of inertia about an axis $Y\overline{Y}$, passing through the centres of gravity S and S_1 of the bases: T

Fig. 255.



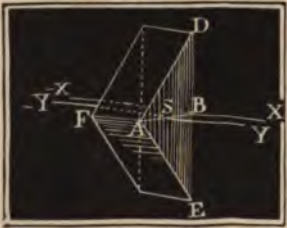
$$= \tfrac{1}{3}\,Md^2 - M\,.\,\overline{CS}^2 = M\left[\frac{d^2}{3} - (\tfrac{1}{3}\,\overline{CB})^2\right] = M\left[\frac{d^2}{3} - \left(\frac{d}{3}\right)^2\right]$$

$= \frac{2}{9}\,Md^2$, and it follows also that the moment of inertia about a side edge BH:

$T_1 = T + M\,.\,\overline{SB}^2 = \frac{2}{9}\,Md^2 + M\,(\frac{2}{3}\,d)^2 = \frac{6}{9}\,Md^2 = \frac{2}{3}\,Md^2$, where d always represents half the hypothenuse of the triangular base.

§ 221. *Prisms and Cylinders.*—For a prism $ADFE$, Fig. 256, with isosceles triangular bases, the moment of inertia about an axis $X\overline{X}$, which connects the middle points of the bases, $T = \frac{2}{3}\,Md^2$, if d represent half the side AD of the surface of the base, because this surface may be decomposed by the line of the height AB into two equal rectangular triangles. If now the height AB of the isosceles triangular base $= h$, we have then for the moment of inertia

Fig. 256.



[illegible]

[illegible] pursued [illegible]

[illegible]

[illegible] rectangular [illegible]

[illegible] the

[illegible] adjacent edges [illegible]

[illegible]

[illegible]

[illegible] and [illegible] $M = V\rho$ [illegible]

[illegible] $B = \frac{1}{5} M_1 (c^2 + a^2)$ and $C = \frac{1}{5} M_1 (a^2 + b^2)$ [illegible]

[illegible]

of this prism about the axis $\overline{YY}$ passing through the centres of gravity of the base:

$$T = \tfrac{2}{3}\, M d^2 - M\left(\frac{h}{3}\right)^2 = M\left(\tfrac{2}{3}\, d^2 - \tfrac{1}{9}\, h^2\right) = \tfrac{1}{3} M\left(2\, d^2 - \tfrac{1}{3}\, h^2\right),$$

and finally, the moment of inertia about the edge passing through the points A and F of the bases:

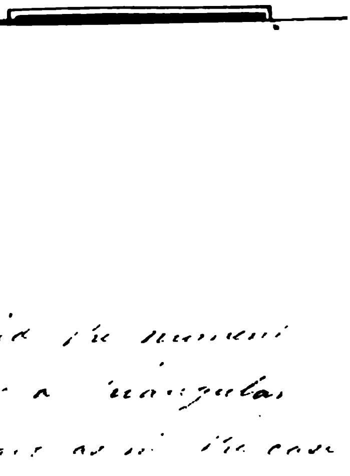
$$T_1 = T + M\left(\tfrac{2}{3} h\right)^2 = M\left(\frac{2\, d^2}{3} - \frac{h^2}{9} + \frac{4\, h^2}{9}\right) = \tfrac{1}{3} M\left(2\, d^2 + h^2\right).$$

Hence the moment of inertia of a right and regular prism revolving about its geometric axis may be found. Let h be the height CA, Fig. 257, of one of the supplementary triangles, $CA = CB = 2\, d = r$ the radius of the base or of a supplementary triangle, and M the mass of the entire prism. We have then by the last formula:

$$T = \tfrac{1}{3}\, M\left(\frac{r^2}{2} + h^2\right).$$

The regular prism becomes a cylinder when $h = r$, hence the moment of inertia [illegible]

Fig. 257.



[illegible]

of this prism about the axis $\overline{YY}$ passing through the centres of gravity of the base:

$$T = \tfrac{2}{3}\, M d^2 - M \left(\frac{h}{3}\right)^2 = M\left(\tfrac{2}{3}\, d^2 - \tfrac{1}{9}\, h^2\right) = \tfrac{1}{3}\, M\left(2\, d^2 - \tfrac{1}{3}\, h^2\right),$$

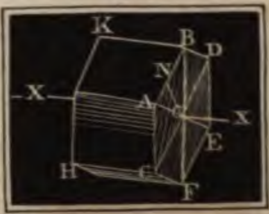
and finally, the moment of inertia about the edge passing through the points A and F of the bases:

$$T_1 = T + M\left(\tfrac{2}{3}h\right)^2 = M\left(\frac{2\, d^2}{3} - \frac{h^2}{9} + \frac{4\, h^2}{9}\right) = \tfrac{1}{3}\, M\left(2\, d^2 + h^2\right).$$

Hence the moment of inertia of a right and regular prism revolving about its geometric axis may be found. Let h be the height CA, Fig. 257, of one of the supplementary triangles, $CA = CB = 2\, d = r$ the radius of the base or of a supplementary triangle, and M the mass of the entire prism. We have then by the last formula:

$$T = \tfrac{1}{3}\, M\left(\frac{r^2}{2} + h^2\right).$$

Fig. 257.

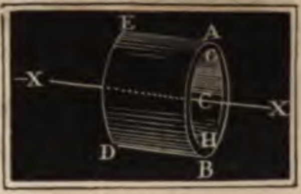


The regular prism becomes a cylinder when $h = r$, hence the moment of inertia of a right cylinder about its geometric axis is:

$$T = \tfrac{1}{3}\, M\left(\frac{r^2}{2} + r^2\right) = \tfrac{1}{2}\, M r^2.$$

The moment of inertia of a cylinder is, therefore, equivalent to the moment of inertia of half the mass of the cylinder collected at its circumference, or equivalent to the moment of inertia of the entire mass, at the distance $r \sqrt{\tfrac{1}{2}} = 0{,}7071 \,.\, r$.

Fig. 258.



If the cylinder $ABDE$ be hollow, Fig. 258, the moment of inertia of the hollow space must be subtracted from that of the solid cylinder. If the outer radius $CA = r_1$, and the inner $CG = r_2$, we then have from what has preceded the moment of inertia of the hollow cylinder:

$$T = \tfrac{1}{2}\left(M_1 r_1^2 - M_2 r_2^2\right) = \tfrac{1}{2}\, \pi\left(r_1^2 \,.\, r_1^2 - r_2^2 \,.\, r_2^2\right) = \tfrac{1}{2}\, \pi\left(r_1^4 - r_2^4\right)$$
$$= \tfrac{1}{2}\, \pi\left(r_1^2 - r_2^2\right)\left(r_1^2 + r_2^2\right) = \tfrac{1}{2}\, M\left(r_1^2 + r_2^2\right),$$

because the volume considered as the mass $= \pi\left(r_1^2 - r_2^2\right)$. If r be the mean radius $\frac{r_1 + r_2}{2}$, and b the breadth of the annular surface, we then have $T = M\left(r^2 + \frac{b^2}{4}\right)$.

§ 222. *Cone and Sphere.*—The moment of inertia of a right cone, as well as that of a sphere, may be calculated from the formula for the moment of inertia of a cylinder. Let ACB, Fig. 259, be a cone revolving about its geometric axis, $DA = DB = r$ the radius of its base, and $CD = h$ its height coinciding with the axis.

If we make n slices parallel to the base at equal distances, we then obtain thin discs of the radii $\frac{r}{n}, 2\frac{r}{n}, 3\frac{r}{n} \ldots n\frac{r}{n}$, and of the common height $\frac{h}{n}$. The half volumes of these discs are $\pi\left(\frac{r}{n}\right)^2 \cdot \frac{h}{2n}, \pi\left(\frac{2r}{n}\right)^2 \cdot \frac{h}{2n}, \pi \cdot \left(\frac{3r}{n}\right)^2 \cdot \frac{h}{2n}$, &c., and hence their moments of inertia:

$$\pi\left(\frac{r}{n}\right)^4 \cdot \frac{h}{2n}, \pi\left(\frac{2r}{n}\right)^4 \frac{h}{2n}, \pi\left(\frac{3r}{n}\right)^4 \frac{h}{2n}, \text{\&c.};$$

the sum of these values gives, finally, the moment of inertia of the entire cone:

$$T = \frac{\pi r^4 h}{2n^5} 1^4 + 2^4 + 3^4 + \ldots + n^4),$$

and as $1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n^5}{5}$, we have

$$T = \frac{\pi r^4 h}{10} = \frac{3}{10} \cdot \frac{\pi r^2 h}{3} \cdot r^2 = \frac{3}{10} Mr^2.$$

Fig. 259.

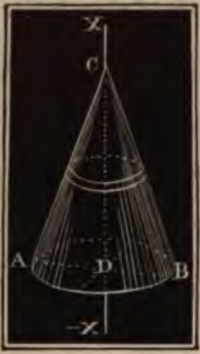
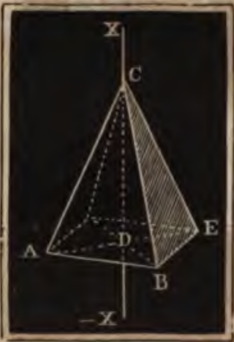


Fig. 260.



For the right pyramid ACE, Fig. 260, with rectangular base, under the same circumstances $T = \frac{1}{6} Md^2$, if d represent the semi-diagonal DA of the base. Also by subtraction of the two moments of inertia, the moment of inertia of a right truncated cone with the radii r_1 and r_2 and the heights h_1 and h_2 may be obtained:

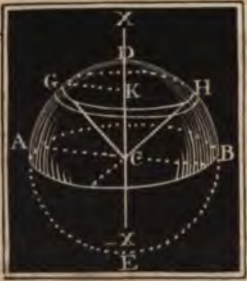
$$T = \frac{\pi}{10}(r_1^4 h_1 - r_2^4 h_2) = \frac{\pi h_1}{10 r_1}(r_1^5 - r_2^5),$$

or, since the mass

$$M = \frac{\pi}{3}(r_1^2 h_1 - r_2^2 h_2) = \frac{\pi h_1}{3 r_1}(r_1^3 - r_2^3),$$

$$T = \tfrac{3}{10} M\left(\frac{r_1^5 - r_2^5}{r_1^3 - r_2^3}\right).$$

Fig. 261.

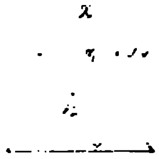


In a similar manner, we find the moment of inertia of a sphere, revolving about one of its diameters $DE = 2r$. Let us divide the hemisphere ADB, Fig. 261, by sections parallel to the base ACB, into n equally thick circular slices, as GKH, &c., and determine their moments. The square $\overline{GK}^2$ of the radius of any such slice is:

$$= \overline{CG}^2 - \overline{CK}^2 = r^2 - \overline{CK}^2,$$

hence its moment of inertia

$$= \tfrac{1}{2}\pi \cdot \frac{r}{n}(r^2 - \overline{CK}^2)^2,$$



$y : r :: x : h$ $y = \frac{rx}{h}$ [illegible] the area $= \pi y^2$ [illegible] elementary cylinder $\pi y^2 dx$. ~~[illegible]~~ $= \frac{[illegible]}{[illegible]}$. The moment of inertia of this elementary cylinder about an axis passing through [illegible] perpendicular to its plane [illegible] $\frac{\pi y^4 dx}{2}$, putting for y its value [illegible] the moment of [illegible]

$= \frac{\int \pi r^4 x^4 dx}{2h^4} = \frac{\pi r^4 h^5}{5 \cdot 2 h^4} = \frac{[illegible]}{10}$ [illegible] of the mass $= \frac{1}{3}\pi r^2 h$ [illegible]

$= \frac{3}{10} r^2 M$

[illegible] $\frac{c}{d} : \frac{c}{d} :: x$ [illegible] $b : \frac{c}{d}$ $= \frac{bc}{dc}$ [illegible]

[illegible]

[illegible] $= \frac{1}{3}$ [illegible] C [illegible] $= \frac{1}{3}$ [illegible]

$C^2 = \frac{1}{3}\int \frac{c c^2 x^2}{h^2 e} dx$ [illegible] $= \frac{1}{3}$ [illegible] $+ \frac{[illegible]}{5 \cdot 4}$ [illegible]

[illegible] $I = \frac{1}{[illegible]} M$ [illegible] the ellipsoid transformed into an infinite number of elliptical cones parallel to the plane yz, and each cone [illegible] divided into an infinite number of parallelopipeds [illegible] the axis of x and terminating at the surface and each parallelopiped into elements of which the dimensions are all infinitely small. The limits [illegible] [illegible] the value of this [illegible] are given by equations [illegible] definite integral will express [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] values of [illegible] transformed to [illegible] [illegible] ellipsoid by the plane xy. [illegible] the [illegible] of [illegible] parallel to the plane yz [illegible]

[illegible] distance x from [illegible] plane [illegible] the [illegible] integral [illegible] respect [illegible] will be taken between the limits $x = -a$ and $x = a$ and will express the moment of the entire ellipsoid. Our first original differential equation is [illegible] $dx\,dy\,dz\,(x^2 + y^2)$. Integrating with respect [illegible] $dx\,dy$. The limits given by [illegible] $z = \pm c\sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}$ the definite integral is then [illegible] $2x^2 c\sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}} + 2cy^2\sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}$. If we put for the sake of abbreviation [illegible] $\frac{x^2}{a^2} - 1$, the integral of the first part [illegible] with respect [illegible] dy [illegible] the equation of the section of the ellipsoid by the plane [illegible] $\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$ [illegible] the integral [illegible] respect [illegible] dy [illegible] the value [illegible] dy [illegible] integrating between the limits [illegible] $I_x = \frac{4\pi a^3 bc}{15}$. By simply changing letters $I_y = \frac{4\pi b^3 ac}{15}$ & $I_z = I_x + I_y = \frac{4\pi abc}{15}(a^2 + b^2)$. The Mass of the ellipsoid $= M$ its volume $V = \frac{4\pi abc}{3}$

[illegible]

[illegible]

[illegible] A [illegible] M [illegible], B [illegible] M [illegible] C [illegible] M [illegible]

[illegible]

[illegible] equal [illegible]

[illegible]

[illegible] $M = \frac{4}{3}\pi a^3$: $I = \frac{2}{5} M r^2$

To get the moment of inertia of the Sphere directly.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

The same [illegible] formula [illegible] hold good.

The calculation of the moment of inertia of a homogeneous body bounded by a surface of revolution reduces to a single integration depending upon the generating curve taken with respect to the axis of the figure. We will then decompose the body into circular rings of a thickness and height infinitely small of which each one shall have its centre in the axis and be comprised on one hand between two planes perpendicular to the axis and on the other hand between two cylindrical surfaces of which the right line is the common axis, calling r the radius of the interior cylinder, r + dr that of the exterior surface and dx the distance of the two planes the volume of the ring will be $\pi(r+dr)^2dx - \pi r^2dx$ or developing $\underline{\pi r^2dx} + 2\pi r\,dr\,dx + \pi dr^2dx - \underline{\pi r^2dx}$ or neglecting the third term as infinitely small compared to the o [illegible] $= 2\pi r\,dr\,dx$.

we [illegible]

value of [illegible]

[illegible] and [illegible]

[illegible]

of a [illegible]

The height [illegible]

in [illegible]

of ABC [illegible] $= \frac{1}{2}\pi y$ [illegible]

[illegible]

$ax^4 + \frac{x^5}{5}) = \frac{1}{2}\pi y$ [illegible]

of [illegible] $\frac{8\pi y a^5}{15}$

If the generating [illegible]

[illegible] A

and [illegible] AB [illegible]

tangent at O [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] $\mu = \frac{1}{10}\pi y h (r_1^4 + r_1^3 r_2 + r_1^2 r_2^2 + r_1 r_2^3 + r_2^4)$. In Cone [illegible] $\mu = \frac{1}{10}\pi y h r^4$ but $M = \frac{1}{3}\pi y h r^2$ ∴ $\mu = \frac{3}{10} Ma^2$. In case of cylinder [illegible], $M = \pi r^2 y h$ ∴ $\mu = \frac{1}{2} Mar^2$

$$= \frac{\pi r}{2n}(r^4 - 2r^2\,\overline{CK}^2 + \overline{CK}^4).$$

Let us put in succession for CK $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, &c., to $\frac{nr}{n}$, and add the results, we shall then have the moment of inertia of the sphere:

$$T = \frac{\pi}{2n}\Big[n\,.\,r^4 - 2r^2\left(\frac{r}{n}\right)^2(1^2 + 2^2 + 3^2 + \,.\,.\, + n^2) + \left(\frac{r}{n}\right)^4(1^4 + 2^4 + 3^4 + \,.\,.\, + n^4)\Big] = \frac{\pi r}{2n}\Big[nr^4 - \frac{2r^4}{n^2}\cdot\frac{n^3}{3} + \left(\frac{r}{n}\right)^4\cdot\frac{n^5}{5}\Big] = \frac{\pi r^5}{2}(1 - \tfrac{2}{3} + \tfrac{1}{5}) = \frac{4\pi r^5}{15}.$$

Now the solid contents of a hemisphere $M = \frac{2}{3}\pi r^3$, hence we may put:

$$T = \tfrac{2}{5}\,.\,\tfrac{2}{3}\,\pi r^3\,.\,r^2 = \tfrac{2}{5}\,Mr^2,$$

and if we take M for the whole sphere, the formula will hold good for the case.

The formula $T = \frac{2}{5}\,Mr^2$ is true also for a spheroid whose equatorial radius $= r$ (§ 117).

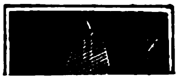
If the sphere revolves about another axis at the distance e from its centre, the moment of inertia is then

$$T = M(e^2 + \tfrac{2}{5}\,r^2).$$

The radius of gyration $= r\sqrt{\tfrac{2}{5}} = 0{,}6324\,.\,r$; two-fifths of the mass of the sphere at a distance from the axis of rotation equal to the radius of the sphere, have the same moment of inertia as the whole sphere.

§ 223. The moment of inertia of a circular disc $ABDE$, Fig. 262, revolving about its diameter BE, is, as for the moment of flexure of a cylinder, (§ 195) $= \frac{\pi r^4}{4} = \frac{Mr^2}{4}$, consequently the radius of gyration $= r\sqrt{\tfrac{1}{4}} = \tfrac{1}{2}\,r$,

Fig. 262.



we must then put $y^2 = 2ax - x^2$ for the value of y^2 which must be substituted in our formula and if we wish the moment of inertia of a spherical segment of which the height is a we must integrate in this formula between the limits of $x = 0$ which gives $\mu = \int_0^a \frac{1}{2}\pi\gamma y^4 dx = \frac{1}{2}\pi\gamma \int_0^a (2ax - x^2)^2 dx = \frac{1}{2}\pi\gamma \int_0^a (4a^2x^2 - 4ax^3 + x^4) dx = \frac{1}{2}\pi\gamma \cdot (\frac{4}{3}a^2x^3 - ax^4 + \frac{x^5}{5}) = \frac{1}{2}\pi\gamma a^3(\frac{4}{3}a^2 - 4.a + \frac{a^2}{5})$. For the [illegible] of the entire sphere $a = 2a$ ∴ $\mu = \frac{8\pi\gamma}{15}$

If the generatrix is a right line

[illegible] $\pi r^2 dx$ = elementary [illegible] [illegible] [illegible]

$\int_0^l \pi r^2 dx = \pi r^2 l$ = volume [illegible]

Then by [illegible] $I = I_1 + Mc^2$ and

$I_1 = \int M \frac{r^2}{4} - [illegible] = \int_0^l \pi r^2 dx \cdot \frac{r^2}{4} = \int_0^l \frac{\pi r^4}{4} dx$ [illegible] Mc^2 [illegible]

$= \int_0^l \pi r^2 dx\, x^2$ ∴ $I = \int_0^l$ [illegible] $r^2x^2 dx = \frac{\pi r^4 x}{4} - \frac{\pi r^2 x^3}{3} = \pi r^2 l($

[illegible] $x^2)$

$$= \frac{\pi r}{2n}(r^4 - 2r^2\,\overline{CK^2} + \overline{CK^4}).$$

Let us put in succession for CK $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, &c., to $\frac{nr}{n}$, and add the results, we shall then have the moment of inertia of the sphere:

$$T = \frac{\pi}{2n}\Big[n \,.\, r^4 - 2r^2\left(\frac{r}{n}\right)^2(1^2 + 2^2 + 3^2 + \ldots + n^2) +$$
$$\left(\frac{r}{n}\right)^4(1^4 + 2^4 + 3^4 + \ldots + n^4)\Big] = \frac{\pi r}{2n}\Big[nr^4 - \frac{2r^4}{n^2} \cdot \frac{n^3}{3} +$$
$$\left(\frac{r}{n}\right)^4 \cdot \frac{n^5}{5}\Big] = \frac{\pi r^5}{2}(1 - \tfrac{2}{3} + \tfrac{1}{5}) = \frac{4\pi r^5}{15}.$$

Now the solid contents of a hemisphere $M = \frac{2}{3}\pi r^3$, hence we may put:

$$T = \tfrac{2}{5} \,.\, \tfrac{2}{3}\pi r^3 \,.\, r^2 = \tfrac{2}{5} M r^2,$$

and if we take M for the whole sphere, the formula will hold good for the case.

The formula $T = \frac{2}{5} Mr^2$ is true also for a spheroid whose equatorial radius $= r$ (§ 117).

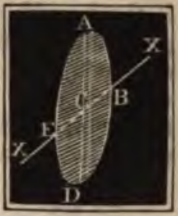
If the sphere revolves about another axis at the distance e from its centre, the moment of inertia is then

$$T = M(e^2 + \tfrac{2}{5} r^2).$$

The radius of gyration $= r\sqrt{\frac{2}{5}} = 0{,}6324 \,.\, r$; two-fifths of the mass of the sphere at a distance from the axis of rotation equal to the radius of the sphere, have the same moment of inertia as the whole sphere.

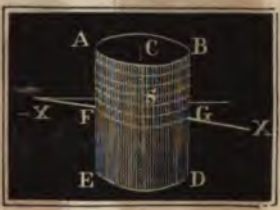
§ 223. The moment of inertia of a circular disc $ABDE$, Fig. 262, revolving about its diameter BE, is, as for the moment of flexure of a cylinder, (§ 195) $= \frac{\pi r^4}{4} = \frac{Mr^2}{4}$, consequently the radius of gyration $= r\sqrt{\frac{1}{4}} = \frac{1}{2} r$, *i. e.* half the radius of the circle.

Fig. 262.



From this we may now find the moment of inertia of a cylinder $ABDE$, Fig. 263, which revolves about a diameter FG, passing through the centre of gravity S. If l be half the height and r the radius of the cylinder, we then have the volume of half the cylinder $= \pi r^2 l$, and if we make equi-distant sections parallel to the base, we decompose this body into n equal parts, each of which $= \frac{\pi r^2 l}{n}$, the first is distant $\frac{l}{n}$ from the centre of gravity S, the second $\frac{2l}{n}$, the third $\frac{3l}{n}$, &c. In virtue of the formula, § 217, the moments of inertia of these circular slices are:

Fig. 263.



$$\frac{\pi r^2 l}{n}\left[\tfrac{1}{4}\, r^2 + \left(\frac{l}{n}\right)^2\right], \frac{\pi r^2 l}{n}\left[\tfrac{1}{4}\, r^2 + \left(\frac{2l}{n}\right)^2\right],$$
$$\frac{\pi r^2 l}{n}\left[\tfrac{1}{4}\, r^2 + \left(\frac{3l}{n}\right)^2\right],$$

&c., their sum gives the moment of inertia of half the cylinder:

$$T = \frac{\pi r^2 l}{n}\left[\frac{n r^2}{4} + \left(\frac{l}{n}\right)^2 (1^2+2^2+3^2+\ldots+n^2)\right] =$$
$$\pi r^2 l \left(\frac{r^2}{4} + \frac{l^2}{n^3}\cdot\frac{n^3}{3}\right) = M\left(\frac{r^2}{4}+\frac{l^2}{3}\right)$$

which holds good likewise for the whole cylinder, if M represents its mass.

We find in like manner for the right cone *ABD*, Fig. 264, whose axis of revolution passes through its centre of gravity, and is perpendicular to the geometrical axis *CD*

$$T = \tfrac{3}{20}\, M\left(r^2 + \frac{h^2}{4}\right).$$

Fig. 264.

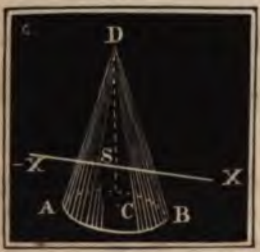
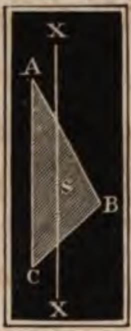


Fig. 265.



For a plate *ABC*, Fig. 265, in the form of a rectangular triangle, the moment of inertia about an axis passing through the centre of gravity *S*, and parallel to the side *AC*, is according to § 193:

$$T = \frac{bh^3}{36} = \frac{bh}{2}\cdot\frac{h^2}{18} = \tfrac{1}{18}\, Mh^2,$$

if the breadth b parallel to the axis of revolution, and h the height perpendicular to it be given. This formula holds good even for an oblique angled triangle, if the axis runs parallel to the base, and h represents the height of the triangle. From this the moment of inertia of a triangular prism *ADEF*, Fig. 266, may be found, if the axis of revolution $\overline{XX}$ passes through its centre of gravity *S*, and is parallel to the side *DE* of the surface of the base, it follows from the same method as that adopted for the cylinder, that

$$T = M\left(\tfrac{1}{18}\, h^2 + \frac{l^2}{3}\right),$$

where l represents half the length of the prism.

Fig. 266.

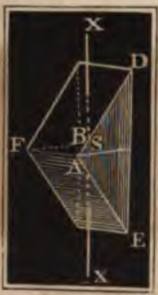
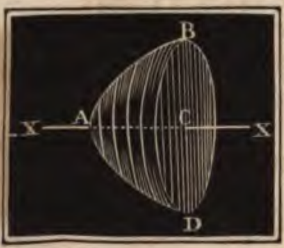


Fig. 267.



§ 224. *Segments.*— The moment of inertia of a paraboloid of revolution, *BAD*, Fig. 267, which turns about its axis of rotation *AC*, is

$\pi y^2 dx$ = element

$I_x = y^2 dx\, x^2$, $y = \frac{bx}{h}$ ∴ $I_x = \int^h \frac{bx^3}{h} dx = \frac{bh^3}{4}$

$I_g = I_x - M\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} - M\frac{4h^2}{9} = M\frac{h^2}{2} - M\frac{4h^2}{9} = M\frac{h^2}{18}$

$\pi y^2 dx$ - element

$\frac{\pi}{2} y^4 dx$ = moment of in-

ertia of that element [illegible]

$\frac{y^2}{a^2} = \frac{x}{h}$ $y^2 = \frac{a^2 x}{h}$

$I_x = \frac{\pi}{2}\int_0^h \frac{a^4 x^2}{h^2} dx = \frac{\pi}{2}\frac{a^4}{h^2}\cdot\frac{h^3}{3} = \frac{\pi a^4 h}{6}$

But $M = \frac{\pi a^2 h}{2}$ ∴ $I_x = \frac{1}{3} M a^2$

So applies to their compound [illegible]

[illegible]

determined in a similar manner to that of a sphere. Let the radius of the base $CB = CD = a$, the height $CA = h$, and the body consist of n slices, each of the height $\frac{h}{n}$, we have then the contents of these

$$= \frac{h}{n} \cdot \pi \cdot \frac{1}{n} a^2, \frac{h}{n} \pi \cdot \frac{2}{n} a^2, \frac{h}{n} \cdot \pi \frac{3}{n} a^2, \text{\&c.},$$

because the squares of the radii are as the heights. From this the moments of inertia are given

$$= \frac{h}{n} \cdot \frac{\pi}{2} \cdot \frac{a^4}{n^2}, \frac{h}{n} \cdot \frac{\pi}{2} \cdot \frac{4\, a^4}{n^2}, \frac{h}{n} \cdot \frac{\pi}{2} \cdot \frac{9\, a^4}{n^2}, \text{\&c.},$$

and hence, finally, it follows that the moment of inertia of the whole paraboloid is

$$T = \frac{\pi a^4 h}{2n^3} (1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{\pi a^4 h}{2n^3} \cdot \frac{n^3}{3} = \frac{\pi a^4 h}{6}$$

$$= \frac{\pi a^2 h}{2} \cdot \frac{a^2}{3} = \tfrac{1}{3} M a^2,$$

because the volume of this body is $M = \frac{\pi a^2 h}{2}$.

The same formula is applicable to a small segment of a sphere, but if the height h is not very small compared with a, we have to put the moment of inertia of slices

$$= \frac{\pi h}{2n} \cdot a^4 = \frac{\pi h}{2n} \cdot h^2 (2\, r - h)^2 = \frac{\pi h}{2n} \cdot (4\, r^2\, h^2 - 4\, r h^3 + h^4),$$

where r represents the radius of the sphere. If now we take successively for h the values $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c., we then obtain for the moment of inertia of a segment of a sphere

$$T = \frac{\pi h}{2n} \left[4 r^2 \left(\frac{h}{n}\right)^2 \cdot \frac{n^3}{3} - 4r \left(\frac{h}{n}\right)^3 \cdot \frac{n^4}{4} + \left(\frac{h}{n}\right)^4 \cdot \frac{n^5}{5} \right]$$

$$= \frac{\pi h^3}{30} (20\, r^2 - 15\, rh + 3\, h^2).$$

The contents of the segment are $M = \pi h^2 (r - \tfrac{1}{3}\, h)$, hence

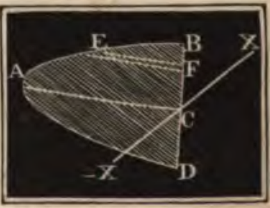
$$T = \pi h^2 (r - \tfrac{1}{3}\, h) \cdot \frac{2h}{3} \left(r - \tfrac{5}{12}\, h + \tfrac{1}{90} \cdot \frac{h^2}{r - \frac{1}{3}\, h} \right)$$

$$= \tfrac{2}{3}\, Mh \left(r - \tfrac{5}{12}\, h + \tfrac{1}{90} \cdot \frac{h^2}{r - \frac{1}{3}\, h} \right).$$

Generally $T = \frac{2}{3}\, Mh\, (r - \frac{5}{12}\, h)$ is sufficiently correct. This formula finds its application in pendulum-bobs.

The moment of inertia of the surface of a parabola ABD, Fig. 268, which revolves about an axis $X\overline{X}$, passing through the middle C of the chord BD, is found if the surface be decomposed into equally broad stripes, such as EF, and their moments added together. Let $AC = l$ be the length, and $CB = b$ half the breadth of the sur-

Fig. 268.



face, $CF = x$ the absciss, and $EF = y$ the ordinate or length of an element. Its moment of inertia is then $= \frac{b}{n} y \left(x^2 + \frac{y^2}{3}\right)$; but as $\frac{x^2}{b^2} = \frac{l-y}{l}$, therefore $y = l\left(1 - \frac{x^2}{b^2}\right)$, it follows that this moment $= \frac{b}{n}\left[lx^2\left(1 - \frac{x^2}{b^2}\right) + l^3\left(1 - \frac{x^2}{b^2}\right)^3\right]$. If now x be successively put $= \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}$, &c., and the results added, we obtain the moment of inertia of half the surface of the parabola:

$$T = bl\left[\frac{l^2}{3n}\left(n - n + \tfrac{3}{5}n - \frac{n}{7}\right) + \frac{b^2}{3} - \frac{b^2}{5}\right]$$

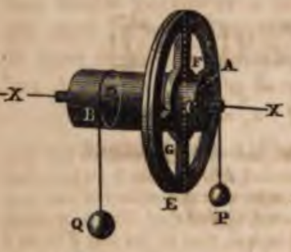
$$= bl\left(\frac{16\, l^2}{3\,.\,35} + \frac{2\, b^2}{3\,.\,5}\right) = \tfrac{2}{3}\, bl\,(\tfrac{8}{35}\, l^2 + \tfrac{1}{5}\, b^2) = \tfrac{1}{5}\, M\,(\tfrac{8}{7}\, l^2 + b^2),$$

because the surface of the parabola is $M = \frac{2}{3}\, bl$.

This formula, which holds good for the entire surface of the parabola, is also applicable to a prism having a parabolic surface for the base, as in vibrating beams.

§ 225. *Wheel and Axle.*—The theory of the moment of inertia finds its most frequent application in machines and instruments, because in these rotatory motions about a fixed axis are those which generally present themselves. Many applications of this doctrine will be met with in the sequel, hence it will suffice to treat of only a few simple cases for the present.

Fig. 269.



If two weights P and Q, act on a wheel and axle $ACDB$, Fig. 269, with the arms $CA = a$ and $DB = b$ through the medium of a perfectly flexible string, and if the radius of the gudgeons be so small that their friction may be neglected, it will remain in equilibrium if the statical moments $P\,.\,CA$ and $Q\,.\,DB$ are equal, and therefore $Pa = Qb$. But if the moment of the weight P is greater than that of Q, therefore, $Pa > Qb$, P will descend and Q ascend; if $Pa < Qb$, P will ascend and Q descend. Let us now examine the conditions of motion in one of the latter cases. Let us suppose that $Pa > Qb$. The force corresponding to the weight Q and acting at the arm b generates at the arm a a force $\frac{Qb}{a}$, which acts opposite to the force corresponding to the weight P, and hence there is a residuary moving force $P - \frac{Qb}{a}$ acting at A. The mass $\frac{Q}{g}$ is reduced by its transference from the distance b to that of a to $\frac{Qb^2}{ga^2}$, hence the mass moved by $P - \frac{Qb}{a}$ is

$M = \left(P + \frac{Qb^2}{a^2}\right) \div g$, or, if the moment of inertia of the wheel and axle $= \frac{Gy^2}{g}$, and, therefore, its inert mass reduced to $A = \frac{Gy^2}{ga^2}$, we have more exactly:

$$M = \left(P + \frac{Qb^2}{a^2} + \frac{Gy^2}{a^2}\right) \div g = (Pa^2 + Qb^2 + Gy^2) \div ga^2.$$

From thence it follows that the accelerated motion of the weight P, together with that of the circumference of the wheel, namely

$$p = \frac{\text{moving force}}{\text{mass}} = \frac{P - \frac{Qb}{a}}{Pa^2 + Qb^2 + Gy^2} \cdot ga^2$$
$$= \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2} \cdot ga;$$

on the other hand, the accelerated motion of the ascending weight Q, or of the circumference of the ~~wheel~~ is: axle

$$q = \frac{b}{a} p = \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2} \cdot gb.$$

The tension of the string by P is $S = P - \frac{Pp}{g} = P\left(1 - \frac{p}{g}\right)$ (§ 73), that of the string by $Q : T = Q + \frac{Qq}{g} = Q\left(1 + \frac{q}{g}\right)$ hence the pressure on the gudgeon is:

$$S + T = P + Q - \frac{Pp}{g} + \frac{Qq}{g} = P + Q - \frac{(Pa - Qb)^2}{Pa^2 + Qb^2 + Gy^2};$$

the pressure, therefore, on the gudgeons for a revolving wheel and axle is here less than in a state of equilibrium. Lastly, from the accelerating forces p and q, the rest of the relations of motion may be found. After t seconds the velocity of P is $v = pt$, of $Q : v_1 = qt$, and the space described by $P : s = \frac{1}{2} pt^2$, by $Q : s_1 = \frac{1}{2} qt^2$.

Example. Let the weight P at the wheel be $= 60$ lbs., that at the axle $Q = 160$ lbs., the arm of the first $CA = a = 20$ inches, that of the second $DB = b = 6$ inches; further, let the axle consist of a solid cylinder of 10 lbs. weight, and the wheel of two iron rings and four arms, the rings of 40 and 12 lbs., the arms, together, of 15 lbs. weight; lastly, let the radii of the greater ring $AE = 20$ and 19 inches, that of the less $FG = 8$ and 6 inches. Required, the conditions of motion of this machine. The moving force at the circumference of the wheel is:

$$P - \frac{b}{a} Q = 60 - \frac{6}{20} \cdot 160 = 60 - 48 = 12 \text{ lbs.},$$

the moment of inertia of the machine, neglecting the masses of the gudgeons and the strings, is equivalent to the moment of inertia of the axle $= \frac{Wb^2}{2} = \frac{10 \cdot 6^2}{2} = 180$, plus the moment of the smaller ring $= \frac{R_1(r_1^2 + r_2^2)}{2} = \frac{12 \cdot (8^2 + 6^2)}{2} = 600$, plus the moment of the larger ring $= \frac{40 \cdot (20^2 + 19^2)}{2} = 15220$, plus the moment of the arms, approximately $= \frac{A(\rho_1^3 - \rho_2^3)}{3(\rho_1 - \rho_2)} = \frac{A(\rho_1^2 + \rho_1\rho_2 + \rho_2^2)}{3} = \frac{15 \cdot (19^2 + 19 \cdot 8 + 8^2)}{3} = 2885$; hence, collectively, $Gy^2 = 180 + 600 + 15220 + 2885 = 18885$, or for foot measure

$= \frac{18885}{144} = 131,14$. The collective mass, reduced to the circumference of the wheel is:

$$= \left(P + \frac{Qb^2 + Gy^2}{a^2}\right) + g = \left[60 + 160\left(\frac{6}{20}\right)^2 + \frac{18885}{20^2}\right]^2 + g$$

$= \left(60 + 160 \,.\, 0,09 + \frac{18885}{400}\right) . 0,031 = 121,61 \times 0,031 = 377$ lbs.

Accordingly, the accelerated motion of the weight P, together with that of the circumference of the wheel, is:

$p = \frac{P - \frac{b}{a}Q}{\frac{P + Qb^2 + Gy^2}{a^2}} . g = \frac{12}{3,77} = 3,183$ feet; on the other hand, that of $Q : q = \frac{b}{a} p$ $= \frac{6}{20} . 3,183 = 0,954$ feet; further, the tension of the string by P is $= \left(1 - \frac{p}{g}\right) P$ $= \left(1 - \frac{3,133}{32,2}\right) . 60 = 54,07$ lbs.; that by Q, on the other hand, $Q = \left(1 + \frac{q}{g}\right) . Q$ $= (1 + 0,925 \,.\, 0,031) \,.\, 160 = 1,030 \,.\, 160 = 164,8$ lbs.; and consequently the pressure on the gudgeons $S + T = 54,06 + 164,80 = 218,86$ lbs., or inclusive of the weight of the machine $= 218,86 + 77 = 295,86$ lbs. After 10 seconds, P has acquired the velocity $pt = 3,084 \,.\, 10 = 30,84$ feet, and described the space $s = \frac{vt}{2} = 30,84 \,.\, 5$ $= 154,2$ feet, and Q has ascended a height $\frac{b}{a} s = 0,3 \,.\, 154,2 = 46,26$ feet.

§ 226. The weight P which communicates to the weight Q the accelerated motion $q = \frac{Pab - Qb^2}{Pa^2 + Qb^2 + Gy^2} . g$ may also be replaced by another weight P_1, without changing the acceleration of the motion Q, if it act at the arm a_1 for which:

$$\frac{P_1 a_1 - Qb}{P_1 a_1^2 + Qb^2 + Gy^2} = \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2}.$$

The magnitude $\frac{Pa^2 + Qb^2 + Gy^2}{Pa - Qb}$, represented by k, and we obtain a_1^2 $- ka_1 = - \frac{Qb(b+k) + Gy^2}{P_1}$, and the arm in question:

$$a_1 = \tfrac{1}{2} k \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{Qb(b+k) + Gy^2}{P_1}}.$$

We may also find by help of the differential calculus, that the motion of Q is most accelerated by the weight P, when the arm of the latter corresponds to the equation $Pa^2 - 2\,Qab = Qb^2 + Gy^2$, therefore,

$$a = \frac{b\,Q}{P} + \sqrt{\left(\frac{b\,Q}{P}\right)^2 + \frac{Qb^2 + Gy^2}{P}}.$$

The formula found above assumes a complicated form if the friction of the gudgeons and the rigidity of the cord are taken into account. If we represent the statical moments of both resistances by Fr, we must then substitute for the moving force $P - \frac{b}{a} Q$, the value P $- \frac{Qb + Fr}{a}$, whence the acceleration of Q comes out

$$q = \frac{(Pa - Fr)\,b - Qb^2}{Pa^2 + Qb^2 + Gy^2} \cdot g \text{ and}$$

$$a = \frac{Qb + Fr}{P} + \sqrt{\left(\frac{Qb + Fr}{P}\right)^2 + \frac{Qb^2 + Gy^2}{P}}.$$

Examples.—1. The weights $P = 30$ lbs. $Q = 80$ lbs. act at the arms $a = 2$ feet, and $b = \frac{1}{2}$ foot of a wheel and axle, and their moments of inertia Gy^2 amount to 60 lbs.; then the accelerated motion of the ascending weight Q is:
$q = \frac{30 \cdot 2 \cdot \frac{1}{2} - 80 \cdot (\frac{1}{2})^2}{30 \cdot 2^2 + 80 \cdot (\frac{1}{2})^2 + 60} \cdot g = \frac{30 - 20}{120 + 20 + 60} \cdot 32{,}2 = \frac{322}{200} = 1{,}61$ feet. But if a weight $P_1 = 45$ lbs. generates the same acceleration in the motion of Q, the arm of P_1 is then:
$a_1 = \frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{80 \cdot \frac{1}{2}(\frac{1}{2} + k) + 60}{45}}$, or as $k = \frac{200}{60 - 40} = 10$, a_1 is $= 5 \pm \sqrt{25 - \frac{32}{3}} = 5 \pm \frac{1}{3}\sqrt{11{,}358} = 5 \pm 3{,}786 = 8{,}786$ feet, or $1{,}214$ feet.—2. The accelerated motion of Q comes out greatest if the arm of the force or radius of the wheel amount to:
$a = \frac{\frac{1}{2} \cdot 80}{30} + \sqrt{\left(\frac{40}{30}\right)^2 + \frac{20 + 60}{30}} = \frac{4}{3} + \sqrt{\frac{16}{9} + \frac{24}{9}} = \frac{4 + \sqrt{40}}{3} = 3{,}4415$ feet,
and q is $= \left(\frac{30 \cdot 1{,}7207 - 20}{30 \cdot (3{,}4415)^2 + 80}\right) g = \frac{31{,}621}{435{,}32} \cdot g = 2{,}339$ feet.—3. The statical moment of the friction, together with the rigidity of the string, is $Fr = 8$; then, instead of Qb, we must put $Qb + Fr = 40 + 8 = 48$; whence it follows that: $a = \frac{48}{30} + \sqrt{\left(\frac{40}{30}\right)^2 + \frac{8}{3}} = 1{,}6 + \sqrt{5{,}227} = 3{,}886$, and the correspondent maximum accelerating force $q = \frac{30 \cdot 1{,}943 - 8 \cdot \frac{1}{2} - 20}{30 \cdot (3{,}886)^2 + 80} \cdot g = \frac{34{,}29}{533} \cdot 32 \cdot 2 = 2{,}071$ feet.

§ 227. *Attwood's Machine.*—The formulæ found in § 225 for the wheel and axle hold good also for the simple fixed pulley, for if $b = a$, the wheel and axle becomes either a pulley or an axle. Retaining

$\frac{Pab - Qb^2}{Pa^2 + Qb^2 + Gy^2} \cdot g$, $\frac{dq}{da} = \frac{(Pa^2 + Qb^2 + Gy^2)\ [illegible]}{(Pa^2 + Qb^2 + Gy^2)^2} = 0$

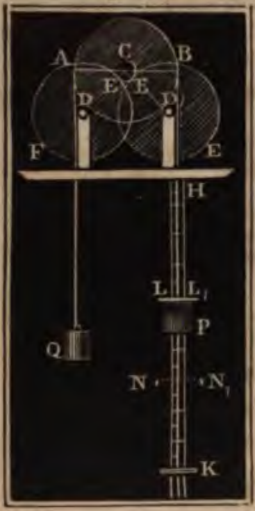
multiply out by the denominator divide by P + $Pa^2b + Qb^3 + Gy^2b - 2Pa^2b + 2Qab^2 = 0$ [illegible]

$Pa^2b - 2Qab^2 = Qb^3 + Gy^2b$ dividing out

$a^2 - \frac{2Qb}{P}a = \frac{Qb^2 + Gy^2}{P}$ ∴ $a = \frac{Qb}{P} \pm \sqrt{\left(\frac{Qb}{P}\right)^2 + \frac{Qb^2 + Gy^2}{P}}$

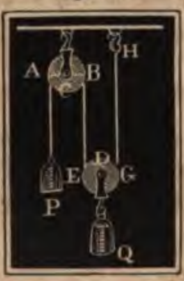
$$g = \frac{(P+Q)\,a^2 + Gy^2 + G_1\,\frac{y_1^{\,2}r^2}{a_1^{\,2}}}{(P-Q)\,a^2 - Far}\cdot p.$$

Fig. 270.



For a small difference of the two weights $P - Q$ the accelerating force p comes out small, hence the motion goes on slowly, and if the resistance opposed to the weight by the air be inconsiderable, with the assistance of experiments upon the descent of weights in such an arrangement, the accelerating force of gravity may be measured with tolerable accuracy, which by a body falling freely it is impossible to do. Experiments of this kind were first instituted by Attwood,* whence this arrangement is known by the name of Attwood's machine. To determine the spaces fallen through, there is a scale *HK* along which the weight *P* descends. From the space fallen through *s*, and the corresponding time *t*, it follows of course that $p = \frac{2\,.\,s}{t^2}$; if, however, during the descent, the moving force be removed, while an equal weight LL_1 forming a hollow ring is taken up by a fixed narrower ring NN_1, the remaining part of the space s_1 will be described with a uniform motion, and having the time observed by a good clock, the velocity will be given $v = \frac{s_1}{t_1}$, and the accelerating force $p = \frac{v}{t} = \frac{s_1}{tt_1}$. If $t_1 = t = 1$, experiment gives directly $p = s_1$, and by putting the value found in the above formula, it will give the accelerating force of gravity.

Fig. 271.



§ 228. The acceleration of the motion of the weights *P* and *Q*, which are suspended to a system consisting of a fixed pulley *AB*, and a movable pulley *EG*, is given in the following manner. Let the weights of the pulleys *AB* and $EG = G$ and G_1, their moments of inertia Gy^2 and $G_1y_1^{\,2}$, and the radii $CA = a$ and $DE = a_1$, therefore the masses reduced to the circumference of the wheel

$$M = \frac{G}{g}\cdot\frac{y^2}{a^2}, \text{ and } M_1 = \frac{G_1}{g}\cdot\frac{y_1^{\,2}}{a_1^{\,2}}.$$

If the weight descend a certain space s, $Q + G_1$ will then ascend by $\frac{1}{2}\,s$ (§ 151), hence the mechanical effect produced will be $Ps - (Q + G_1)\,\frac{s}{2}$; if by this descent *P* acquires the velocity v, then

---

* Attwood's Treatise on Rectilinear and Rotary Motion.

$Q + G_1$ takes the velocity $\frac{v}{2}$, and the pulley *AB* at its circumference the velocity v, and the pulley *EG*, since in rolling motion the progressive and rotary motion are equal to each other, at its circumference the velocity $\frac{v}{2}$. The sum of the *vis viva* corresponding to these masses and their velocities is $\frac{P}{g} \cdot v^2 + \frac{Q + G_1}{g} \cdot \left(\frac{v}{2}\right)^2 + \frac{Gy^2}{a^2} \cdot v^2 + \frac{G_1 y_1}{a_1^2} \cdot \left(\frac{v}{2}\right)^2$, and if their halves be equated to the mechanical effect expended, we shall obtain the equation:

$$\left(P - \frac{(Q + G_1)}{2}\right) s = \left(P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}\right) \frac{v^2}{2g}.$$

Hence, the velocity corresponding to the space s described by P:

$$v = \sqrt{\frac{2\, gs \left(P - \frac{Q + G_1}{2}\right)}{P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}}}$$

For the acceleration $ps = \frac{v^2}{2}$; hence, here

$$p = \left(\frac{P - \frac{Q + G_1}{2}}{P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}}\right) g.$$

The acceleration of $Q + G_1$ is $= \frac{p}{2}$, and the rotary acceleration of G_1 is equal to it.

The tension of the string *BE*, connecting both pulleys, is $S = P - \left(P + \frac{Gy^2}{a^2}\right) \frac{p}{g}$, because the force $\left(P + \frac{Gy^2}{a^2}\right) \frac{p}{g}$ is expended upon the acceleration of the motions of P and G. The tension of the fixed string *GH*, on the other hand:

$$S_1 = S - \frac{G_1 y_1^2}{a_1^2} \cdot \frac{p}{2g},$$

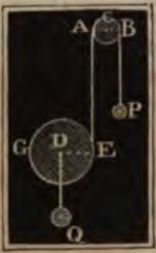
because the pulley *EG* is put into rotation by the difference $S - S_1$ of the tensions of the strings.

Example. In a system of pulleys, Fig. 271, the weights $P = 40$ lbs. and $Q = 66$ lbs. are suspended, and each of the solid pulleys weighs 6 lbs.; required, the acceleration of the motions of these weights. The moving force is $P - \frac{Q + G_1}{2} = 40 - \frac{66 + 6}{2} = 4$ lbs., the mass of a pulley reduced to its circumference is: $\frac{Gy^2}{ga^2} = \frac{G_1 y_1^2}{g a_1^2} = \frac{G}{2g} = \frac{6}{2g} = \frac{3}{g}$ (§ 221), and the aggregate of the inert masses:

$$= \left(P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4\, a^2}\right) \div g = \left(40 + \frac{72}{4} + 3 + \frac{3}{4}\right) \div g = \frac{247}{4\, g},$$

hence the accelerated motion of the descending weight is: $p = \frac{4}{247} \cdot 4\,g = \frac{16 \cdot g}{247} = \frac{16 \cdot 32{,}2}{247} = \frac{515{,}2}{247} = 2{,}086$ feet; on the other hand, the accelerated motion of the ascending weight: $\frac{p}{2} = 1{,}043$ feet. The tension of the string BE is: $S = P - \left(P + \frac{G}{2}\right) \frac{p}{g} = 40 - 43 \cdot \frac{2{,}086}{32{,}2} = 40 - 2{,}782 = 37{,}218$ lbs.; that of the string $GH, = S - \frac{G}{2} \cdot \frac{p}{2g} = 37{,}218 - 3 \cdot \frac{1{,}043}{32{,}2} = 36{,}247$ lbs.

Fig. 272.



§ 229. The motion is more complicated, if the pulley EG, Fig. 272, be suspended only by a string passing round it. Let us assume that P descends with the accelerating force p, and Q ascends with the accelerating force q, we then obtain the acceleration of the rotary motion at the circumference of the loose pulley $q_1 = p - q$ (§ 42). Let us now put the tension of the strings at $AE = S$, we then obtain $P - S = \left(P + \frac{Gy^2}{a^2}\right) \frac{p}{g}$; further, $S - (Q + G_1) = (Q + G_1) \frac{q}{g}$, since from § 214, it may be assumed that S acts at the centre of gravity D of EG; and lastly, $S = \frac{G_1 y_1^2}{a_1^2} \cdot \frac{q_1}{g}$, since it may also be assumed that the centre of gravity D is fixed, and the pulley put into rotation by S. The last three formulæ give the accelerating force

$$p = \frac{P - S}{P + \frac{Gy^2}{a^2}} g,\ q = \left(\frac{S - (Q + G_1)}{Q + G_1}\right) g, \text{ and } q_1 = \frac{Sa_1^2}{G_1 y_1^2}\, g, \text{ and}$$

all three being put into the equation $q_1 = p - q$, we obtain

$$\frac{Sa_1^2}{G_1 y_1^2} g = \frac{P - S}{P + \frac{Gy^2}{a^2}} g - \frac{S - (Q + G_1)}{Q + G_1} g,$$

whence the tension of the string follows

$$S = \frac{2\,Pa^2 + Gy^2}{\left(\frac{a_1^2}{G_1 y_1^2} + \frac{1}{Q + G_1}\right)(Pa^2 + Gy^2) + a^2}.$$

The accelerating forces are given from the value of S by the application of the above formulæ.

If we neglect the mass G of the fixed pulley, and also put $Q = 0$, we obtain simply:

$$S = \frac{2\,Pa^2 \cdot G_1 y_1^2}{P\,(a_1^2 + y_1^2)\,a^2 + Ga_1^2 y_1^2} = \frac{2\,PG_1 y_1^2}{G_1 y_1^2 + P\,(a_1^2 + y_1^2)}.$$

If the extremity of the string AE, instead of passing over the pulley AB, is fixed, we have then the accelerating force $p = 0$, hence $g_1 = -q$, and consequently the tension

$$S = \frac{(Q + G_1)\, G_1 y_1^2}{(Q + G_1)\, a_1^2 + G\, y_1^2};\ \text{for } Q = 0:$$

$S=\frac{8PG_1y_1^2}{G_1y_1^2+P_1a_1^2y^2}$; placing [illegible]

by different form $=\frac{8PG}{G+P\frac{a_1^2}{y_1^2}}$ [illegible]

$\frac{Gy_1^2}{a_1^2}=3G_1$ ∴ $\frac{y_1^2}{a_1^2}=3$ ∴ $S=\frac{8PG}{G+P3}=\frac{8PG}{3P+G}$

$\frac{G_1 y_1^2}{a_1^2}$

the

p is

y_1^2, that G_1 may descend, it is necessary that $S < G_1$, therefore $\frac{G_1}{P_1} > 1 - \frac{a_1^2}{y_1^2}$.

Example. If when in a system of pulleys, Fig. 271, the string GH suddenly breaks, the string BE at the commencement becomes stretched by the force

$$S = \frac{2\,P + \frac{Gy^2}{a^2}}{\left(\frac{a_1^2}{G_1\, y_1^2} + \frac{1}{Q + G_1}\right)\left(P + \frac{Gy^2}{a^2}\right) + 1} = \frac{2\,.\,40 + 3}{\left(\frac{1}{3} + \frac{1}{72}\right)(40 + 3) + 1}$$

$$= \frac{83\,.\,72}{25\,.\,43 + 72} = \frac{5976}{1147} = 5{,}210 \text{ lbs.}$$

The accelerated motion of the descending weight P will be:

$$p = \left(\frac{P - S}{P + \frac{Gy^2}{a^2}}\right) g = \left(\frac{40 - 5{,}210}{40 + 3}\right).\,32{,}2 = \frac{34{,}79}{43}\,.\,32{,}2 = 26{,}023 \text{ feet.}$$

Further, that of the descending pulley:

$$q = \left(\frac{Q + G_1 - S}{Q + G_1}\right) g = \left(\frac{72 - 5{,}210}{72}\right) 32{,}2 = \frac{66{,}79}{72}\,.\,32{,}2 = 30{,}0 \text{ feet;}$$

and the acceleration of rotation of this pulley:

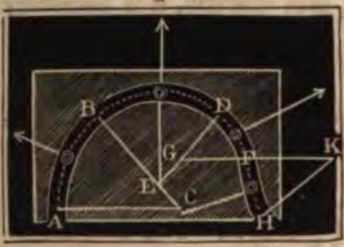
$$q_1 = \frac{S a_1^2}{G_1\, y_1^2} \cdot g = \frac{5{,}210}{3}\,.\,32{,}2 = 55{,}75 \text{ feet.}$$

CHAPTER II.

CENTRIFUGAL FORCE.

§ 230. *Normal Force.*—When a material point moves in a curved line, it has in every point of its path an accelerated motion, deviating from that of the direction of motion, which we have become acquainted with in phoronomics under the name of the ***normal accelerating force.*** If the radius of curvature at any place of the path of the moving point $= r$, and its velocity $= v$, we have then for the normal accelerating force $p = \frac{v^2}{r}$ (§ 41). Let now the mass of the point $= M$, the normal accelerating force will then be $Mp = \frac{Mv^2}{r}$, which we must regard as the first cause of the point changing its direction of motion at each position. If no ~~other tangential~~ force but the normal act upon the point, its velocity v will be invariable and $= c$, and hence the normal force $P = \frac{Mc^2}{r}$ will be dependent only on the curvature at each moment ~~and~~ or on the radius of curvature, and will be greater, the greater the curvature ~~or its radius~~; for double the radius of curvature, for instance, the normal force is only half as great as for a single radius of curvature. If a material point M is constrained by a horizontal path, Fig. 273, to describe a curve $ABDFH$, it will have, disregarding the friction at all places, the same velocity c, and will exert at each place a pressure equivalent to the normal force against the concave surface. During the description of the arc AB, the pressure $= \frac{Mc^2}{\overline{CA}}$, during that of $BD = \frac{Mc^2}{\overline{EB}}$, for the arc DF it $= \frac{Mc^2}{\overline{GD}}$, and for the arc $FH = \frac{Mc^2}{\overline{KF}}$, if CA, EB, GD, and KF are the radii of curvature of the portions of the path AB, BD, DF, and FH.

Fig. 273.



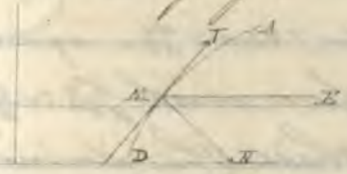
§ 231. *Centripetal and Centrifugal Force.*—When a material point or body moves in The circumference of a a circle, the normal force acts radially inwards, whence it is called the *centripetal force;* whilst the force which the body opposes, by virtue of its inertia, *i. e.*, which acts radially outwards, has received the name of *centrifugal force.* The centripetal force is that which acts directly upon the body; the centrifugal is the reacting force of the body. Each is equal in amount and opposite in direction to the other (§ 62).

In the revolution of the planets about the sun, the attractive force of the sun is the centripetal; but were the body constrained by a directrix, Fig. 273, to describe its ~~circular~~ eccentric orbit, this directrix would act, by its rigidity, as a centripetal force, and opposed to the centrifugal force of the body.

If, lastly, the revolving body be connected by a thread or by a rod with the centre of revolution, the elasticity of the rod will then be in equilibrium with the centrifugal force of the body, and thereby act as a centripetal force.

Let G be the weight of the revolving body, therefore its mass $M = \frac{G}{g}$, the radius of the circle in which it revolves $= r$, and the velocity of revolution $= v$; from the last § we have the centrifugal force:

To prove generally our first fundamental expression for Centrifugal force. Let us take



any curve AMD upon which we suppose the body to move being impelled by a tangential force T and a normal one N. This curve is not necessarily in a plane but if it is a curve of double curvature the infinitely small part of which we take cognizance may be regarded as in a plane so that this case is perfectly general and the relations deduced are equally applicable to curves of either double or single curvature. The acceleration of the body in the direction of the axis of x will we know be expressed by $\frac{d^2x}{dt^2}$ and this acceleration is due to the components of N and T i

this direction. This component
will be $N \cos NME = N \sin TME = N\frac{dy}{ds}$.
while its component parallel to Y i
$N \sin NME = N \cos TME = N \cos \frac{dx}{ds}$. Decomposin
similarly T we have its componen
$= T \cos TME = T\frac{dx}{ds}$ and $T \sin TME = T\frac{dy}{ds}$; equa
our accelerating forces with $\frac{d^2x}{dt^2}$ & $\frac{d}{dt}$
respectively with ~~these~~ the sums
of these components we obtain o
first two equations $\frac{d^2x}{dt^2} = N\frac{dy}{ds} + T\frac{dx}{ds}$ an
$\frac{d^2y}{dt^2} = -N\frac{dx}{ds} + T\frac{dy}{ds}$ the N being taken ne
in the second equation because
acts vertically downwards acting
-ly to decrease the effect of the com
-ponent of T parallel to this axis
Let us eliminate T between these two
equations to do which multiply t
first by $\frac{dy}{ds}$ and the second by $\frac{dx}{ds}$ th
will then become $\frac{d^2x}{dt^2}\cdot\frac{dy}{ds} = N\frac{dy^2}{ds^2} + T\frac{dx}{ds}\frac{d}{d}$
also $\frac{dx}{ds}\cdot\frac{d^2y}{dt^2} = -N\frac{dx^2}{ds^2} + T\frac{dx}{ds}\cdot\frac{dy}{ds}$. Subtracting
recollecting that $\frac{dy^2}{ds^2} + \frac{dx^2}{ds^2} = \frac{ds^2}{ds^2} = 1$ w
have $N = \frac{d^2x}{dt^2}\cdot\frac{dy}{ds} - \frac{d^2y}{dt^2}\cdot\frac{dx}{ds}$ We must
bear in mind that our v

for the acceleration $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ supposes that t is the independent variable and that therefore dt is a constant. If however we suppose s the independent variable then in order to introduce this condition in our discussion $\frac{d^2x}{dt^2}$ under the form $\frac{ds}{dt}.\frac{d\frac{dx}{dt}}{ds}$ which is but indicating the differentiation of $\frac{dx}{dt}$ and multiplying and dividing by ds. Performing the indicated differentiation in the numerator $\frac{d^2x}{dt^2}=\frac{ds}{dt}.\frac{d^2x\,dt-dx\,d^2t}{ds.dt^2}$ $=\frac{d^2x}{dt^2}-\frac{dx\,d^2t}{dt^3}$ we wish to have $\frac{ds^2}{dt^2}$ outside the parenthesis, as a common factor therefore multiply and divide the above expression by that quantity it becomes $\frac{ds^2}{dt^2}\left(\frac{d^2x}{ds^2}-\frac{dx.d^2t}{dt.ds^2}\right)=\frac{d^2x}{dt^2}$ similarly $\frac{d^2y}{dt^2}=\frac{ds^2}{dt^2}\left(\frac{d^2y}{ds^2}-\frac{dy\,d^2t}{dt.ds^2}\right)\therefore$ Substitu $\frac{dy}{ds}.\frac{d^2x}{dt^2}-\frac{dx}{ds}.\frac{d^2y}{dt^2}=\frac{ds^2}{dt^2}\left(-\frac{d^2y}{ds^2}.\frac{dx}{ds}+\frac{dy\,dx\,d^2t}{dt.ds^3}+\frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{dx\,dy\,d^2t}{dt.ds^3}\right)=\frac{ds^2}{dt^2}\left(\frac{d^2x\,dy-d^2y\,dx}{ds^3}\right)$ but we already know from the calculus that the expression for the Radius of

Curvature is $\frac{ds^3}{d^2x\,dy-d^2y\,dx}$
and that therefore the sec-
-ond member of our last eq
-tion is $\frac{ds^2}{dt^2}$ & multiplied by
(calling ρ the radius of cur
-vature) $\frac{1}{\rho}$ but $\frac{ds}{dt}=v$ therefo
the second member is $\frac{v^2}{\rho}$. The
first member is [illegible]
moreover equal to N so that
now our general relation com
out that $N=\frac{v^2}{\rho}$. We may obta
this last relation yet more
simply by substituting for
dt in our equation $N=\frac{d^2x}{dt^2}\cdot\frac{dy}{ds}-$
$\frac{d^2y}{dt^2}\cdot\frac{dx}{ds}$ its value taken from
the relation $\frac{ds}{dt}=v$ or $dt=\frac{ds}{v}$. Un
-der this substitution we ha

$$N=\frac{d^2x}{\frac{ds^2}{v^2}}\cdot\frac{dy}{ds}-\frac{d^2y}{\frac{ds^2}{v^2}}\cdot\frac{dx}{ds}=v^2\,\frac{d^2x\,dy-d^2y\,dx}{ds^3}=\frac{v}{\rho}$$

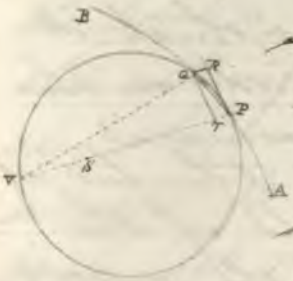
This last method was propo
by Mr Wm Watson; whether
original with him or not
"deponent sayeth not" Depo-
nent hath si stated such a
be the case.

Instead of eliminating T in our first two equations let us eliminate N by multiplying the first by $\frac{dx}{ds}$ and the second $\frac{dy}{ds}$ and they will become respectively $\frac{d^2x}{dt^2} \cdot \frac{dx}{ds} = N\frac{dy}{ds}\frac{dx}{ds} + T\frac{dx^2}{ds^2}$ and $\frac{d^2y}{dt^2}\frac{dy}{ds} = -N\frac{dx}{ds}\frac{dy}{ds} + T\frac{dy^2}{ds^2}$ and adding $\frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds} = T\left(\frac{dy^2}{ds^2} + \frac{dy^2}{ds^2}\right) = T$ but the numerator of the first term is the differential of $\frac{dx^2+dy^2}{2}$ and putting this value in the above $T = \frac{d(dx^2+dy^2)}{3dt^2ds} = \frac{d(ds^2)}{2dt^2ds} = \frac{2ds\,d^2s}{2ds\,dt^2}$ This is no new property for since the only force acting are the tangential and normal the former is the only one which can produce acceleration of velocity and is therefore from our very first fundamental relations $= \frac{d^2s}{dt^2}$. This is indeed no more than what we assume in our very first value of $\frac{d^2x}{dt^2}$. It can in indeed be called nothing more than a kind of proof of of the correctness of our work.

Let us consider now the particular case in which ~~when~~ the normal force only acts upon the body that is when the body has received an initial impulse and then by virtue of its inertia is kept in a state of uniform mot In this case the body moves upo the circumference of a circle and the acceleration $= \frac{d^2x}{dt^2} = -N \cos a$, N being taken negative because it produces a decrease in the ordinates x; but w have $r \cos a = x$ and therefore x cos a substituting $\frac{d^2x}{dt^2} = -N\frac{x}{r}$ similarly we h $\frac{d^2y}{dt^2} = -N\frac{y}{r}$ but from the general equat of the circle $x^2+y^2=r^2$ differentiati with respect to time we will ha $\frac{x\,dx}{dt} + \frac{y\,dy}{dt} = 0$ and differentiating ag $\frac{dx^2}{dt^2} + \frac{x\,d^2x}{dt^2} + \frac{dy^2}{dt^2} + \frac{y\,d^2y}{dt^2} = 0$ (1) but from our two equations $\frac{d^2x}{dt^2} = -N\frac{x}{r}$ & $\frac{d^2y}{dt^2} = -N\frac{y}{r} \therefore \frac{x\,d^2x}{dt^2}$ & $\frac{y\,d^2y}{dt^2} = -N\frac{y^2}{r}$ adding $\frac{x\,d^2x}{dt^2} + \frac{y\,d^2y}{dt^2} = -N\frac{x^2+y^2}{r} =$ $= -Nr$ substituting this in equat

$$(1)\ \frac{dx^2+dy^2}{dt^2} - Nr = 0 \text{ or } Nr = \frac{dx^2+dy^2}{dt^2} = \frac{ds^2}{dt^2} = v^2 \therefore N = \frac{v^2}{r}$$

There is yet another method of obtaining this relation of a less analytical method than the above. Let a material body of



small dimensions move in any curvilinear path AB impelled continually toward a given point S by a given force whose am^t when the body has arrived at the p^t P of its path is P. Let PQ be an exceedingly small part of the path of the body and conceive P to remain constant and parallel to itself whilst this portion of the path is being describe then if PR be a tangent at P and QR be drawn parallel to SP, PR is the space which the body would have traversed in the time of describing PQ if it had moved with its velocity of projection from P alone and had not been deflected.

S, and RQ or PT (QT being drawn parallel to RP) is the space through which it would have fallen by its attraction toward S alone or if it had not been projected at all from P. Let v represent the velocity which it would have acquired on this last supposition when it reached the point T. Therefore if G represent the weight of the body, $P_1 \times \overline{PT} = \frac{1}{2}\frac{G}{g}v^2$ Now the velocity v which the body would have acquired by falling freely throu the distance PT by the action of the constant force P_1 is equal to double that which would cau it to describe the same space uniformly in the same time If therefore we represent by V the actual velocity of the body in itn its path at P we have $\frac{\frac{1}{2}v}{V} = \frac{PT}{PR} \therefore v = 2V\frac{PT}{PR}$ Substituting this value of V in the preceding equation $P_1 \times \overline{PT} = 2\frac{G}{g}V^2\left(\frac{PT}{PR}\right)^2 \therefore P_1 = 2\frac{G}{g}V^2\frac{Q}{(PR}$

Let the circumference PQV be described having a common tangent with the curve AB at P and passing through the point Q. Produce PS to intersect this circumference in V and join QV then the triangles PQV and PQR are similar for the angle RQP is equal to the angle QPV (QR and VP being parallel) and the angle QPR is equal to the angle QRVP in the alternate segment of the circle. Therefore $\frac{QR}{PQ} = \frac{PQ}{QVP}$. therefore $QR = \frac{(PQ)^2}{PV}$. Substituting this value of QR in the last equation we have $P_1 = 2\frac{G}{g}\frac{V^2}{PV}\left(\frac{PQ}{RP}\right)^2$ This is true however much PQ is diminished. Let it be made infinitely small, the supposed constant amount and parallel direction of P_1 will then coincide with the actual case of the ~~supposed~~ a variable amount and ~~direction~~ inclination of that force, and since for this cas the tangent and chord may be taken as equal. The ratio $\frac{PQ}{PR}$ ~~may~~ be ~~taken~~ will become a ratio of equality and the circle PQV will become the circle of curvature at P and PV that chord of the circle of curvature which being drawn from P passes through S. Let this chord of the circle of curvature be represented by C $\therefore P_1 = 2\frac{G}{g}\frac{V^2}{C}$. When the body revolves in a circular path the circle of curvature to the path at any one point evidently coincides with it throughout and the chord of curvature becomes one of its diameters; let the radius of the circle which the body thus describes be represented by R then $C = 2R \therefore P_1 = \frac{G}{g}\frac{V^2}{R}$. if ω be the angular velocity of the body's revolution about the centre of its circle of curvature then $V = \overset{\omega}{\cancel{\alpha}} R$ & therefore

$P_1 = \frac{G}{g}\delta\omega^2 R$

$P = \frac{Mv^2}{r} = \frac{Gv^2}{gr} = 2 \,.\, \frac{v^2}{2g} \,.\, \frac{G}{r}$, therefore also $P : G = 2 \,.\, \frac{v^2}{2g} : r$, i. e., *the centrifugal force is to the weight of the body as double the height due to the velocity is to the radius of revolution.*

If the motion be uniform, which always takes place if no other ~~force~~ (tangential force) than the centripetal act upon the body, the velocity $v = c$ may be expressed by the time of revolution T, if we put $c = \frac{\text{path}}{\text{time}} = \frac{2\pi r}{T}$, and hence we shall obtain for the centrifugal force:

$$P = \left(\frac{2\pi r}{T}\right)^2 \frac{M}{r} = \frac{4\pi^2}{T^2} \,.\, Mr = \frac{4\pi^2}{gT^2} \,.\, Gr.$$

Since $4\pi^2 = 39{,}4784$, and for the measure in feet $\frac{1}{g} = 0{,}031$, we then have more conveniently for calculation

$$P = \frac{39{,}4784}{T^2} \,.\, Mr = 1{,}224 \,.\, \frac{Gr}{T^2}.$$

The number n of revolutions per minute is often given, and therefore T is replaced by $\frac{60''}{n}$, whence it follows:

$$P = \frac{39{,}4784}{3600}\, n^2\, Mr = 0{,}010966\, n^2\, Mr = 0{,}000331\, n^2\, Gr.$$

Hence, for equal times of revolution, or for an equal number of revolutions in a given time, the centrifugal force increases as the product of the mass and radius of revolution, and is inversely proportional, other circumstances being alike, to the squares of the times of revolution, or directly proportional to the squares of the number of revolutions. As $\frac{2\pi}{T}$ is the angular velocity ω, we may finally put: $P = \omega^2 \,.\, Mr$.

Examples.—1. If a body of 50 lbs. weight describe a circle of 3 feet radius 400 times per minute, its centrifugal force will then be:

$$P = 0{,}000331 \,.\, 400^2 \,.\, 50 \,.\, 3 = 52{,}96 \,.\, 50 \,.\, 3 = 7944 \text{ lbs.}$$

If this body be connected with an axis by a hempen cord, and the modulus of strength of the cord (§ 186) be 7000 lbs., it will follow that $7944 = 7000 \,.\, F$; hence the section of this cord will be : $F = \frac{7944}{7000} = 1{,}1342$ square inches, and its radius $D = \sqrt{\frac{4\,F}{\pi}} = \sqrt{\frac{4{,}5468}{3{,}1416}} = \sqrt{1{,}44} = 1{,}2$ inch. But for a threefold security D must be taken $= 1{,}2\,\sqrt{3} = 1{,}2 \,.\, 1{,}732 = 2{,}077$ inches.—2. From the earth's radius $r = 20\frac{1}{4}$ millions of feet, and the time of rotation or length of a day $T = 24$ h. $= 24 \,.\, 60 \,.\, 60 = 86400''$, the centrifugal force of a body at the earth's equator is $P = 1{,}224 \,.\, \frac{G \,.\, 20{,}250000}{86400^2} = \frac{2478}{864^2} \cdot G = \frac{1}{300} \,.\, G$ nearly; but were the length of the day $\frac{1}{17}$th part only, then: $\frac{24}{17} = 1$ h. 24 m. 42 sec. this force would be $17^2 = 289$ times as great, therefore equal to about the weight of the body. At the equator, therefore, the centrifugal force would be equivalent to that of gravity, and the body would neither rise nor fall. In the revolution of the moon about the earth, its centrifugal force is counteracted by the attraction of the earth. Let G be the weight of the moon, r its distance from the earth, and T its time of revolution; the centrifugal force of this heavenly body $= 1{,}224 \,.\, \frac{Gr}{T^2}$. Let a be the radius of the earth, and let us assume that the force of gravity at different distances from its centre increases inversely as a power of these distances; we have then the gravity of the moon or the attractive force of the earth $= G\left(\frac{a}{r}\right)^n$, and if we equate both forces to each other, we then obtain $\left(\frac{a}{r}\right)^n = 1{,}224 \,.\, \frac{r}{T^2}$.

Now $\frac{a}{r} = \frac{1}{60}$, $r = 1250$ million feet, and $T = 27$ days, 7 hours, 42 minutes $= 39342$ minutes $= 39342 \,.\, 60$ seconds; hence it follows:

$$\left(\frac{1}{60}\right)^n = \frac{1{,}229 \,.\, 1250}{393{,}4^2 \,.\, 36} = \frac{1}{3600}, \text{ nearly } = \left(\frac{1}{60}\right)^2$$

and hence, $n = 2$; *i. e.*, the gravitating force of the earth is in an inverse ratio to the square of the distance.

§ 232. *Centrifugal Forces of Extended Masses.*—For any system of masses, or for a mass of finite extension, the formula above found for the centrifugal force is not directly applicable, because we know not beforehand what radius of [illegible] ~~gyration~~ we have to introduce into the calculation. To find this, we proceed in the following manner. In Fig. 274, let CZ be the axis of revolution, CX and CY its two rectangular co-ordinate axes, further let M be a particle M_1, and $MK = x$, $MM = y$, and $MN = z$, its distances from the co-ordinate planes YZ, XZ and XY. As the centrifugal force P acts radially, its point of application may be transferred to its point of intersection O with the axis of revolution. If now we resolve this force in the direction of the axes CX and CY, we shall obtain the component forces $OQ = Q$ and $OR = R$, for which $OQ : OP = OL : OM$, and $OR : OP = OK : OM$, whence $Q = \frac{x}{r} P$ and $R = \frac{y}{r} P$, where r represents the distance OM of the particle from the axis of revolution. Let us proceed in a similar manner with all the particles, and we shall obtain two systems of parallel forces, one in the plane XZ, and the other in the plane YZ, but each acting perpendicularly to the axis CZ. For distinction, let us avail ourselves of the index numbers 1, 2, 3, &c., and, therefore, put for the particles of the mass, M_1, M_2, M_3, and for their distances x_1, x_2, x_3, &c., we shall obtain the resultant of the one system, Fig. 275,

Fig. 274.

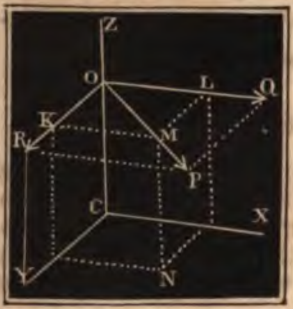
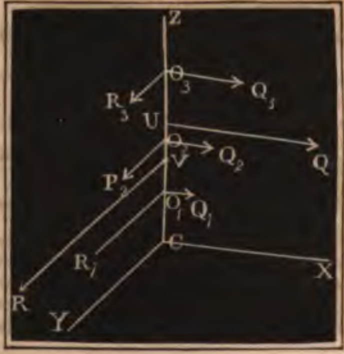


Fig. 275.



$$Q = Q_1 + Q_2 + Q_3 + \ldots$$
$$= \frac{P_1 x_1}{r_1} + \frac{P_2 x_2}{r_2} + \frac{P_3 x_3}{r_3} + \ldots =$$

$\omega^2 . (M_1x_1 + M_2x_2 + \ldots)$ and that of the other $R = R_1 + R_2 + \ldots = \omega^2 . (M_1y_1 + M_2y_2 + \ldots)$. Let us finally put the distances of the particles from the plane XY, CO_1, CO_2, &c., $= z_1, z_2$, &c., we shall obtain for the points of application of these resultants the distances $CU = u$, and $CV = v$ by the equations $(Q_1 + Q_2 + \ldots) u = Q_1z_1 + Q_2z_2 + \ldots$ and $(R_1 + R_2 + \ldots) v = R_1z_1 + R_2z_2 + \ldots$, whence it follows:

$$u = \frac{Q_1z_1 + Q_2z_2 + \ldots}{Q_1 + Q_2 + \ldots} = \frac{M_1x_1z_1 + M_2x_2z_2 + \ldots}{M_1x_1 + M_2x_2 + \ldots}, \text{ and}$$

$$v = \frac{R_1z_1 + R_2z_2 + \ldots}{R_1 + R_2 + \ldots} = \frac{M_1y_1z_1 + M_2y_2z_2 + \ldots}{M_1y_1 + M_2y_2 + \ldots}.$$

Hence, therefore, in general, the centrifugal forces of a system of bodies, or an extended body, may be reduced to two forces, which, so long as u and v are unequal, cannot be resolved into a single one.

Example. The masses of a system are
$M_1 = 10$ lbs., $M_2 = 15$ lbs., $M_3 = 18$ lbs., $M_4 = 12$ lbs.,
and their distances $x_1 = 0$ inches, $x_2 = 4$ inches, $x_3 = 2$ inches, $x_4 = 6$ inches,
$y_1 = 3$ " $y_2 = 1$ " $y_3 = 5$ " $y_4 = 3$ "
$z_1 = 2$ " $z_2 = 3$ " $z_3 = 3$ " $z_4 = 0$ "
we have then the following mean centrifugal forces
$Q = \omega^2 . (10 . 0 + 15 . 4 + 18 . 2 + 12 . 6) = 168 . \omega^2$, and
$R = \omega^2 . (10 . 3 + 15 . 1 + 18 . 5 + 12 . 3) = 171 . \omega^2$, and the distances of their points of application from the origin C:

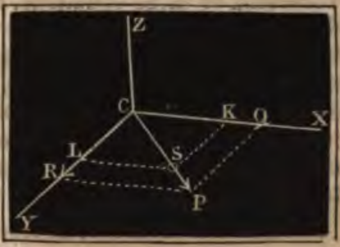
$$u = \frac{10 . 0 . 2 + 15 . 4 . 3 + 18 . 2 . 3 + 12 . 6 . 0}{10 . 0 + 15 . 4 + 18 . 2 + 12 . 6} = \frac{288}{168} = \frac{12}{7} = 1{,}714 \text{ inches and}$$

$$v = \frac{10 . 3 . 2 + 15 . 1 . 3 + 18 . 5 . 3 + 12 . 3 . 0}{10 . 3 + 15 . 1 + 18 . 5 + 12 . 3} = \frac{375}{171} = \frac{125}{57} = 2{,}193 \text{ inches.}$$

The difference of these two values of u and v shows that the centrifugal forces cannot be replaced by a single force.

§ 233. If the particles of the mass lie in a plane at right angles to the axis of revolution, Fig. 276, their centrifugal forces may be reduced to a single force, because their directions intersect at a point in the axis. Retaining the denominations of the former §, we shall obtain the resultant centrifugal force in this case:

Fig. 276.



$P = \sqrt{Q^2 + R^2} = \omega^2 \sqrt{[(M_1x_1 + M_2x_2 + \ldots)^2 + (M_1y_1 + M_2y_2 + \ldots)^2]}$.
Now $CK = x$, and $CL = y$, are the co-ordinates of the centre of gravity of the system $M = M_1 + M_2 + \ldots$, we then have:

$$M_1x_1 + M_2x_2 + \ldots = Mx \text{ and } M_1y_1 + M_2y_2 + \ldots = My,$$

hence the centrifugal force:

$$P = \omega^2 \sqrt{M^2x^2 + M^2y^2} = \omega^2 M \sqrt{x^2 + y^2} = \omega^2 Mr,$$

provided further that $r = \sqrt{x^2 + y^2}$ represent the distance CS of the centre of gravity from the axis of revolution CZ.

For the angle $PCX = \alpha$, which this force includes with the axis

CX, *tang.* $\alpha = \frac{R}{Q} = \frac{My}{Mx} = \frac{y}{x}$; *hence the direction of the centrifugal force passes through the centre of gravity of the system, and centrifugal force is exactly the same as if the collective masses were united at the centre of gravity.*

Fig. 277.

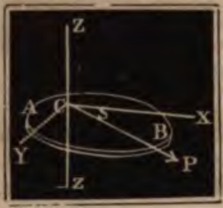
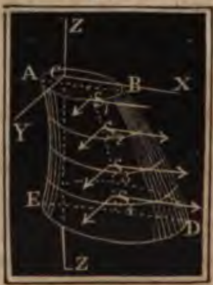


Fig. 278.



For a disc AB at right angles to the axis of revolution $\overline{ZZ}$, Fig. 277, the centrifugal force is from this $= \omega^2 Mr$, if M represents its mass, and r the distance CS of its centre of gravity S from the axis. To find the centrifugal force of another body $ABDE$, Fig. 278, we must decompose it into elementary discs by planes at right angles to the axis $\overline{ZZ}$, find their centres of gravity S_1, S_2, &c., and determine the centrifugal forces by help of these last, decompose each of them in the direction of the axes CX and CY into component forces, and reduce the forces in the plane ZCX to a resultant Q, and those in the plane ZCY to a resultant R.

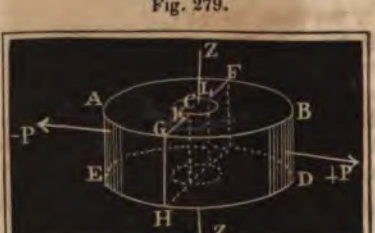
If the centres of gravity of the aggregate discs lie in a line parallel to the axis of revolution, $x = x_1 = x_2$, &c., and $y = y_1 = y_2$, &c., and therefore also $r = r_1 = r_2$, &c., hence the centrifugal force of the whole body $P = \omega^2 (M_1 r + M_2 r + \ldots) = \omega^2 Mr$, and the distance of its point of application from the plane XY:

$$z = \frac{(M_1 z_1 + M_2 z_2 + \ldots)\, r}{(M_1 + M_2 + \ldots)\ \ r} = \frac{M_1 z_1 + M_2 z_2 + \ldots}{M_1 + M_2 + \ldots}.$$

According to these equations, the centrifugal force of a body, whose elements are in a line parallel to the axis, is equivalent to the centrifugal force of the mass of this body, reduced to its centre of gravity, and its point of application and centre of gravity coincide. From this the centrifugal forces of all rotary bodies, whose geometric axes run parallel with the axis of rotation may be found. If the geometric axis of any such body coincide with the axis of rotation, the centrifugal force is equal to nothing.

Example.—The dimensions, the density, and strength of a millstone, $ABDE$, Fig. 279, are given; it is required to find the angular velocity ω, in consequence of which rupture will take place in virtue of centrifugal force. If we put the radius of the millstone $= r_1$, the radius CK of its eye $= r_2$, the height $AE = GH = l$, the density $= \gamma$, and the modulus of strength $= K$, we obtain the force required for rupture $= 2 (r_1 - r_2) lK$, the weight of the stone $G = \pi (r_1^2 - r_2^2) l\gamma$, and the radius of gyration of each half of the stone, *i. e.* the distance of its centre of gravity from the axis of rotation (§ 109), $r = \frac{4}{3\pi} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_1^2}$. At the

Fig. 279.



moment of rupture, the centrifugal force of half the stone is equivalent to the strength; we hence obtain the equation of condition $\omega^2 \cdot \frac{1}{2} \frac{Gr}{g} = 2\,(r_1 - r_2)\, lK$, i. e. $\omega^2 \cdot \frac{2}{3}\,(r_1^3 - r_2^3)\, \frac{l\gamma}{g} = 2\,(r_1 - r_2)\, lK$, or leaving out $2\,l$ on both sides, it follows that

$$\omega = \sqrt{\frac{3\,g\,(r_1 - r_2)\,K}{(r_1^3 - r_2^3)\,\gamma}} = \sqrt{\frac{3\,g\,K}{(r_1^2 + r_1\,r_2 + r_2^2)\,\gamma}}.$$

If $r_1 = 2$ feet $= 24$ inches, $r_2 = 4$ inches, $K = 750$ lbs., and the specific gravity of the millstone $= 2{,}5$, therefore the weight of a cubic inch of its mass $= \frac{62{,}5 \times 2{,}5}{1728}$ $= 0{,}0903$ lbs., it follows that the angular velocity at the moment of rupture is:

$$\omega = \sqrt{\frac{3\,.\,12\,.\,32{,}2\,.\,750}{688\,.\,0{,}0903}} = \sqrt{\frac{869400}{62{,}1264}} = 112{,}1 \text{ inches.}$$

If the number of rotations per minute $= n$, we have then $\omega = \frac{2\,\pi\,n}{60}$; hence, inversely, $n = \frac{30\,\omega}{\pi}$, but here $= \frac{30\,.\,112{,}1}{\pi} = 1070$. The average number of rotations of such a millstone is only 120, therefore 9 times less.

§ 233. If the collective particles M_1, M_2, of a system of masses, Fig. 280, or the centres of gravity of the elements of a body lie in a plane passing through their axis of revolution, the centrifugal forces will then form a system of parallel forces, and these may be reduced according to the rule to a single force. The distances of the particles or the elements from the axis of revolution $\overline{ZZ}$, are $O_1M_1 = r_1$, $O_2M_2 = r_2$, &c., we obtain for their centrifugal forces:

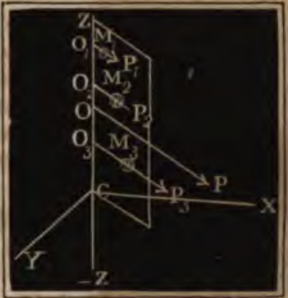
$$P_1 = \omega^2\, M_1 r_1,\ P_2 = \omega^2\, M_2 r_2, \text{ \&c.},$$

and hence the resultant centrifugal force:

$$P = \omega^2 (M_1 r_1 + M_2 r_2 + \ldots) = \omega^2\, Mr,$$

r representing the distance of the centre of gravity of the mass M from the axis of revolution. Therefore, here also the distance of the centre of gravity from the axis of revolution must be regarded as the radius of ~~gyration~~. But to find the point of application O of the resultant centrifugal force, let us put the distances of the particles of the mass from the normal plane: $CO_1 = z_1$, $CO_2 = z_2$, &c., into the formula:

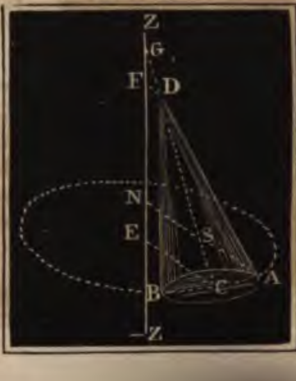
Fig. 280.



$$CO = z = \frac{M_1 r_1 z_1 + M_2 r_2 z_2 + \ldots}{M_1 r_1 + M_2 r_2 + \ldots}.$$

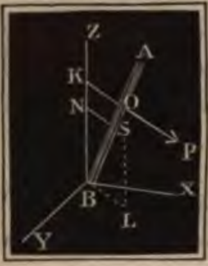
By help of the formula $P = \omega^2\, Mr$, the centrifugal forces of rotary bodies and other geometric bodies may be found, when their axes and the axis of revolution lie in one plane. The centrifugal force of a right cone ADB, Fig. 281, may be found, if the distance SN of its centre of gravity S from the axis of revolution $\overline{ZZ}$ be put as r into the formula. If the height of the cone $CD = h$, the distance DF of the point D from the axis

Fig. 281.



of revolution $= a$, and the angle CGE, by which the geometrical axis deviates from the axis of revolution $\overline{ZZ} = \alpha$, we have then $r = a + \frac{3}{4} h \sin. \alpha$. For a rod AB, Fig. 282, whose length $AB = l$, and angle of inclination ABZ to the axis of revolution $BZ = \alpha$, we have $r = SN = \frac{1}{2} l \sin. \alpha$, therefore, the centrifugal force:

Fig. 282.



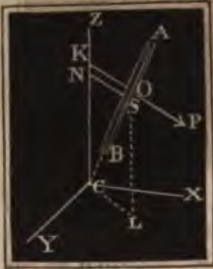
$P = \omega^2 . \frac{1}{2} M l \sin. \alpha$; but to find the point of application O of this force, in the expression $\omega^2 . \frac{M}{n} x \sin. \alpha . x \cos. \alpha = \omega^2 . \frac{M}{n} x^2 \sin. \alpha \cos. \alpha$ for the moment of the element $\frac{M}{n}$ of the rod, let us put for x successively the values $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c., and add the results, in this manner we shall obtain the moment of the entire rod:

$$Pz = \omega^2 \frac{M}{n} \sin. \alpha \cos. \alpha \frac{l^2}{n^2} (1^2+2^2+3^2+ \ldots + n^2)$$

$$= \tfrac{1}{3} \omega^2 M l^2 \sin. \alpha \cos. \alpha,$$

$z = \frac{1}{3} \omega^2 M l^2 \sin. \alpha \cos. \alpha \div \frac{1}{2} \omega^2 M l \sin. \alpha = \frac{2}{3} l \cos. \alpha$, and the distance of the point of application O from the extremity of the rod B lying in the axis, $BO = \frac{2}{3} l$.

Fig. 283.



If the rod AB, Fig. 283, does not reach the axis, we then have: $P = \frac{1}{2} \omega^2 F l_1^2 \sin. \alpha - \frac{1}{2} \omega^2 F l_2^2 \sin. \alpha = \frac{1}{2} \omega^2 F \sin. \alpha (l_1^2 - l_2^2)$, and the moment:

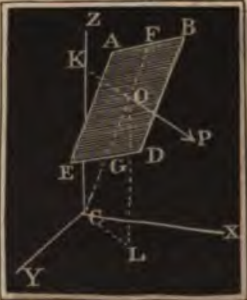
$$Pz = \tfrac{1}{3} \omega^2 F \sin. \alpha \cos. \alpha (l_1^3 - l_2^3),$$

because the mass of CA, $=$ the cross section into the length, $= Fl_1$, and the mass of $CB = Fl_2$, hence it follows that the distance of the point of application O from its intersection with the axis C is:

$$CO = \tfrac{2}{3} \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} = l + \frac{(l_1 - l_2)^2}{12 l},$$

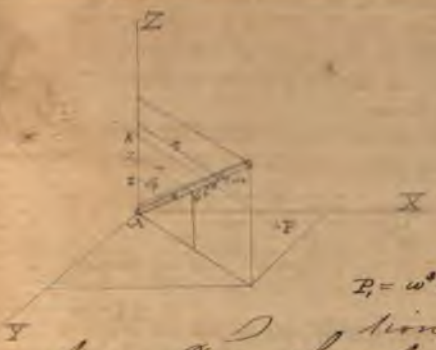
where l expresses the distance CS of the centre of gravity, but $l_1 - l_2$ the length of the rod AB.

Fig. 284.



This formula is also applicable to a rectangular plate $ABDE$, Fig. 284, which is divided into two congruent right angles by the plane of the axis COZ, because the centrifugal force acts at the middle of each of the elements, which are obtained by sections normal to CZ. Therefore the distances CF and CG of the two bases AB and DE from the point C of the axis, are l_1 and l_2, we have here also $CO = \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2}$.

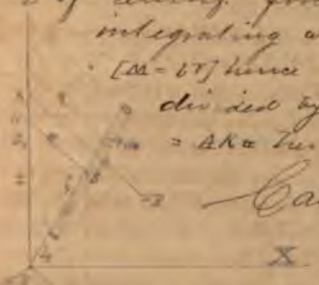
§ 234. In the case where the particles of the body neither lie in a



Rod

Fig 282

$P_1 = w^2 M_1 r_1$ .. General equation for centrifugal force. For the element, $F dx = M_1$, $r_1 = x \sin a$ then $P_1 = w^2 F dx\, x \sin a$ integrating for entire Rod $P = w^2 F \sin a \int_0^l x\, dx = w^2 F \sin a \frac{l^2}{2}$ But Fl = mass of the whole rod therefore $P = \frac{1}{2} w^2 F l \sin a$ ∴ $r = \frac{1}{2} l \sin a$ and $AS = \frac{1}{2} l$ To find the point of application of this force. $P_1 z_1 = w^2 F dx\, x \sin a\, x \cos a$ = moment of centrif. force of elem.t [since $r_1 = x \sin a$ & $z_1 = x \cos a$] integrating $w^2 F \sin a \cos a \int_0^l x^2 dx = w^2 F \sin a \cos a \frac{l^3}{3}$ [$M = lF$] hence arm of lever z = moment divided by force $= \frac{\frac{1}{3} w^2 F \sin a \cos a\, l^3}{\frac{1}{2} w^2 F \sin a\, l^2} = \frac{2}{3} l \cos a = AK$ hence $AO = \frac{2}{3} l$ vide Fig 282.



Case 2d Fig 283.

from above $P = w^2 F \sin a\, x\, dx$

[limits l_1 & l_2] $P = w^2 F \sin a \int_{l_2}^{l_1} x\, dx = w^2 F \sin a \frac{l_1^2 - l_2^2}{2}$. To find pt. of application $Pz = w^2 F \sin a \cos a \int_{l_2}^{l_1} x^2 dx = w^2 F \sin a \cos a \frac{l_1^3 - l_2^3}{3}$ ∴ dividing $z = \frac{\frac{1}{3} w^2 F \sin a \cos a (l_1^3 - l_2^3)}{\frac{1}{2} w^2 F \sin a (l_1^2 - l_2^2)} = \frac{2}{3} \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} \cos a = AK$ hence $AO = \frac{2}{3} \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} = \frac{2}{3}\left(a + b + \frac{ab^2 - a^2 b}{a^2 - b^2}\right) = \frac{2}{3}\left(a + b + \frac{(b-a)ab}{(b+a)(a-b)}\right) = \frac{2}{3}\left(a + b - \frac{ab}{a+b}\right)$ But $a_1 = b + \frac{a-b}{2} = \frac{a+b}{2}$ ∴ $AO = \left(2a_1 - \frac{ab}{2a_1}\right)\frac{2}{3} = \left(\frac{4}{3} a_1 - \frac{ab}{3a_1}\right) = \frac{4a_1^2 - ab}{3a_1}$ but since $2a_1 = a + b$, $a = 2a_1 - b$ substituting $AO = \frac{4a_1^2 - 2a_1 b + b^2}{3a_1} = \frac{3a_1^2 + (a_1^2 - 2a_1 b + b^2)}{3a_1} = \frac{3a_1^2 + (a_1 - b)^2}{3a_1}$ but $a_1 = \frac{a+b}{2}$ ∴ $AO = a_1 + \frac{\left(\frac{a+b}{2} - b\right)^2}{3a_1} = a_1 + \frac{\left(\frac{a-b}{2}\right)^2}{3a_1} = a_1 + \frac{(a-b)^2}{12a_1}$ In this ab and a_1 represent respectively l, l_2

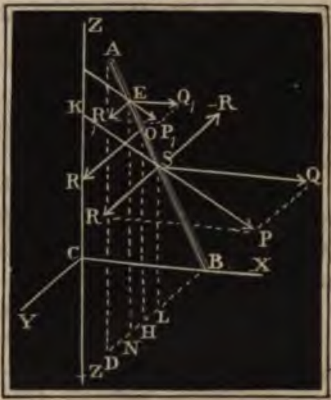
of revolution $= a$, and the angle CGE, by which the geometrical axis

§ 234. In the case where the particles of the body neither lie in a plane normal to, nor in a plane passing through the axis of revolution, the resultant centrifugal forces,
$Q = \omega^2 (M_1 x_1 + M_2 x_2 + \ldots)$ and $R = \omega^2 (M_1 y_1 + M_2 y_2 + \ldots)$ cannot be reduced to a single force, nevertheless it is possible to replace these forces by a force acting at the centre of gravity:

$$P = \sqrt{Q^2 + R^2} = \omega^2 \, Mr,$$

and by a couple composed of Q and R. If, namely, we apply to the centre of gravity S, four forces $+ Q$ and $- Q$, $+ R$ and $- R$ balancing each other, the positive parts will give a resultant $P = \omega^2 \sqrt{Q^2 + R^2}$, and the negative parts on the other hand, $- Q$ and $- R$, will form the couples $(Q, - Q)$ and $(R, - R)$ with the centrifugal forces applied at U and V, which may be reduced to a single couple. In order to make ourselves acquainted with this reduction of the centrifugal forces of a rotary body, let us take the following simple case. Let the bar AB, Fig. 285, which turns about the axis $\overline{ZZ}$, lie parallel to the plane YZ, and rest with its extremity B on the axis CX. Let the length of this bar $= l$, its weight $= G$, the angle BAD at which it is inclined to the axis of rotation $= \alpha$, and its distance CB from the plane YZ, which is also its shortest distance from the axis $\overline{ZZ} = a$. Let now E be an element $\frac{M}{n}$ of the bar, and $BE = x$ its distance from the extremity B, we shall then have the projection $BN = x \sin. \alpha$, and hence the components of the centrifugal force P_1 of this element:
$Q_1 = \omega^2 . \frac{M}{n} . CB = \omega^2 . \frac{M}{n} a$ and $R_1 = \omega^2 . \frac{M}{n} . BN = \omega^2 . \frac{M}{n} x \sin. \alpha$, and their moments about the principal plane XCY:
$Q_1 z_1 = \omega^2 . \frac{M}{n} . CB . EN = \omega^2 . \frac{M}{n} ax \cos. \alpha$ and $R_1 z_1 = \omega^2 . \frac{M}{n} x^2 . \sin. \alpha \cos. \alpha$. The several components parallel to the plane XZ give the resultant $Q = Q_1 + Q_2 + \ldots = n . \omega^2 . \frac{M}{n} a = \omega^2 . Ma$ and its moment $Qu = Q_1 z_1 + Q_2 z_2 + \ldots = \omega^2 . \frac{M}{n} a \cos. \alpha (x_1 + x_2 + \ldots)$, or, as x_1 is to be taken $= \frac{l}{n}$, $x_2 = 2 \frac{l}{n}$, $x_3 = \frac{3l}{n}$, &c.,
$Qu = \omega^2 . \frac{M}{n} a \cos. \alpha . \frac{l}{n} (1 + 2 + 3 + \ldots + n) = \omega^2 . \frac{M}{n} a \cos. \alpha$.

Fig. 285.



$\frac{l}{n} \cdot \frac{n^2}{2} = \frac{1}{2} \omega^2 . Mal\ cos.\ \alpha$; the distance, therefore, of the point of application of this component from the plane XY is:

$$LS = u = \frac{\frac{1}{2}\,\omega^2\,Mal\ cos.\ \alpha}{\omega^2\,Ma} = \tfrac{1}{2}\,l\ cos.\ \alpha,$$

i. e. it coincides with the centre of gravity of the bar. The components, which act parallel to YZ, give the resultant $R = R_1 + R_2 + \ldots = \omega^2 . \frac{M}{n}\ sin.\ \alpha\,(x_1 + x_2 + \ldots) = \omega^2 . \frac{M}{n}\ sin.\ \alpha\ \frac{l}{n} . \frac{n^2}{2} = \frac{1}{2}\,\omega^2\,Ml\ sin.\ \alpha$ with the moment $\omega^2 . \frac{M}{n}\ sin.\ \alpha\ cos.\ \alpha\,(x_1^2 + x_2^2 + \ldots) = \omega^2 . \frac{M}{n}\ sin.\ \alpha\ cos.\ \alpha . \left(\frac{l^2}{n^2} + \frac{4l^2}{n^2} + \ldots\right) = \omega^2 . \frac{M}{n}\frac{l^2}{n^2}\ sin.\ \alpha\ cos.\ \alpha\ (1 + 4 + 9 + \ldots + n^2) = \omega^2 . \frac{M}{n} . \frac{l^2}{n^2}\ sin.\ \alpha\ cos.\ \alpha . \frac{n^3}{3} = \frac{1}{3}\,\omega^2\,M\,l^2\ sin.\ \alpha\ cos.\ \alpha$; the distance of the point of application of the force from the plane XY is : $HO = v = \frac{\frac{1}{3}\,\omega^2\,M\,l^2\ sin.\ \alpha\ cos.\ \alpha}{\frac{1}{2}\,\omega^2\,M\,l\ sin.\ \alpha} = \frac{2}{3}\,l\ cos.\ \alpha$, *i. e.* this point lies about $(\frac{2}{3} - \frac{1}{2})\,l\ cos.\ \alpha = \frac{1}{6}\,l\ cos.\ \alpha = \frac{1}{6}$ of the projection AD parallel to the axis above the centre of gravity S of the bar.

From the forces $Q = \omega^2\,M\,a$, and $R = \frac{1}{2}\,\omega^2\,M\,l\ sin.\ \alpha$, the final resultant applied at the centre of gravity of the bar follows: $P = \sqrt{Q^2 + R^2} = \omega^2\,M\,\sqrt{a^2 + \frac{1}{4}\,l^2\ sin.\ \alpha^2}$, and the couple $(R_1 — R)$ with the moment

$$R . \overline{SO} = \tfrac{1}{2}\,\omega^2\,M\,l\ sin.\ \alpha . \tfrac{1}{6}\,l = \tfrac{1}{12}\,\omega^2\,M\,l^2\ sin.\ \alpha.$$

§ 235. *Free Axes.*—In general, the centrifugal forces of a body revolving uniformly about an axis, exert a pressure upon the axis; it is, nevertheless, possible that these forces mutually counteract each other, and for this reason the axis will have no pressure to sustain. This case presents itself, for instance, in every solid of rotation revolving about its geometric axis, or its axis of symmetry, and especially in the wheel and axle, and in the water-wheel, &c. If, under these circumstances, no external forces act upon a rotary body, or upon such a system, the body will remain for ever in this state of revolution, without its being necessary that the axis of revolution should be fixed. This axis is called, for this reason, a *free axis*. From the preceding, the conditions immediately follow by which an axis of revolution becomes a *free axis*. It is requisite not only that the resultants P and Q of the components of the centrifugal forces acting parallel to the planes of the axes XZ and YZ, but that the sum of the statical moments of each of the two systems of forces, should $= 0$. Hence, from this:

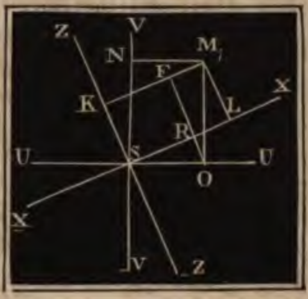
1. $M_1x_1 + M_2x_2 + \ldots = 0$,
2. $M_1y_1 + M_2y_2 + \ldots = 0$, further
3. $M_1x_1z_1 + M_2x_2z_2 + \ldots = 0$, and
4. $M_1y_1z_1 + M_2y_2z_2 + \ldots = 0$.

The two first equations require that the free axis pass through the centre of gravity of the body or system. The two last afford the elements for determining the position of this axis. It may, besides, be proved that every body or system has at least three *free axes*, and that these axes meet at right angles in the centre of gravity of the system.

The higher mechanics distinguish other axes besides the free, which run parallel to these and intersect each other in a point of the system, and are called the ***principal axes***. It may be also proved that the moment of inertia of a body about one of the principal axes is a maximum, about a second axis a minimum, and about a third neither one nor the other.

§ 236. If the particles of a mass lie in one plane, for instance, if the mass forms a thin plate or plane figure, then the straight line passing through the centre of gravity of the entire mass, and normal to its plane, is a free axis of the mass; for in this case the mass has no radius of ~~gyration~~ centrifugal force, and hence the only possible centrifugal force is = 0. To find the other two free axes, let us proceed in the following manner. Let S, Fig. 286, be the centre of gravity of a mass, and let $U\overline{U}$ and $V\overline{V}$ be two co-ordinate axes in the plane of the mass; let us determine the molecules by co-ordinates parallel to these axes, viz. the molecules M by the co-ordinates $M_1N = u_1$ and $M_1O = v_1$. Let $X\overline{X}$, on the other hand, be a free axis, $Z\overline{Z}$ an axis perpendicular to it; further, let the angle to be determined XSU, which the free axis makes with the co-ordinate axis SU, $= \phi$, and let the co-ordinates of the particles referred to the axes $X\overline{X}$ and $Z\overline{Z}$: be $x_1, x_2 \,.\,., z_1, z_2 \,.\,.$, therefore for the particle $M_1 : M_1K = x_1$ and $M_1L = z_1$. From this we easily obtain:

Fig. 286.



$x_1 = M_1K = SR + RL = SO \cos.\phi + OM_1 \sin.\phi = u_1 \cos.\phi + v_1 \sin.\phi$
$z_1 = M_1L = -OR + OF = -SO \sin.\phi + OM_1 \cos.\phi = -u_1 \sin.\phi + v_1 \cos.\phi$; and hence the product:

$$x_1z_1 = (u_1 \cos.\phi + v_1 \sin.\phi)(-u_1 \sin.\phi + v_1 \cos.\phi)$$
$$= -(u_1^2 - v_1^2) \sin.\phi \cos.\phi + u_1v_1 (\cos.\phi^2 - \sin.\phi^2)$$

or, since $\sin.\phi \cos.\phi = \frac{1}{2} \sin. 2\phi$ and $\cos.\phi^2 - \sin.\phi^2 = \cos. 2\phi$, $x_1z_1 = -\frac{1}{2}(u_1^2 - v_1^2) \sin. 2\phi + u_1v_1 \cos. 2\phi$, and hence the moment of the particle M_1:

$M_1x_1z_1 = -\frac{M_1}{2}(u_1^2 - v_1^2) \sin. 2\phi + M_1u_1v_1 \cos. 2\phi$. The moment of the particle M_2:

$M_2x_2z_2 = -\frac{M_2}{2}(u_2^2 - v_2^2) \sin. 2\phi + M_2u_2v_2 \cos. 2\phi$, &c., and the sum of the moments of all the particles, or the moment of the entire mass:

$$M_1x_1z_1 + M_2x_2z_2 + \ldots = -\tfrac{1}{2}\, sin.\, 2\,\phi\, [\,(M_1u_1^2 + M_2u_2^2 + \ldots) - (M_1v_1^2 + M_2v_2^2 + \ldots)] + cos.\, 2\,\phi\, (M_1u_1v_1 + M_2u_2v_2 + \ldots).$$

That $\overline{XX}$ may become a free axis, its moment from the former paragraph must be $= 0$; hence we must put

$$\tfrac{1}{2}\, sin.\, 2\,\phi\, [\,(M_1u_1^2 + M_2u_2^2 + \ldots) - (M_1v_1^2 + M_2v_2^2 + \ldots)] - cos.\, 2\,\phi\, (M_1u_1v_1 + M_2u_2v_2 + \ldots) = 0,$$

and from this we obtain the equation of condition:

$$tang.\, 2\,\phi = \frac{sin.\, 2\,\phi}{cos.\, 2\,\phi} = \frac{2\,(M_1u_1v_1 + M_2u_2v_2 + \ldots)}{(M_1u_1^2 + M_2u_2^2 + \ldots) - (M_1v_1^2 + M_2v_2^2 + \ldots)}$$

$$= \frac{\text{twice the moment of the centrifugal force}}{\text{difference of the moments of inertia}}.$$

By this formula, two values for $2\,\phi$ are given, which vary 180° from each other, and therefore also two values of ϕ, which vary 90° from each other; on this account, not only is the axis $X\overline{X}$ determined by this angle ϕ, a free axis, but also the axis $Z\overline{Z}$ perpendicular to it.

§ 237. The free axes of many surfaces and bodies are known without any calculation. In symmetrical figures, for instance, the axis of symmetry is a free axis, the perpendicular to the centre of gravity is a second, and the axis perpendicular to the plane of the figure a third free axis. The axis of rotation $Z\overline{Z}$ of a rotary body *AB*, Fig. 287, is a free axis, so is every normal $X\overline{X}$, $Y\overline{Y}$ . . to this, passing through the centre of gravity *S*. Every diameter of a sphere is a free axis, the axes $X\overline{X}$, $Y\overline{Y}$, $Z\overline{Z}$, of a right parallelopiped *ABD*, Fig. 288, bounded by six rectangles, passing through the centre of gravity *S*, and normal to the sides *BD*, *AB*, and *AD*, or running parallel with the edges, are free axes.

Fig. 287.

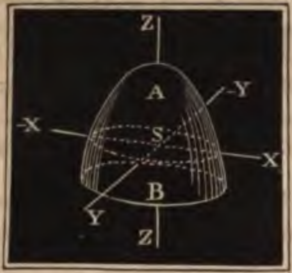
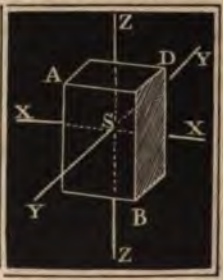
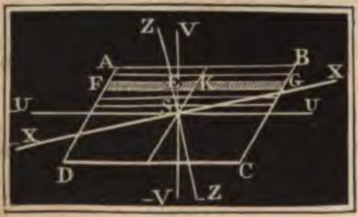


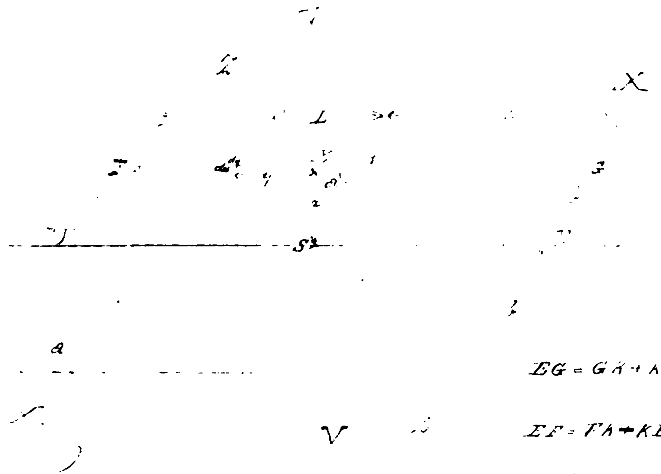
Fig. 288.



Let us now determine the free axes of an acute angled parallelogram *ABCD*, Fig. 289. Let us draw through its centre of gravity *S*, the co-ordinate axes $U\overline{U}$ and $V\overline{V}$ at right angles to each other, so that one of the sides *AB* of the

Fig. 289.





$EG = GK + KE = a + x\cot a$

$EF = FK + KE = a - x\cot a$

The moment of centrifugal force about UV

$(a + x\cot a)\,dx\,\tfrac{1}{2}(a + x\cot a)\rho$ = total moment centrifugal force on EG

[illegible] $x\cot a\,dx$ [illegible] " " " " FE.

$P = \tfrac{1}{2}\int_0^{b\sin a}(a + x\cot a)^2 dx - \tfrac{1}{2}\int_0^{b\sin a}(a - x\cot a)^2 x\,dx$, [illegible]

$= \tfrac{1}{2}\int_0^{b\sin a}(a^2 + 2ax\cot a + x^2\cot^2 a)\,x\,dx - \tfrac{1}{2}\int_0^{b\sin a}(a^2 - 2ax\cot a + x^2\cot^2 a)\,x\,dx$

$\tfrac{1}{2}\left(\frac{a^2b^2\sin^2 a}{2} + 2a\cot a\frac{x^3\sin^3 a}{3} + \cot^2 a\frac{b^4\sin^4 a}{4} - \frac{a^2b^2\sin^2 a}{2} + 2a\cot a\frac{b^3\sin^3 a}{3} - \cot^2 a\frac{b^4\sin^4 a}{4}\right)$

$= 2a\cot a\frac{b^3\sin^3 a}{3} = \tfrac{2}{3}\,a\cos a\,b^3\sin^2 a$ Putting for $\cot a$, $\frac{\cos a}{\sin a}$

$P = \tfrac{2}{3}\,a\frac{\cos a}{\sin a}\,b^3\sin^3 a = \tfrac{2}{3}\,ab^3\sin^2 a\cos a$. The moment of

inertia about UV. $x\,dx\,dy\,y^2$ = moment of [illegible] $dx\,dy$ about UV.

$y = a + x\cot a,\ y = a - x\cot a$

$\int x\,dy\,y^2 + \int dx\,dy\,y^2$ for both sides of VU. Integrating

$= \frac{(a + x\cot a)^3}{3}dx + \frac{(a - x\cot a)^3}{3}dx$ = moment of inertia of EG

$\int_0^{x = b\sin a}\frac{(a + x\cot a)^3}{3}dx + \int_0^{b\sin a}\frac{(a - x\cot a)^3}{3}dx$ = total [illegible]

$= \tfrac{1}{3}\int_0^{b\sin a}(a^3 + 3a^2x\cot a + 3ax^2\cot^2 a + x^3\cot^3 a + a^3 - 3a^2x\cot a + 3ax^2\cot^2 a$

$- x^3\cot^3 a)\,dx = \tfrac{1}{3}\int_0^{b\sin a}(2a^3 + 6ax^2\cot^2 a)\,dx = \tfrac{1}{3}(2a^3b\sin a +$

$6a\cot^2 a\frac{b^3\sin^3 a}{3})$ i.e. $I_0 = \tfrac{2}{3}\,ab\sin a(a^2 + b^2\cos^2 a)$ [illegible]

The moment of inertia about [illegible]

the
ent

pa

an
ta

$I_a = \int_0^{b\sin a} 2a\,dx\,x^2 = 2a\frac{b^3\sin^3 a}{3} = \frac{2}{3}ab^3\sin^3 a$. The

difference of moments of Inertia

$I_1 - I_a = \frac{4}{3}ab\sin a(a^2+b^2\cos^2 a) - \frac{4}{3}ab^3\sin^3 a = \frac{4}{3}$

$a^2+b^2(\cos^2 a - \sin^2 a)) = \frac{4}{3}ab\sin a(a^2+b^2\cos 2a)$

fore our formula of the previous article

$\tan 2\theta = \frac{2P}{I_1-I_a} = \frac{2\cdot\frac{2}{3}ab^3\sin^2 a\cos a}{\frac{4}{3}ab\sin a(a^2+b^2\cos 2a)} = \frac{2b^2\sin}{a^2+b}$

$= \frac{b^2\sin 2a}{a^2+b^2\cos 2a}$ For an equilateral plate a

$\tan 2\theta = \frac{\sin 2a}{1+\cos 2a} = \frac{2\sin a\cos a}{1+\cos^2 a-\sin^2 a} = \frac{2\sin a\cos}{\cos^2 a + \cos}$

$= \tan a$.

Chas. F.

parallelogram may run parallel to it, and let us decompose the parallelogram by parallel lines into $2n$ equal strips, such as FG. If, now, one side $AB = 2\,a$, the other $AD = 2\,b$, and the angle ADC between the two sides $= \alpha$, we then obtain for the strip FG, distant from $U\overline{U}$, $SE = x$, the length of one part:

$$EG = KG + EK = a + x \; cotg. \; \alpha,$$

and that of the other $EF = a - x \; cotang. \; \alpha$, and since $\frac{b}{n} \; sin. \, \alpha$ is the breadth of both, the area of these strips $= \frac{b \; sin. \; \alpha}{n} (a + x \; cotg. \; \alpha)$ and $\frac{b \; sin. \; \alpha}{n} (a - x \; cotg. \; \alpha)$; the measure of the centrifugal forces about the axis $V\overline{V}$ is therefore:

$$= \frac{b \; sin. \; \alpha}{n} (a + x \; cotg. \; \alpha) \, . \, \tfrac{1}{2} (a + x \; cotg. \, \alpha) = \frac{b \; sin. \; \alpha}{2\,n} (a + x \; cotg. \; \alpha)^2$$

and $\frac{b \; sin. \; \alpha}{2n} (a - x \; cotg. \; \alpha)^2$, and their moments about the axis $U\overline{U}$: $\frac{b \; sin. \; \alpha}{2\,n} (a + x \; cotg. \, \alpha)^2 \, x$, and $\frac{b \; sin. \; \alpha}{2\,n} (a - x \; cos. \; \alpha)^2 \, x$. As both the forces about $V\overline{V}$ act opposite to each other, the uniting of their moments gives the difference:

$$\frac{b \; x \; sin. \; \alpha}{2\,n} [(a + x \; cotg. \; \alpha)^2 - (a - x \; cotg.)^2] = \frac{2}{n} \, abx^2 \; cos. \; \alpha.$$

If we substitute in this formula for x the values:

$$\frac{b \; sin. \; \alpha}{n}, \; \frac{2 \; b \; sin. \; \alpha}{n}, \; \frac{3 \; b \; sin. \; \alpha}{n}, \text{ \&c.,}$$

successively, and add the results, we shall obtain the measure of the moment of the centrifugal force of half the parallelogram:

$$\frac{2\,ab}{n} \; cos. \; \alpha \, . \, \frac{b^2 \, sin. \, \alpha^2}{n^2} (1^2 + 2^2 + 3^2 + \, . \, . + n^2) = 2 \; ab^3 \, sin. \, \alpha^2 \; cos. \, \alpha \, . \, \frac{n^3}{3n^3}$$

$= \frac{2}{3} \, ab^3 \; sin. \; \alpha^2 \; cos. \; \alpha$, and, therefore, for the whole parallelogram, or

$M_1 u_1 v_1 + M_2 u_2 v_2 + \, . \, . \, . = \frac{4}{3} \, ab^3 \; sin. \; \alpha^2 \; cos. \; \alpha$. The moment of inertia of a strip FG about the axis $V\overline{V}$ is:

$$= \frac{b \; sin. \; \alpha}{n} \left(\frac{(a + x \; cotg. \; \alpha)^3}{3} + \frac{(a - x \; cotg. \; \alpha)^2}{3} \right)$$

$= \frac{2 \; b \; sin. \; \alpha}{3\,n} (a^3 + 3 \; ax^2 \; cotg. \; \alpha^2) = \frac{2}{3} \frac{ab}{n} \; sin. \; \alpha \, (a^2 + 3 \; x^2 \; cotg. \; \alpha^2)$; if now we substitute in succession for x:

$$\frac{b \; sin. \; \alpha}{n}, \; \frac{2 \; b \; sin. \; \alpha}{n}, \; \frac{3 \; b \; sin. \; \alpha}{n}, \text{ \&c.,}$$

and sum the resulting values, we shall have the moment of inertia of a half $= \frac{2}{3} \, ab \; sin. \; \alpha \, (a^2 + b^2 \; cos. \; \alpha^2)$, and hence that of the whole

$= \frac{4}{3}\, ab\ sin.\ \alpha\ (a^2+b^2\ cos.\ \alpha^2)$. On the other hand, the moment of inertia of the parallelogram about the axis of revolution $\overline{UU}$ is $= 4\ ab\ sin.\ \alpha\ .\ \frac{b^2 sin.\ \alpha^2}{3} = \frac{4}{3}\, ab^3\ sin.\ \alpha^3$ (§ 220); hence the difference of the moments of inertia sought, *i. e.*

$$(M_1u_1^2+M_2u_2^2+\ldots)—M_1v_1^2+M_2v_2^2+\ldots),$$

$$= \frac{4}{3}\, ab\ sin.\ \alpha\ (a^2+b^2\ cos.\ \alpha^2) — \frac{4}{3}\, ab^3\ sin.\ \alpha^3$$

$$= \frac{4}{3}\, ab\ sin.\ \alpha\ [a^2+b^2\ (cos.\ \alpha^2—sin.\ \alpha^2)]$$

$$= \frac{4}{3}\, ab\ sin.\ \alpha\ (a^2+b^2\ cos.\ 2\ \alpha).$$

Lastly for the angle $USX=\phi$, which the *free axis* $\overline{XX}$ makes with the co-ordinate axis $\overline{UU}$ or the side AB from § 236:

$$tang.\ 2\ \phi = \frac{2\ (M_1u_1v_1+M_2u_2v_2+\ldots)}{(M_1u_1^2+M_2u_2^2+\ldots)—(M_1v_1^2+M_2v_2^2+\ldots)}$$

$$= \frac{2\frac{4}{3}\, ab^3\ sin.\ \alpha^2\ cos.\ \alpha}{\frac{4}{3}\, ab\ sin.\ \alpha\ (a^2+b^2\ cos.\ 2\ \alpha)} = \frac{b^2\ sin.\ 2\ \alpha}{a^2+b^2\ cos.\ 2\ \alpha}.$$

In the rhombus $a = b$, hence

$$tang.\ 2\ \phi = \frac{sin.\ 2\ \alpha}{1+cos.\ 2\ \alpha} = \frac{2\ sin.\ \alpha\ cos.\ \alpha}{1+cos.\ \alpha^2—sin.\ \alpha^2}$$

$$= \frac{2\ sin.\ \alpha\ cos.\ \alpha}{2\ cos.\ \alpha^2} = tang.\ \alpha,$$

therefore $2\ \phi = \alpha$, and $\phi = \frac{\alpha}{2}$. As this angle gives the direction of the diagonal, it follows that the diagonals are free axes of the rhombus.

Example. The sides of the acute angled parallelogram, $ABCD$, Fig. 289, $AB = 2\ a = 16$ inches, and $BC = 2b = 10$ inches, and the angle of the perimeter $ABC = \alpha = 60°$, what directions have its free axis?

$$tang.\ 2\ \phi = \frac{5^2\ .\ sin.\ 120°}{8^2+5^2\ .\ cos.\ 120°} = \frac{25\ .\ sin.\ 60°}{64—25\ cos.\ 60°} = \frac{25\ .\ 0{,}86603}{64—25\ .\ 0{,}5} = 0{,}42040$$

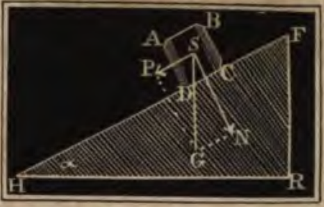
$= tang.\ 22°\ 48'$, or $tang.\ 202°\ 48'$. From this it follows, that $\phi = 11°\ 24'$ and $101°\ 24'$ are the angles of inclination of the two free axes to the side AB. The third free axis stands at right angles to the plane of the parallelogram. These angles determine also the free axes of a right parallelopiped with rhomboidal bases.

CHAPTER III.

OF THE ACTION OF GRAVITY ON MOTIONS ALONG CONSTRAINED PATHS.

§ 238. *Inclined Plane.*—A heavy body may be impeded in various ways from falling freely, and in the following we shall consider only two cases, the one where a body is supported on an inclined plane, and the other where it revolves about a horizontal axis. In both cases the paths of the body are contained in a vertical plane. If the body rests on an inclined plane, its weight may be resolved into two components, of which the one is directed normal to the plane and resisted by it, and the other parallel to the plane, and acts upon the body as a moving force. If G be the weight of the body $ABCD$, Fig. 290, and α the inclination of the inclined plane FHR to the horizon, we shall then have from § 134, for the normal pressure; $N = G\ cos.\ \alpha$, and the moving force $P = G\ sin.\ \alpha$. The motion of the body may be either sliding or rolling, let us next consider the first only. In this case all the parts of the body equally participate in its motion, and hence have a common motion of acceleration p, which is given by the known formula:

Fig. 290.



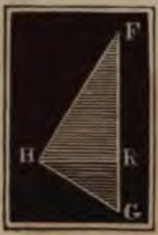
$$p = \frac{\text{force}}{\text{mass}}, = \frac{P}{M} = \frac{G\,sin.\ \alpha}{G} \cdot g = g\ sin.\ \alpha.$$

Therefore $p : g = sin.\ \alpha : 1$, *i. e.* the *accelerated motion of a body on an inclined plane is to the accelerated motion of free descent as the sine of the angle of descent to unity.* In consequence of the friction which takes place, this formula is rarely sufficiently accurate, hence it is necessary in many cases of application to take this into account.

If the body moves on a curved surface, the accelerating force is variable, and at each place equal to the accelerating force, which corresponds with the plane of contact to the curved surface.

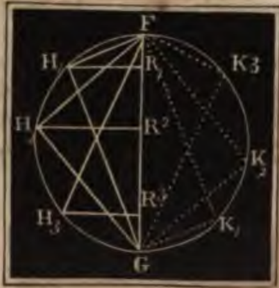
§ 239. A body slides with the initial velocity 0 down an inclined plane, without friction, from § 10 the final velocity after t seconds is: $v = g\ sin.\ \alpha\ .\ t = 32{,}2\ sin.\ \alpha\ .\ t$ ft., and the space described: $s = \frac{1}{2}\ g\ sin.\ \alpha\ .\ t^2 = 16{,}1\ sin.\ \alpha\ .\ t^2$ ft. In free descent $v_1 = gt$, and $s_1 = \frac{1}{2}\ gt^2$, hence we may put: $v : v_1 = s : s_1 = sin.\ \alpha : 1$, *i. e.* the ~~*final*~~ [illegible] *velocity and the space of descent down the inclined plane are to the final velocity and space of free descent as the sine of the angle of inclination of the inclined plane to unity.*

Fig. 291.



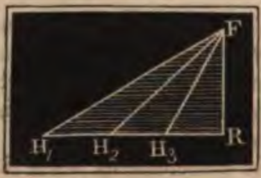
The perpendicular FH of a right-angled triangle, FGH, Fig. 291, with vertical hypothenuse FG, $= FG \ sin. \ FGH = FG \ sin. \ FHR = FG \ sin. \ \alpha$, if α is the angle of inclination of this perpendicular to the horizon, hence $FH : FG = sin. \ \alpha : 1$, and a body describes the vertical hypothenuse FG, and the inclined side FH in one and the same time. The space of free descent corresponding to the space of descent down the inclined plane may be found from this, and the latter from the former by construction. Since the angles of the periphery FH_1G, FH_2G, &c., on the diameter FG, Fig. 292, are right angles, the semicircle on FG cuts off from all the inclined planes, commencing at F, the spaces described with this diameter, and therefore, in equal times, FH_1, FH_2, &c., hence it is asserted, *the chords of a circle and its diameter will descend simultaneously or isochronously.* This isochronism is besides true, not only for the chords FH_1, FH_2, &c., which have their origin at the highest point F of the circle, but also for the chords K_1G, K_2G, &c., which commence at the lowest point G, for chords FK_1, FK_2, &c., may be drawn through F, which have like positions and equal lengths with the chords GH_1, GH_2, &c.

Fig. 292.



§ 240. From the equation $s = \frac{v^2}{2p} = \frac{v^2}{2 g \ . \ sin. \ \alpha}$, it follows that $s \ sin. \ \alpha = \frac{v^2}{2g}$, and, inversely, $v = \sqrt{2 \ gs \ sin. \ \alpha}$. But now $s \ sin. \ \alpha$ is the height FR of the inclined plane or the vertical projection s_1 of the space $FH = s$ upon it, hence the final velocities of bodies which descend with an initial velocity 0 down planes of equal heights F_1H_1, F_2H_2, &c., and of different inclination, Fig. 293, are equal, and also equal to the velocity which a body would acquire if it fell freely from the height FR of these planes.

Fig. 293.



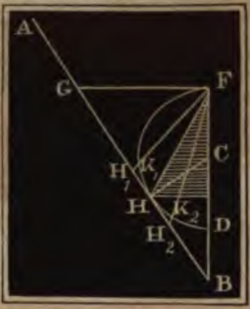
From the equation $s = \frac{1}{2} g \ sin. \ \alpha \ . \ t^2$ follows the formula for the time.

$$t = \sqrt{\frac{2 \ s}{g \ sin. \ \alpha}} = \frac{1}{sin. \ \alpha} \sqrt{\frac{2 \ s \ sin. \ \alpha}{g}} = \frac{1}{sin. \ \alpha} \cdot \sqrt{\frac{2 \ . \ FR}{g}}.$$

But for a free descent through the height FR the time is:

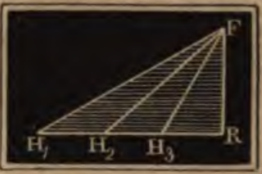
$t_1 = \sqrt{\frac{2 \ FR}{g}}$, it follows accordingly $t : t_1 = 1 : sin. \ \alpha = FH_1 : FR$, *the time of descent down the inclined plane is to the time of free descent* *from the height of this plane as the length of the inclined plane is to its height.*

Fig. 294.



Examples.—1. The initial point F of an inclined plane FH, Fig. 294, is given, and the final point H in a given line AB; required to determine the descent down the plane so that it may take place in the shortest time. If the horizontal line FG be drawn through F to its intersection with AB, and GH be made $= GF$, we shall obtain in H the point sought, and, therefore, in FH the plane of quickest descent; for if through F and H a circle tangent to FG and FH be carried, its isochronously described chords FK_1, FK_2, &c., will be shorter than the lengths FH_1, FH_2, &c., of the corresponding inclined planes; consequently, therefore, the time of descent for these chords will be less than for these lengths, and the time of descent for the inclined plane FH, which coincides with a chord, will be the shortest.

Fig. 295.



2. Required the inclination of that inclined plane FH_1, Fig. 295, down which a body would fall in the same time as if it originally fell freely from the height FR, and then proceeded with the acquired velocity horizontally to H_1. The time of falling down from the vertical height $FR = s_1$ is $t_1 = \sqrt{\frac{2\,s_1}{g}}$, and the acquired velocity at $R: v = \sqrt{2\,gs_1}$. If now no loss of velocity ensue in transition from the vertical to the horizontal motion, which would follow if the corner R were rounded, the space $RH_1 = s_1\, cotg.\, \alpha$ will be uniformly described, and in the time $t_2 = \frac{s_1\, cotg.\, \alpha}{v} = \frac{s_1\, cotg.\, \alpha}{\sqrt{2\, gs_1}} = \frac{1}{2}\, cotg.\, \alpha \sqrt{\frac{2\,s_1}{g}}$. The time of descent down the inclined plane is $t = \frac{1}{sin.\, \alpha} \sqrt{\frac{2\,s_1}{g}}$; hence, if we put $t = t_1 + t_2$, we shall obtain the equation of condition $\frac{1}{sin.\, \alpha} = 1 + \frac{1}{2}\, cotg.\, \alpha$, whose solution will give $tang.\, \alpha = \frac{3}{4}$. In the corresponding inclined plane, accordingly, the height is to the base and to the length as 3 is to 4 is to 5, and the angle of inclination is $\alpha = 36°\, 52'\, 11''$.—3. The time for sliding down an inclined plane of a given base a is

$$t = \sqrt{\frac{2\,s}{g\, sin.\, \alpha}} = \sqrt{\frac{2\,a}{g\, sin.\, \alpha\, cos.\, \alpha}} = \sqrt{\frac{4\,a}{g\, sin.\, 2\, \alpha}};$$

hence the descent is quickest when $sin.\, 2\, \alpha$ is a maximum, *i. e.* $= 1$; therefore $2\, \alpha° = 90°$, or $\alpha° = 45°$. Hence, water falls down in the shortest time from roofs of 45° inclination.

§ 241. If the motion on an inclined plane proceeds with a certain initial velocity c, we shall then have to apply the formulæ found in § 13 and § 14. According to these the terminal velocity of a body ascending an inclined plane is $v = c - g\, sin.\, \alpha\, .\, t$, and the space described $s = ct - \frac{1}{2}\, g\, sin.\, \alpha\, .\, t^2$; on the other hand, for a body falling down the inclined plane:

$$v = c + g\, sin.\, \alpha\, .\, t, \text{ and } s = ct + \tfrac{1}{2}\, g\, sin.\, \alpha\, .\, t^2.$$

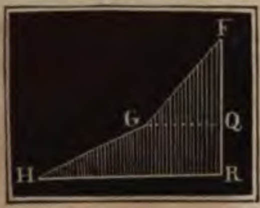
In both cases of motion the formula is true:

$$s = \frac{v^2 - c^2}{2\, g\, sin.\, \alpha}, \text{ or } s\, sin.\, \alpha = \frac{v^2 - c^2}{2\, g} = \frac{v^2}{2g} - \frac{c^2}{2g}.$$

The vertical projection, therefore, (s *sin.* α) *of the space* (s) *described along the inclined plane is always equal to the difference of the heights due to the velocity.*

If two inclined planes *FGQ* and *GHR*, Fig. 296, meet each other in a rounded edge, no impulse will take place in the passage from one plane to the other, and for this reason, no loss of velocity ensue; the rule for the descent of a body down this combination of two planes is also true, *the height of descent* (*FR*) *is equal to the difference of the heights due to the velocity.* It is easy to ascertain that this rule is correct also for the ascent or descent on any system of any number of planes, and for the ascent or descent on curved lines or surfaces. (Compare § 82.)

Fig. 296.



Examples.—1. A body ascends with a 21 feet initial velocity an inclined plane of 22° inclination, what is the amount of its velocity and its space described in $1\frac{1}{2}$ seconds? The velocity is:
$v = 21 - 32{,}2\ sin.\ 22° \,.\, 1{,}5 = 21 - 32{,}2 \,.\, 0{,}3746 \,.\, 1{,}5 = 2{,}906$ feet; and the space:

$$s = \frac{c+v}{2} . t = \frac{21 + 2{,}906}{2} \cdot \frac{3}{2} = \frac{23{,}906 \,.\, 3}{4} = 17{,}928 \text{ feet.}$$

2. How high does a body, with an initial velocity of 36 feet, ascend an inclined plane of 48° acclivity? The vertical height is $s_1 = \frac{v^2}{2g} = 0{,}01550\ v^2 = 0{,}0155 \,.\, 36^2 = 21{,}638$ feet; hence the whole space up the inclined plane: $s = \frac{s_1}{sin.\ \alpha} = \frac{21{,}638}{sin.\ 48°} = 28{,}494$ feet. The time required is:

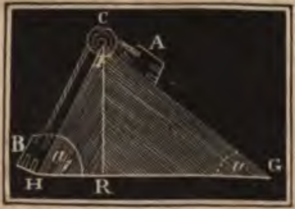
$$t = \frac{2 \,.\, s}{v} = \frac{2 \,.\, 28{,}499}{36} = \frac{28{,}494}{18} = 1{,}583 \text{ seconds.}$$

§ 242. Sliding friction exerts a considerable influence upon the ascent and descent of a body along an inclined plane. From the weight *G* of the body, and from the angle of inclination α of the inclined plane, the normal pressure follows, $N = G\ cos.\ \alpha$, and again from this, the friction $F = f\ N = f\ G\ cos.\ \alpha$. If we subtract this from the force $P = G\ sin.\ \alpha$, with which gravity urges the body down the plane, there then remains for the moving force $= G\ sin.\ \alpha - f\ G\ cos.\ \alpha$, and the accelerating force of the body sliding down the plane is known:

$$p = \frac{\text{the force}}{\text{the mass}} = \frac{G\ sin.\ \alpha - f\ G\ cos.\ \alpha}{G} g = (sin.\ \alpha - f\ cos.\ \alpha)\ g.$$

The moving force of the body ascending the inclined plane is negative and $= G\ sin.\ \alpha + f \,.\, G\ cos.\ \alpha$, hence also the accelerating force p is negative and $= -(sin.\ \alpha + f\ cos.\ \alpha)\ g$.

Fig. 297.



If two bodies are supported on different planes *FG* and *FH*, Fig. 297, by perfectly flexible strings connected with each other, passing over a roller *C*, it is then possible for one of the two bodies to descend and pull up the other. If we represent the weights of these bodies by *G* and G_1, and the angles of inclination of the inclined planes along which they move by

α and α_1, and if we assume that G descends and draws G_1 upwards, we shall then obtain as the moving force:
$G\ sin.\ \alpha - G_1\ sin.\ \alpha_1 - f\ G\ cos.\ \alpha - f\ G_1\ cos.\ \alpha_1 = G\ (sin.\ \alpha - f\ cos.\ \alpha) - G_1\ (sin.\ \alpha_1 + f\ cos.\ \alpha_1)$, and the mass moved $= \frac{G + G_1}{g}$, hence the accelerated motion with which G descends and G_1 ascends:

$$p = \frac{G\ (sin.\ \alpha - f\ cos.\ \alpha) - G_1\ (sin.\ \alpha_1 + f\ cos.\ \alpha_1)}{G + G_1} \cdot g.$$

Since friction as a ~~resisting force~~ resistance can generate no motion, it is requisite for the fall of G and the rise of G_1, that

$G\ (sin.\ \alpha - f\ cos.\ \alpha)$ be $> G_1\ (sin.\ \alpha_1 + f\ cos.\ \alpha_1)$, therefore
$\frac{G}{G_1} > \frac{sin.\ \alpha_1 + f\ cos.\ \alpha_1}{sin.\ \alpha - f\ cos.\ \alpha}$. If, on the other hand, G_1 descend, and G be drawn up, then must:

$$\frac{G_1}{G} \text{ be} > \frac{sin.\ \alpha + f\ cos.\ \alpha}{sin.\ \alpha_1 - f\ cos.\ \alpha_1}, \text{ or, } \frac{G}{G_1} < \frac{sin.\ \alpha_1 - f\ cos.\ \alpha_1}{sin.\ \alpha + f\ cos.\ \alpha}.$$

So long, however, as $\frac{G}{G_1}$ lies within the limits:

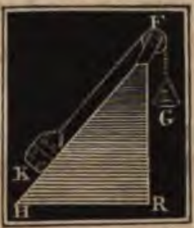
$$\frac{sin\ \alpha_1 + f\ cos.\ \alpha_1}{sin.\ \alpha - f\ cos.\ \alpha} \text{ and } \frac{sin\ \alpha_1 - f\ cos.\ \alpha_1}{sin.\ \alpha + f\ cos.\ \alpha},$$

so long will the friction resist motion.

Examples.—1. A sledge moves down an inclined snow plane, 150 feet long and 20° inclination, and when arrived at the bottom, proceeds along a horizontal one until friction brings it to rest. If the co-efficient of friction between the snow and the sledge be taken $= 0{,}03$ feet, what space will the sledge describe along the horizontal plane, neglecting the resistance of the air? The accelerating force $p = (sin.\ \alpha - f\ cos.\ \alpha)\ g = (sin.\ 20^\circ) \,.\, 32{,}2 = (0{,}3420 - 0{,}03 \,.\, 0{,}9397) \,.\, 32{,}2 = 0{,}3138 \,.\, 32{,}2 = 10{,}104$ feet; hence, the final velocity of descent is,
$v = \sqrt{2\,ps} = \sqrt{2 \,.\, 10{,}104 \,.\, 150} = \sqrt{3031{,}2} = 55{,}54$ feet. On the horizontal plane the accelerating force is $p_1 = -fg = -0{,}03 \,.\, 32{,}2 = 0{,}966$ feet; hence, the space $s_1 = \frac{v^2}{2fg} = \frac{3031{,}2}{1{,}932} = 1630$ feet. The time of descent is $t = \frac{2s}{v} = \frac{300}{55{,}54} = 5{,}22$ seconds, and that for sliding onward $t_1 = \frac{2s_1}{v} = \frac{3260}{55{,}54} = 58{,}6$ seconds; hence the whole time of the course $t + t_1 = 63{,}82$ seconds $= 1^{k}\ 3{,}82''$.—2. A filled tub K, Fig. 298, of 250 lbs. clear weight, is drawn up an inclined plane FH, 70 feet long and of 50° inclination, by a descending weight G of 260 lbs.; what will be the time required for this, if the co-efficient of friction of the tub along its path amount to 0,36. The moving force is $= G - (sin.\ \alpha + f\ cos.\ \alpha)\ K = 260 - (sin.\ 50^\circ + 0{,}36 \,.\, cos.\ 50^\circ) \,.\, 250 = 260 - 0{,}9974 \,.\, 250 = 10{,}6$ lbs.; hence, the accelerating force $p = \frac{10{,}6}{250 + 260} = \frac{10{,}6}{510} = 0{,}0208$ lbs.; further, the time
$t = \sqrt{\frac{2s}{p}} = \sqrt{\frac{140}{0{,}0208}} = \sqrt{6731} = 82{,}04$ seconds $= \overset{m}{1^{k}}\ \overset{s}{22''}$,
and the final velocity $v = \frac{2s}{t} = \frac{140}{82} = 1{,}70$ feet.

m s Vide Smyth's physical Outlines of [illegible]

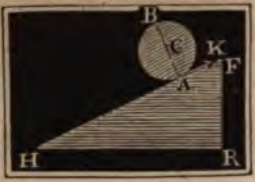
Fig. 298.



§ 243. *Rolling Motion.*—When a carriage rolls down an inclined plane, the friction of the axle chiefly acts in opposition to the accelerating force; if r be the radius of the axle, and a that of the wheel, the friction will amount to $\frac{fr}{a} N = \frac{fr}{a} G \cos. \alpha$, and hence the accelerating force $p = (\sin. \alpha - \frac{fr}{a} \cos. \alpha)\, G$.

If a round body AB, a cylinder or sphere, for example, roll down an inclined plane FH, Fig. 299, we have to consider a progressive and a rotatory motion at the same time. Generally the acceleration of the progression is equal to that of the rotation (§ 156); since if we put the moment of inertia of the rolling body $= G y^2$, and the radius of the cylinder $= a$, we shall then obtain for the force $AK = K$, with which the cylinder is set into revolution by virtue of the penetration of its parts into those of the inclined plane:* $K = p \,.\, \frac{G y^2}{g a^2}$. But the force K acts opposed to the force of descent $G \sin. \alpha$, hence it follows that the moving force for progressive motion $= G \sin. \alpha - K$, and the accelerating force $p = \frac{G \sin. \alpha - K}{G} \,.\, g$. If we eliminate K from both equations, we shall obtain $G p = G g \sin. \alpha - \frac{G y^2}{a^2} \,.\, p$, consequently the accelerating force sought:

Fig. 299.



$$p = \frac{g \sin. \alpha}{1 + \frac{y^2}{a^2}}.$$

For the case of a homogeneous rolling cylinder $y^2 = \frac{1}{2} a^2$ (§ 221), hence $p = \frac{g \sin. \alpha}{1 + \frac{1}{2}} = \frac{2}{3} g \sin. \alpha$; but for a sphere $y^2 = \frac{2}{5} a^2$ (§ 222), hence $p = \frac{g \sin. \alpha}{1 + \frac{2}{5}} = \frac{5}{7} g \sin. \alpha$; therefore, the accelerating force of the rolling cylinder is only $\frac{2}{3}$, that of a rolling sphere only $\frac{5}{7}$ that of a body sliding without friction.

The force of rotation is:

$$K = \frac{g \sin. \alpha}{1 + \frac{y^2}{a^2}} \cdot \frac{G y^2}{g a^2} = \frac{G y^2 \sin. \alpha}{a^2 + y^2}.$$

As long as this is less than the sliding friction $f G \cos. \alpha$, the body descends rolling perfectly down the plane. But if

$$K \text{ is } > f G \cos. \alpha, \text{ i. e. } \operatorname{tang}. \alpha > f\left(1 + \frac{a^2}{y^2}\right),$$

the friction is no longer sufficient to communicate to the body a velocity of rotation equal to its velocity of progression; hence the acceleration of progression, as for sliding friction, is:

---

* See above, pages, 169 to 175, note.

$$p = \frac{G\ sin.\ \alpha - f\ G\ cos.\ \alpha}{G} \cdot g = (sin.\ \alpha - f\ cos.\ \alpha)\ g,$$

and that of rotation:

$$p_1 = \frac{f\ G\ cos.\ \alpha}{Gy^2 \div a^2} \cdot g = f \frac{a^2}{y^2}\ g\ cos.\ \alpha.$$

For a carriage of the weight G with wheels of the radius a, and with the moment of inertia $G_1 y^2$, we have:

$$K = p\, \frac{G_1 y^2}{g\, a^2} \text{ and } p = \frac{G\ sin.\ \alpha - f \frac{r}{a}\ G\ cos.\ \alpha - K}{G} \cdot g, \text{ i. e.}$$

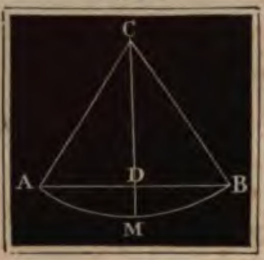
$$p = \frac{g\ (sin.\ \alpha - f \frac{r}{a}\ cos.\ \alpha)}{1 + \frac{G_1 y^2}{Ga^2}}.$$

Examples.—1. A loaded wagon of 3600 lbs. weight, with wheels 4 feet high, and moment of inertia 2000 ft. lbs., rolls down an inclined plane of 12° inclination, what will be its accelerated motion, if the co-efficient of axle friction $= 0{,}15$, and the thickness of the axles of the wheels amounts to 3 inches?

It is $\frac{G_1 y^2}{Ga^2} = \frac{2000}{3600\ .\ 2^2} = \frac{5}{36} = 0{,}139$, and $f \frac{r}{a} = 0{,}15\ .\ \frac{1}{4\ .\ 4} = 0{,}0094$ hence the accelerating force sought is $p = \frac{32{,}2\ (sin.\ 12^\circ - 0{,}0094\ .\ cos.\ 12^\circ)}{1 + 0{,}139} = \frac{32{,}2\ (0{,}2079 - 0{,}0094\ .\ 0{,}978)}{1{,}139} = \frac{32{,}2\ .\ 0{,}1987}{1{,}139} = 6{,}398$ feet.—2. What will be the accelerating forces of a solid cylinder rolling down an inclined plane of a 40° angle of descent? The co-efficient of the sliding friction of the cylinder on the plane $= 0{,}24$, we have then $f\left(1 + \frac{a^2}{y^2}\right) = 0{,}24\ (1 + 2) = 0{,}72$; but now the *tang.* $40^\circ = 0{,}839$, hence the *tang.* α is greater than $f\left(1 + \frac{a^2}{y^2}\right)$, and the acceleration of the rolling motion less than that of the progressive. The last is $p = (sin.\ \alpha - f\ cos.\ \alpha)\ g = (0{,}6428 - 0{,}24\ \cdot\ 0{,}7660)\ .\ 32{,}2 = 0{,}459\ .\ 32{,}2 = 14{,}78$ feet, but the first only $p_1 = 0{,}24\ .\ 2\ .\ 32{,}2\ cos.\ 40^\circ = 11{,}85$ feet.

§ 244. *Circular Pendulum.*—Equilibrium subsists in a body suspended to a horizontal axis so long as its centre of gravity lies vertically below the axis; but if its centre of gravity be drawn out of the vertical plane containing the axis, and the body be left to itself, it will take an oscillatory motion; that is, it will move up and down in a circle. In general, however, a body oscillating about a horizontal axis is called a *pendulum.* If the oscillating body is a material point, and its connection with the axis of revolution be made by a line devoid of weight, we then have the *mathematical* or *simple pendulum;* but if the pendulum consists of a body having dimensions, or of several bodies, we have then a *compound*, *physical*, or *material pendulum.* Such a pendulum may be regarded as a connection of simple pendulums oscillating about a common axis. The simple pendulum is an imaginary one only, but its assumption possesses great advantage, because it is easy to reduce the theory of the motion of the compound to that of the simple pendulum.

Fig. 300.

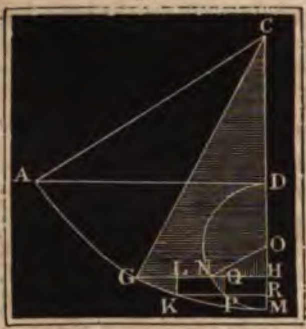


If the pendulum suspended at C, Fig. 300, be drawn out of its vertical position CM into the position CA, and then left to itself, it will go back by virtue of its gravity with an accelerated motion towards CM, and its mass will arrive at its lowest point M with a velocity v, whose height $\frac{v^2}{2g}$ is equal to the height of descent DM. In virtue of this velocity, it will now describe on the other side the arc $MB = MA$, and will thereby ascend to the height DM. From B it will again fall back to M and A, and so it will go on successively describing the circular arc AB. If the resistance of the air and friction were entirely set aside, this oscillation of the pendulum would go on indefinitely; but because these resistances can never be done away with, the amplitude of the oscillation will become smaller and smaller, and the pendulum come at last to a state of rest.

The motion of the pendulum from A to B is called an *oscillation*, the arc AB the *amplitude*, the angle measuring half the amplitude by which the pendulum is distant from either side of the vertical CM, *the angle of elongation* or *angle of deviation*. Lastly, the time in which the pendulum makes an oscillation, is called the *time of oscillation*.

§ 245. On account of the frequent application of the pendulum to the purposes of life, to clocks namely, it is of consequence to know the times of oscillation, hence the determination of these is one of the principal problems in mechanics. With the view of solving this problem, let us put the length of the pendulum $AC = MC = r$, Fig. 301, and the height of ascent or descent corresponding to a complete oscillation $MD = h$. Let us assume that the pendulum has fallen from A to G, and let the height of fall $DH = x$ correspond to this motion, we may then put the acquired velocity $v = \sqrt{2gx}$, and the particle of time in which the particle of space GK is described, $\tau = \frac{GK}{v} = \frac{GK}{\sqrt{2gx}}$. If, now, from the centre O of $MD = h$ and the radius $OM = OD = \frac{1}{2}h$, we describe the semicircle MND, we then have a portion of this arc NP of the height $PQ = KL = RH$ equal to GK, which is in a simple ratio to this particle of space GK. From the similarity of the triangles GKL and CGH, $\frac{GK}{KL} = \frac{CG}{GH}$, and from the similarity of the triangles NPQ and ONH,

Fig. 301.



$\frac{\mathcal{N}P}{PQ} = \frac{O\mathcal{N}}{\mathcal{N}H}$; hence, if we divide these two equations by each other, and bear in mind that $KL = PQ$, we then obtain the ratio of the said portion of arc: $\frac{GK}{\mathcal{N}P} = \frac{CG \, . \, \mathcal{N}H}{GH \, . \, O\mathcal{N}}$. From the properties of the circle, and from the theorem of the mean proportional, $\overline{GH}_2 = MH\,(2\,CM - MH)$ and $\overline{\mathcal{N}H}^2 = MH \, . \, DH$; hence it follows:

$$\frac{GK}{\mathcal{N}P} = \frac{CG \, . \, \sqrt{DH}}{O\mathcal{N} \, . \, \sqrt{2\,CM - MH}} = \frac{r\sqrt{x}}{\frac{1}{2}\,h\sqrt{2\,r - (h - x)}},$$

and the time for describing an element of space is:

$$\tau = \frac{r\sqrt{x}}{\frac{1}{2}\,h\sqrt{2\,r - (h - x)}} \, . \, \frac{\mathcal{N}P}{\sqrt{2\,gx}} = \frac{2\,r}{h\sqrt{2\,g\,[2\,r - (h - x)]}} \, . \, \mathcal{N}P$$

$$= \sqrt{\frac{r}{g}} \, . \, \frac{\mathcal{N}P}{h\sqrt{1 - \frac{h - x}{2r}}}.$$

In most cases of application, a small angle of deviation is given to the pendulum, and for this reason $\frac{h}{2r}$, as also $\frac{x}{2r}$, and, therefore, also $\frac{h - x}{2r}$ is so small a quantity that we may neglect it as well as its powers, and now put $\tau = \sqrt{\frac{r}{g}} \, . \, \frac{\mathcal{N}P}{h}$. The duration of a semi-oscillation, or the time in which the pendulum describes the arc AM, is equal to the sum of all the particles of time corresponding to the elements GK or $\mathcal{N}P$, or as $\frac{1}{h} \, . \, \sqrt{\frac{r}{g}}$ is a constant factor, equal to $\frac{1}{h}\sqrt{\frac{r}{g}}$ times the sum of all the elements forming the semicircle $D\mathcal{N}M$; *i. e.* $= \frac{1}{h}\sqrt{\frac{r}{g}}$ times the semicircle $\left(\frac{\pi h}{2}\right)$ itself, therefore

$$= \frac{1}{h}\sqrt{\frac{r}{g}} \, . \, \frac{\pi h}{2} = \frac{\pi}{2}\sqrt{\frac{r}{g}}.$$

The pendulum, however, requires the same time for ascending, for here the velocities are the same, and only opposite in direction, and for this reason the duration of a complete oscillation is twice as great,

$$i.\,e.\ t = \pi\sqrt{\frac{r}{g}}.$$

§ 246. To determine the duration of an oscillation with greater accuracy, which is necessary where the angles of oscillation are large, let us transform the expression:

$\frac{1}{\sqrt{1-\frac{h-x}{2r}}} = \left(1-\frac{h-x}{2r}\right)^{-\frac{1}{2}}$ into the series

$$1+\tfrac{1}{2}\cdot\frac{h-x}{2r}+\tfrac{3}{8}\cdot\left(\frac{h-x}{2r}\right)^2+\ldots,$$

and we shall obtain the time for an element of space

$$\tau = \left[1+\tfrac{1}{2}\cdot\frac{h-x}{2r}+\tfrac{3}{8}\left(\frac{h-x}{2r}\right)^2+\ldots\right]\sqrt{\frac{r}{g}}\cdot\frac{NP}{h}.$$

If we put the angle NOM, subtended at the centre by NM, $=\phi$, we shall then also obtain

$$MH = h-x = NO\,(1-cos.\,\phi) = \tfrac{1}{2}h\,(1-cos.\,\phi);\text{ hence:}$$

$$\tau = \left(1+\tfrac{1}{2}\cdot\frac{h\,(1-cos.\,\phi)}{4r}+\ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{NP}{h}.$$

If we divide the semicircle DNM into n equal parts, and if we put each $= NP = \frac{\pi h}{2n}$, we shall obtain

$$\tau = \left(1+\tfrac{1}{2}\cdot\frac{h\,(1-cos.\,\phi)}{4r}+\ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{\pi}{2n};$$

by substituting successively for $\phi = \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}\ldots$ to $\frac{n\pi}{n}$, and adding the results, we shall then obtain half the time of an oscillation:

$$t = \left(n+\frac{h}{8r}(n - \text{the sum of all the cosines})+\ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{\pi}{2n}.$$

But the sum of the cosines of all the angles from $\phi = 0$ to $\phi = \pi$ is $= 0$; hence, we have more correctly : $t = \left(1+\frac{h}{8r}\right)\cdot\pi\sqrt{\frac{r}{g}}$.

If we have regard to more members of the series, we shall obtain:

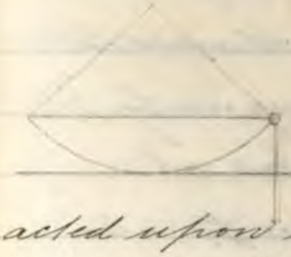
$$t = \left[1+\left(\tfrac{1}{2}\right)^2\cdot\frac{h}{2r}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2r}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\cdot\left(\frac{h}{2r}\right)^3+\ldots\right]\cdot\pi\sqrt{\frac{r}{g}},$$

the last formula but one is, however, generally sufficient. If the pendulum oscillates in a semicircle, we then have $h = r$; hence the duration of an oscillation:

$$t = \left(1+\tfrac{1}{8}+\frac{9}{256}+\frac{225}{18432}+\ldots\right)\pi\sqrt{\frac{r}{g}} = 1{,}180\ldots\pi\sqrt{\frac{r}{g}}.$$

From the angle of elongation α, it follows that $cos.\,\alpha = \frac{r-h}{r} = 1-\frac{h}{r}$, therefore, $\frac{h}{r} = 1 - cos.\,\alpha$; and hence, $\frac{h}{8r} = \frac{1}{4}\cdot\frac{1-cos.\,\alpha}{2} = \frac{1}{4}\left(sin.\,\frac{\alpha}{2}\right)^2$; from this, consequently, the correction for the time of oscillation corresponding to a given angle of elongation may be found. If, for example, this angle $= 15^\circ$, we have $\frac{h}{8r} = \frac{1}{4}\left(sin.\,\frac{15^\circ}{2}\right)^2 = 0{,}00426$; on the other hand, for $\alpha = 5^\circ$: $\frac{h}{8r}$

The Simple Pendulum



Let M be a free material point acted upon by two forces N which constantly acts normally to its path and g the force of gravity. Let $\alpha\ \beta$ & γ be the angles made by N with the three coördinate axes. The accelerations in the directions of these axes are represented by $\frac{d^2x}{dt^2}$ $\frac{d^2y}{dt^2}$ & $\frac{d^2z}{dt^2}$ and the components of N in the directions of these axes will be $N\cos\alpha$, $N\cos\beta$ and $N\cos\gamma$. We have moreover the components of g in these several directions $g\begin{cases}0\\-g\end{cases}$ Since g acts vertically downwards thereby decreasing the values of z. We have then the accelerating forces along each of the three coördinate

$$\frac{d^2x}{dt^2} = N\cos\alpha - 0,\ \frac{d^2y}{dt^2} = N\cos\beta - 0\ \&\ \frac{d^2z}{dt^2} = N\cos\gamma - g$$

stant $t = \frac{1}{2}\sqrt{\frac{r}{g}}\cdot\left[\pi - \text{versin}^{-1}\frac{x}{\frac{1}{2}h}\right]$ This is the general expression for the time which it would take a pendulum of small angles of deviation to oscillate from its extreme pt. to the pt. whose ordinate is x; but if we wish to find the time for a semi-oscillation, that is the time which it would take the pendulum to pass from the highest to the lowest point of its path we substitute for x its value at that pt. i.e. $x=0$ therefore the time of a semi-oscillation of a Pendulum of a small angle of deviation will be $t = \frac{1}{2}\sqrt{\frac{r}{g}}\cdot\pi$ or the Time for the total Oscillation $T = \pi\sqrt{\frac{r}{g}}$. For Pendulums of large angles of deviation we cannot in the expression $-dt = \frac{r\,dx}{\sqrt{2g(h-x)(2rx-x^2)}}$ neglect the x^2 in the denomin.^r but we may instead write it $-dt = \frac{r\,dx}{\sqrt{2g(hx-x^2)}}(2r-x)^{-\frac{1}{2}}$ and develope the last term [illegible] binomial theorem to degree [illegible]

$= 0{,}00047$; for the latter angle of elongation, therefore, the time of oscillation is $t = 1{,}00047 \, . \, \pi \sqrt{\frac{r}{g}}$.

We may, therefore, for a deviation under 5°, put tolerably accurately the time of oscillation:

$$t = \pi \sqrt{\frac{r}{g}} = \frac{\pi}{\sqrt{g}} \sqrt{r} = 0{,}562 \sqrt{r}.$$

§ 247. As the angle of deviation does not appear in the formula $t = \pi \sqrt{\frac{r}{g}}$, it follows that the small times of oscillation of pendulums are independent of this angle, and therefore that pendulums of equal length, but of different angles of deviation, vibrate isochronously, or perform their oscillations in equal times. A pendulum deviating 4° has the same time of oscillation as a pendulum deviating 1°.

If we compare the time of oscillation t with the time of free descent, we shall then arrive at the following. The time of free descent from the height r will be t_1

$$= \sqrt{\frac{2r}{g}} = \sqrt{2} \, . \sqrt{\frac{r}{g}}, \text{ hence } t : t_1 = \pi : \sqrt{2};$$

the time of an oscillation is, therefore, to the time in which a body of the length of the pendulum freely descends, as π to the square root of 2, or since t_1 is also $= \sqrt{\frac{4 \, . \, \frac{1}{2} r}{g}} = 2 \sqrt{\frac{\frac{1}{2} r}{g}}$, the *time of oscillation is to the time of descent of half the length of the pendulum as π is to 2.*

If we put the times of oscillation t and t_1, corresponding to the lengths of the pendulum r and r_1, we then obtain $t : t_1 = \sqrt{r} : \sqrt{r_1}$; therefore, *for one and the same acceleration of gravity, the times of oscillation are as the square roots of the lengths of the pendulum.* On the other hand, if n be the number of oscillations which a pendulum makes in a certain time, one minute, and n_1 the number which another pendulum makes in the same time, we then have $t : t_1 = \frac{1}{n} : \frac{1}{n_1}$, hence, inversely, $n : n_1 = \sqrt{r_1} : \sqrt{r}$, i. e. *the number of oscillations is in an inverse ratio to the square roots of the lengths of pendulums.* A pendulum four times the length gives, therefore, half the number of oscillations.

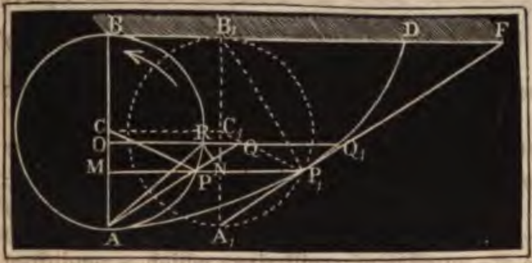
A pendulum is called a *seconds pendulum*, when its time of oscillation is one second. If we put $t = 1$ into the formula $t = \pi \sqrt{\frac{r}{g}}$, we obtain the length of the seconds pendulum $r = \frac{g}{\pi^2} = 39{,}13929$ inches $= 0{,}9938$ meters.

From the formula $t = \pi \sqrt{\frac{r}{g}}$ it follows by inversion that $g = \left(\frac{\pi}{t}\right)^2 r$; the acceleration of gravity may be found, therefore, from the length of a pendulum, and from its time of oscillation t. This method is both simpler and safer than that of Attwood's machine.

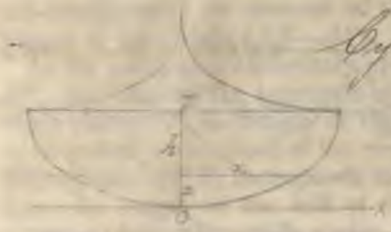
Remark. The diminution of gravity from the poles to the equator has been proved by pendulum observations, and its quantity determined. This diminution is due to the effect of the centrifugal force, which is generated by the diurnal rotation of the earth about its axis, and to the increase of the earth's radius from the poles to the equator. The centrifugal force at the equator diminishes gravity by $\frac{1}{290}$ of its value (§ 231), whilst at the poles it is null. If β be the latitude of the place of observation, the accelerating force of gravity from pendulum observations will be $g = 32{,}2\ (1 - 0{,}00259 \cos. 2\ \beta)$, therefore at the equator where $\beta = 0$; therefore $\cos. 2\ \beta = 1$, $g = 32{,}2\ (1 - 0{,}00259) = 32{,}11$ feet, and at the poles, where $\beta = 90°$; therefore $\cos. 2\ \beta = \cos. 180° = -1$; $g = 32{,}2 \,.\, 1{,}00259 = 32{,}283$ ft. For the rest g is less on mountains and in mines than at the level of the sea.

§ 248. *Cycloid.*—We may in an infinite (?) number of ways set a body into vibration, or into an oscillating motion, and we call every body in this condition of motion a *pendulum*, and distinguish accordingly several kinds of pendulums, for example, the *circular pendulum*, which we have already considered, and the *cycloidal*, where the body, by virtue of gravity, oscillates to and fro in a cycloidal arc, and the *torsion pendulum*, where the body vibrates by virtue of the torsion of a thread, or wire, &c. We shall here speak only of the *cycloidal pendulum.*

Fig. 302.



The *cycloid AD*, Fig. 302, is a curved line described by a point A of a circle APB which rolls along a straight line BD. If this generating circle has rolled forward $BB_1 = CC_1$, and, therefore, come into the position A_1B_1, it has then also revolved through the arc $AP = A_1P_1 = BB_1 = PP_1$, consequently the ordinate corresponding to any absciss MP_1 = ordinate MP of the circle plus the arc of revolution AP. In this rolling the generating circle revolves about the point of contact at each instant with the base line, if, therefore, it be in A_1B_1, it will then revolve about B_1, and describe thereby the elementary arc P_1Q_1 of the cycloid; consequently the chord B_1P_1 will be the direction of the normal, and the chord A_1P_1 that of the tangent to the cycloid at the point P_1. The prolongation PQ of the chord AP reaching to the ordinate OQ_1 is, therefore, equal to the element of the cycloid P_1Q_1, as further the space of revolution is equal to the space RQ of



Cycloidal Pendulum

In the case in which the body moves in the arc of a cycloid we may take, from of our first discussion of a body acted upon by two forces one constantly normal to its path and the other constantly vertical, $v^2 = -\int 2g\,dz = \frac{ds^2}{dt^2} = \frac{dx^2+dy^2}{dt^2} = -\int 2g\,dz = -2g\,dz + c$ and as for the circular pendulum in order to find the value of this constant we pass to the farthest point for which $v=0$ & $\overset{z}{x}=h$ and therefore $c = 2gh$ putting in this value $\frac{dx^2+dz^2}{dt^2} = 2g(h-z)$ But the differential equation of the Cycloid is $\frac{dz}{dx} = \sqrt{\frac{z}{2a-z}}$ in which a is the Radius of the Generating circle. Squaring $\frac{dz^2}{dx^2} = \frac{z}{2a-z}$ ∴

$dx^2 = \frac{2a-z}{z}dz^2 = \frac{2a}{z}dz^2 - dz^2$ ∴ $dx^2 + dz^2 = \frac{2a}{z}dz^2$

$\frac{2a}{z}\frac{dz^2}{dt^2} = \frac{dx^2+dz^2}{dt^2} =$ (from (1)) $2g(h-z)$ hence

$dt^2 = \frac{\frac{2a}{z}dz^2}{2g(h-z)} = \frac{a}{gz(h-z)}dz^2$ ∴ $dt = \pm\sqrt{\frac{a}{g}}\cdot\frac{dz}{\sqrt{hz-z^2}}$

whence by integration we have $t=-\sqrt{\frac{a}{g}}\ \text{versin}^{-1}\frac{z}{\frac{1}{2}h}+C$. To find the value of this constant pass to the extremity by making $t=0$ when $z=h$ and therefore $C=\pi\sqrt{\frac{a}{g}}$ Substituting this abo $t=\sqrt{\frac{a}{g}}\left(\pi-\text{versin}^{-1}\frac{z}{\frac{1}{2}h}\right)$. This is the expression for the time which the pendulum woul require to go from the highest or extre pt to the & point whose ordinate i z. But for the semi-oscillation we must place $z=0$ when the expression for the time becomes $T=\pi\sqrt{\frac{a}{g}}$. which being independent of h show that all arcs of the same cycloid are isochronously describes. For entire oscillation $T=2\pi\sqrt{\frac{a}{g}}=\pi\sqrt{\frac{4a}{g}}$. 4a is the radius of curvature at O.

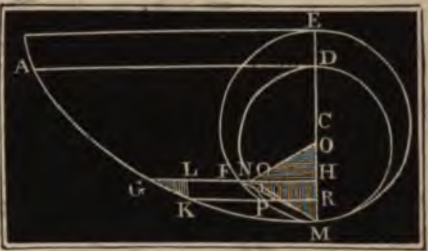
J. T. W

progression, PQ is then the base line of an isosceles triangle PRQ, and equal to double the line PN, which the perpendicular RN cuts off, but PN is the difference of the two contiguous chords AP, AR, and consequently the element of the cycloid P_1Q_1 = twice the difference of the chords ($AR—AP$).

As the continuous elementary arcs make up together the whole arc AP_1, and likewise the aggregate of the differences of the chords, the whole chord AP, the length of the cycloidal arc AP_1, is, from this, equal to double the chord of the circle AP, appertaining to it. To the semi-cycloid AP_1D, corresponds the diameter as a chord of a circle, hence the length of the half of the cycloid is equal to double the diameter of the generating circle.

§ 249. *Cycloidal Pendulum.*—From the above known properties of the cycloid, the theory of the cycloidal pendulum, or the formula for the time of oscillation of a body vibrating in a cycloidal arc, may be easily developed. Let AKM, Fig. 303, be the half of the cycloidal arc in which a body ascends or descends, or oscillates, and ME the generating circle, therefore, $CE = CM = r$ its radius. If the body has described the arc AG, it has, therefore, fallen from the height $DH = x$ (§ 246), it has then acquired the velocity $v = \sqrt{2gx}$, with which it describes the elementary arc CK in the time $\tau = \frac{GK}{v} = \frac{GK}{\sqrt{2gx}}$.

Fig. 303.



But from the similarity of the triangles GLK and FHM, $\frac{GK}{KL} = \frac{FM}{MH}$, or as $\overline{FM^2} = MH \,.\, ME$, $\frac{GK}{KL} = \frac{\sqrt{MH \,.\, ME}}{MH} = \frac{\sqrt{ME}}{\sqrt{MH}}$; from the similarity of the triangles NPQ and ONH, $\frac{NP}{PQ} = \frac{ON}{NH}$, or since $\overline{NH^2} = MH \,.\, DH$, $\frac{NP}{PQ} = \frac{ON}{\sqrt{MH \,.\, DH}}$. Now $KL = PQ$, hence it follows by division:

$$\frac{GK}{NP} = \frac{\sqrt{ME}}{\sqrt{MH}} \cdot \frac{\sqrt{MH \,.\, DH}}{ON} = \frac{\sqrt{ME \,.\, DH}}{ON},$$

or, since ON is half the height of descent $= \frac{h}{2}$, $ME = 2r$, and $DH = x$:

$$\frac{GK}{NP} = \frac{\sqrt{2\,rx}}{\frac{1}{2}\,h} = \frac{2\sqrt{2\,rx}}{h}.$$

By putting $GK = \frac{2\sqrt{2rx}}{h} \,.\, NP$ into the formula $\tau = \frac{GK}{\sqrt{2gx}}$ we obtain:

$$\tau = \frac{2\sqrt{2rx}}{\sqrt{2gx}\,.\,h} \,.\, NP = \frac{2}{h}\sqrt{\frac{r}{g}} \,.\, NP.$$

The time of falling from A to M is the sum of all the values of τ, which are obtained; if for NP all the particles of the semicircle DNM be successively substituted, therefore, $= \frac{2}{h}\sqrt{\frac{r}{g}}$ times the semicircle $DNM \left(\frac{\pi}{2}h\right)$. In this manner we obtain the time for falling through the arc AM,

$$= \frac{\pi}{2}h \,.\, \frac{2}{h}\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{r}{g}},$$

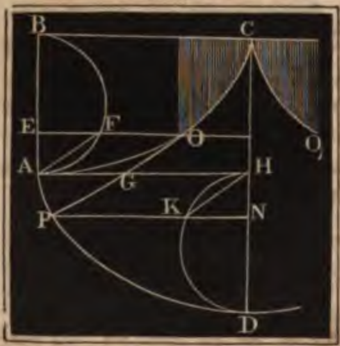
and as the time for ascending the arc MB is likewise as great, the time of oscillation, or the time of describing the whole arc AMB:

$$t = 2\pi\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{4r}{g}}.$$

As this quantity is quite independent of the length of the arc, it follows that, mathematically speaking, the times of oscillation for all arcs of one and the same cycloid are equal, the oscillations of the cycloidal pendulum are, therefore, perfectly isochronous. If we compare this formula with the formula for the time of oscillation of a circular pendulum, it follows that the times of oscillation for both kinds of pendulums are equal, if the length of the circular pendulum is equal to four times the radius of the generating circle of the cycloidal pendulum.

Remark 1. It may be proved by the higher calculus that the cycloid has, besides the property of *isochronism* or *tautochronism*, also that of *brachistochronism*, which is that line between two given points in which a body falls in the shortest time from one point to the other.

Fig. 304.

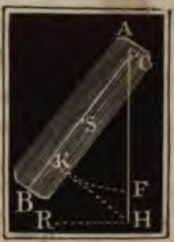


Remark 2. In order to make a body, suspended to a perfectly flexible thread, vibrate in a cycloidal arc, and thereby represent the cycloidal pendulum, we suspend the body between two cycloidal arcs CO and $CO_{,,}$ Fig. 304, so that the thread for every deviation unwinds from the one arc and winds round the other. By this winding and unwinding of the thread COP, its extremity P describes a curve similar to the given cycloid, and it may be similarly represented that the evolute of the cycloid is a similar cycloid in an inverse position. As the length of half the cycloid $COA = CD = 2\,AB$, we have likewise the arc $=$ to the straight line evolved OP; but the arc $OA = 2$ chord $AF = 2\,GO$, hence also $PG = GO = AF$, and $HN = AE$. If now we describe upon DH a semicircle DKH and draw the ordinate NP, we then have $KH = PG$; and hence also $PK = GH = AH - AG = AH - FO =$ arc $AFB -$ arc $AF =$ arc $BF =$ arc DK; and lastly, the ordinate $NP =$ the ordinate NK of the circle, plus the corresponding arc DK; therefore NP is the ordinate of a cycloid, and DPA the cycloid corresponding to the generating circle DKH.

For the application of the cycloidal pendulum to clocks, see *Jahrbücher des polytech. Institutes in Wien.*, vol. xx. art. 2.

§ 250. *Compound Pendulum.*—To find the time of oscillation of the compound pendulum, or that of any other body AB, Fig. 305, oscillating about a horizontal axis C, let us first seek the centre of oscillation, *i. e.* that point K of the body, which if it oscillates of itself about C, or forms a mathematical pendulum, has the same time of oscillation as the whole body. It is easily seen from this explanation that there are several centres of oscillation in a body, but in general, that point only is meant which lies with the centre of gravity, in one and the same perpendicular to the axis of revolution.

Fig. 305.



From the variable angle of deviation $KCF = \phi$, the accelerating force of the isolated point K, $= g \sin. \phi$, because we may suppose that it slides down an inclined plane of the inclination $KHR = KCF$. But if My^2 be the moment of inertia of the entire body or set of bodies AB, Ms will be its statical moment, *i. e.* the product of the mass, and the distance $CS = s$ of its centre of gravity S from the axis of revolution C, and r the distance CK of the centre of oscillation K from the axis of revolution, or the length of the simple pendulum which vibrates isochronously with the material pendulum AB, we have then the mass reduced to $K = \frac{My^2}{r^2}$, and the force of revolution reduced to this $\frac{s}{r} M g \sin. \phi$; consequently the accelerating force $= \frac{\text{force}}{\text{mass}} = \frac{s}{r} M g \sin. \phi \div \frac{My^2}{r^2} = \frac{Msr}{My^2} . g \sin. \phi$. That this pendulum may have the same time of oscillation as a mathematical one, it is requisite that both should have their motion in every position equally accelerated, that therefore, $\frac{Msr}{My^2} . g \sin. \phi = g \sin. \phi$. Now this equation gives:

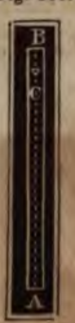
$$r = \frac{My^2}{Ms} = \frac{\text{moment of inertia}}{\text{statical moment}}.$$

We, therefore, find the distance of the centre of oscillation from the centre of gyration, or the length of the simple pendulum, which has a time of oscillation equal to that of the compound one, if we divide the moment of inertia of the compound pendulum by its statical moment.

If we substitute this value in the formula $t = \pi \sqrt{\frac{r}{g}}$, we obtain for the time of oscillation of the compound pendulum the formula $t = \pi \sqrt{\frac{M y^2}{M g s}} = \pi \sqrt{\frac{y^2}{gs}}$, or more accurately $= \left(1 + \frac{h}{8r}\right) \pi \sqrt{\frac{y^2}{gs}}$. Inversely, the moment of inertia may be found from the time of oscillation of a suspended body, if we put:

$$My^2 = \left(\frac{t}{\pi}\right)^2 . Mgs, \text{ or } y^2 = \left(\frac{t}{\pi}\right)^2 gs.$$

Fig. 306.

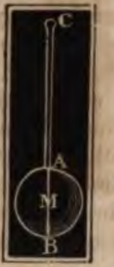


Examples.—1. For a uniform prismatic rod *AB*, Fig. 306, whose centre of oscillation is distant $CA = l_1$ and $CB = l_2$ from the extremities *A* and *B*, we have (§ 219) for the moment of inertia: $My^2 = \frac{1}{3} F\,(l_1^3 + l_2^3)$, and the statical moment $Ms = \frac{1}{2} F\,(l_1^2 - l_2^2)$; hence, the length of the mathematical pendulum which vibrates isochronously with this rod is $r = \frac{My^2}{Ms} = \frac{2}{3} \cdot \frac{l_1^3 + l_2^3}{l_1^2 - l_2^2} = \frac{l^2 + 3d^2}{6d}$, if l represent the sum $l_1 + l_2$, and d the difference $l_1 - l_2$. If this rod beats half seconds, we have $r = \frac{1}{4} \cdot \frac{g}{\pi^2} = \frac{1}{4} \,.\, 0{,}10132 \,.\, 32{,}2 = 0{,}8156$ feet $= 9{,}737$ inches, but if the entire length l of the rod amount to 12 inches, we must then put: $9{,}737 = \frac{144 + 3\,d^2}{6\,d} = d^2 - 19\,d = -48$ nearly; hence it follows:

$d = \frac{19 - \sqrt{169}}{2} = 3$ nearly; and from this

$l_1 = \frac{l + d}{2} = \frac{15}{2} = 7\frac{1}{2}$ inches, and $l_2 = \frac{l - d}{2} = \frac{9}{2} = 4\frac{1}{2}$ inches.

Fig. 307.



—2. For a pendulum with a spherical lenticular bob *AB*, Fig. 307, if *G* be the weight and l the length *CA* of the rod or thread; *K*, on the other hand, the weight of the bob, and ρ its radius $MA = MB$:

$$r = \frac{\frac{1}{3}\,Gl^2 + K\,[(l + \rho)^2 + \frac{2}{5}\,\rho^2]}{\frac{1}{2}\,Gl + K\,(l + \rho)}.$$

If, now, the wire is 0,05 lbs., the bob 1,5 lbs., further, the length of the rod 1 foot, and the radius of the bob 1,15 inches, we then have the distance of the centre of oscillation of this pendulum from the axis of rotation:

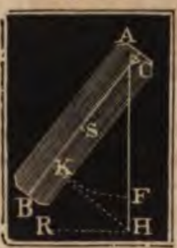
$$r = \frac{\frac{1}{3} \,.\, 0{,}05 \,.\, 12^2 + 1{,}5 \,.\, (13{,}15^2 + \frac{2}{5} \,.\, 1{,}15^2)}{\frac{1}{2} \,.\, 0{,}05 \,.\, 12 + 1{,}5 \,.\, 13{,}15} = \frac{2{,}4 + 260{,}177}{0{,}3 + 19{,}725} = \frac{262{,}577}{20{,}025}$$

$= 13{,}112$ inches. Disregarding the rod, r would $= \frac{262{,}577}{19{,}725} = 13{,}312$ inches; and the inert mass of the bob being reduced to its centre, r would $= 13{,}15$ inches. The time of oscillation of this bob is:

$$t = \pi \sqrt{\frac{r}{g}} = 0{,}562 \sqrt{\frac{13{,}112}{12}} = 0{,}562 \sqrt{1{,}0926\,..} = 0{,}5874 \text{ seconds.}$$

§ 251. The centre of suspension and centre of oscillation of a material pendulum are reciprocal, *i. e.* the one may be interchanged with the other, and the pendulum may be suspended at the centre of oscillation, without the time of oscillation being altered. The proof of this proposition may be given by aid of § 217, in the following manner. If T be the moment of inertia of the compound pendulum *AB*, Fig. 308, oscillating about the centre of gravity *S*, we have then for a revolution about the axis *C*, distant $CS = s$ from the centre of gravity *S*, $T_1 = T + Ms^2$, hence the distance of the centre of oscillation *K* from the axis of revolution *C*:

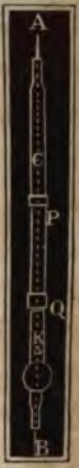
Fig. 308.



$$r = \frac{T_1}{MS} = \frac{T + Ms^2}{Ms} = \frac{T}{Ms} + s.$$

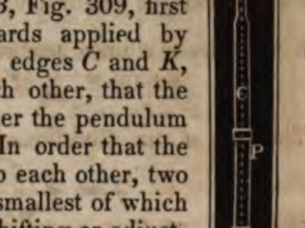
If now we represent the distance $KS = r - s$ of the centre of oscillation from the centre of gravity by s_1, we then obtain the simple equation $ss_1 = \frac{T}{M}$, in which s and s_1 appear in a similar manner, and hence may be substituted one for the other. This formula is not only true for the descent, if *s* represents the distance of the centre of oscillation from the centre of gravity, but also inversely, if *s* expresses the distance of the centre of oscillation, and s_1 that of the centre of gyration from the centre of gravity, and *C* will therefore serve for the centre of oscillation if *K* serve for the centre of suspension. We avail ourselves of this property in the so-called *convertible pendulum AB*, Fig. 309, first proposed by Bohnenberger, and afterwards applied by Kater, which is furnished with two knife edges *C* and *K*, which are so situated with regard to each other, that the times of oscillation remain the same whether the pendulum oscillates about one or the other axis. In order that the axes may not be displaced with regard to each other, two sliding weights *P* and *Q* are applied, the smallest of which is attached by a fine screw. If by the shifting or adjustment of these weights, the time of oscillation comes to be the same, the pendulum may be suspended at *C* or at *K*, we shall then obtain in the distance *CK* of the two edges, the length *r* of the simple pendulum which vibrates synchronously with the convertible pendulum, and we shall now obtain the time of oscillation by the formula $t = \pi \sqrt{\frac{r}{g}}$.

Fig. 309.



§ 252. The swinging or rocking of a body with cylindrical base may be compared with the oscillations of a pendulum. This rocking, like every other rolling motion, is composed of a progressive and a rotary motion, but it may be assumed that it consists of a simple rotary motion with a variable axis of rotation. This axis of rotation is the point of support *P*, by which the vibrating body *ABC*, Fig. 310, rests on the horizontal base *HR*. If $CD = CP$ is the radius of the rolling base $ADB = r$, and the distance *CS* of the centre of gravity of the entire body from the centre *C* of this base $= s$, we have then for the distance corresponding to the angle of rotation $SCP = \varphi$, $SP = y$ of the centre of gravity from the centre of gyration:

Fig. 310.



$$y^2 = r^2 + s^2 - 2\,r\,s\,\cos.\,\varphi = (r - s)^2 + 4\,r\,s\left(\sin.\,\frac{\varphi}{2}\right)^2;$$

hence, if further we represent the moment of inertia of the entire body about the centre of gravity *S* by Mk^2, we shall obtain the moment of inertia about the point of support *P*:

$$T = M\,(k^2 + y^2) = M\left[k^2 + (r - s)^2 + 4\,r\,s\left(\sin.\,\frac{\varphi}{2}\right)^2\right],$$

which for small angles of vibration may be put

$$= M\,[k^2 + (r - s)^2 + r\,s\,\varphi^2],$$

or only $M\,[k^2 + (r - s)^2]$. Since now the moment of force $= G\,.\,SN = Mg\,.\,CS\,sin.\,\phi = M\,g\,s\,sin.\,\phi$, it follows that the angular acceleration for the rotation about P:

$$\pi' = \frac{\text{moment of force}}{\text{moment of inertia}} = \frac{M\,g\,s\,sin.\,\phi}{M\,[k^2+(r-s)^2]} = \frac{g\,s\,sin.\,\phi}{k^2+(r-s)^2}.$$

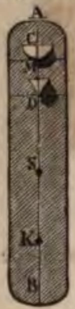
For the simple pendulum it is $= \frac{g\,sin.\,\phi}{r_1}$, if r_1 represent is length; if both are to vibrate isochronously, it is necessary that:

$$\frac{g\,s\,sin.\,\phi}{k^2+(r-s)^2} = \frac{g\,sin.\,\phi}{r_1};\ i.\,e.\ r_1 = \frac{k^2+(r-s)^2}{s}.$$

The time of the vibration of the balance is from this:

$$t = \pi\sqrt{\frac{r_1}{g}} = \pi\sqrt{\frac{k^2+(r-s)^2}{gs}}.$$

Fig. 311.



This theory may also be applied to a pendulum AB, Fig. 311, with a rounded axis of rotation CM, if for r the radius of curvature CM of the axis be substituted. If, instead of the rounded axis, a knife edge D be applied, the time of vibration will then be

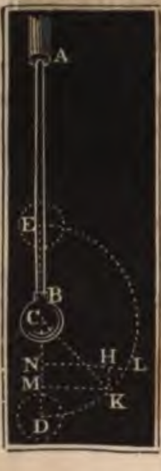
$$t_1 = \pi\sqrt{\frac{k^2+\overline{DS}^2}{g\,DS}} = \pi\sqrt{\frac{k^2+(s-x)^2}{g\,(s-x)}},$$

the distance CD of the edge from the centre of the round axis being represented by x. Both pendulums have equal times of vibration if

$$\frac{k^2+(s-x)^2}{s-x} = \frac{k^2+(r-s)^2}{s},\ \text{or}\ \frac{k^2}{s-x} - x = \frac{k^2+r^2}{s} - 2\,r.$$

If we write $\frac{k^2}{s-x} = \frac{k^2}{s} + \frac{k^2\,x}{s^2}$ approximately, and neglect r^2, we shall then obtain $x = \frac{2\,r\,s^2}{s^2-k^2}$.

Fig. 312.



Remark 1. In the Second Part, under the article "Regulator," the conical pendulum will be mentioned.

Remark 2. *Elastic Pendulum.*—Bodies are likewise very often set into vibratory motion by elasticity. A string, or fine wire, AB, Fig. 312, is stretched by a weight $G = Mg$. If this weight is carried from the point of repose C to D, the string is thereby stretched $CD = r$; and if the weight be afterwards left to itself, it will, by virtue of the elasticity of the string, be raised again to C; it will arrive there with a certain velocity, and ascend by its *vis viva* to E, from whence it will again fall to D and C. In this manner the weight will oscillate a certain time in the space $DE = 2\ CD = 2\,r$ to and fro, and the question now is, as to its duration of oscillation. From the length $AB = l$, transverse section F and modulus of elasticity E of the string, it follows, § 183, that the force to extend it a length $CM = x$ is $P = \frac{x}{l}\,.\,FE$; hence, the mechanical effect required to extend it the length $\frac{r}{n}$ is $= \frac{Pr}{n} = \frac{x}{l}\,.\,\frac{r}{n}\,F\,E$. Let us now put successively $x = \frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, &c., and add the corresponding mechanical effects, we shall then obtain the whole mechanical effect for the extension of the string.

$CD = r : L = \frac{r}{nl} \cdot FE \left(\frac{r}{n} + \frac{2r}{n} + \ldots\right) = \frac{r^2}{n^2 l} FE\,(1 + 2 + \ldots + n)$ $= \frac{r^2}{n^2 l} FE \cdot \frac{n^2}{2} = \frac{r^2}{2l} \cdot FE$; and for the extension $CM = x : L_1 = \frac{x^2}{2l} \cdot FE$. If now, inversely, the string be contracted by DM, therefore the weight D ascend from D to M, *i. e.* $r - x$, it will give the mechanical effect $L - L_1 = \left(\frac{r^2 - x^2}{2l}\right) FE$, and communicate to the weight G a velocity v corresponding to the *vis viva* $v_2\, M = \frac{v^2}{g} \cdot G$; whence we shall have to put $\frac{v^2}{2g} G = \left(\frac{r^2 - x^2}{2l}\right) FE$, and the variable velocity of oscillation will be $v = \sqrt{\frac{FE}{Ml}} \sqrt{r^2 - x^2}$. But now $\sqrt{r^4 - x^2}$ may be put equal to the ordinate $MK = y$ of a semicircle described upon DE; hence it follows, more simply, that $v = \sqrt{\frac{FE}{Ml}} \cdot y$, and the instant for describing the particle of space $MN : \tau = \frac{MN}{y} \cdot \sqrt{\frac{Ml}{FE}}$. From the similarity of the triangles KLH and KCM, $\frac{KH}{KL} = \frac{KM}{KC}$, *i. e.* $\frac{MN}{KL} = \frac{y}{r}$, or $\frac{MN}{y} = \frac{KL}{r}$; hence, it follows that $\tau = \frac{KL}{r} \sqrt{\frac{Ml}{FE}}$; and lastly, the whole time of oscillation, or the time of describing the space DE is : $t = \frac{1}{r} \sqrt{\frac{Ml}{FE}}$ times the sum of all the elements of the semicircle $= \frac{1}{r} \sqrt{\frac{Ml}{FE}}$ times the semicircle

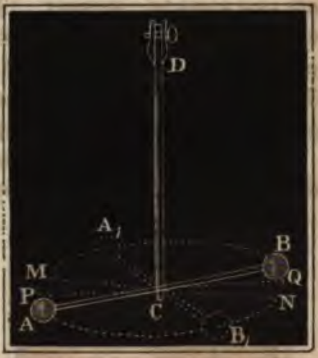
$$\pi r = \frac{\pi r}{r} \sqrt{\frac{Ml}{FE}} = \pi \sqrt{\frac{Ml}{FE}} = \frac{\pi}{\sqrt{g}} \sqrt{\frac{Gl}{FE}}.$$

If, for example, an iron wire, 20 feet long and 0,1 inch thick, be stretched by a weight $G = 100$ lbs. and set into longitudinal vibration, the duration of the oscillations will then be, since from § 186 $E = 26000000$,

$$t = \frac{\pi}{\sqrt{g}} \cdot \sqrt{\frac{100 \cdot 20}{(0{,}1)^2 \cdot \frac{\pi}{4} \cdot 26000000}}$$

$$= 0{,}553 \sqrt{\frac{2}{65 \cdot \pi}} = 0{,}05464 \text{ seconds.}$$

Fig. 313.



Remark 3. We have a torsion pendulum if a string or wire CD, Fig. 313, turns about an arm AB and is brought out of its natural position MN into the position AB, and then left to itself. The rod or arm AB is set into vibration by virtue of the torsion of the string, which extends to an equal distance on both sides of MN, so that $AM = A_1M$. If we put the force of torsion for the distance (1) and for the arc of vibration $(1) = K$, it will then be, for the angle of vibration $MCP = \phi^\circ, = K\phi$, and the corresponding mechanical effect $\frac{K\phi^2}{2}$; on the other hand, for the entire angle of elongation $MCA = \alpha = L_1 = \frac{K\alpha^2}{2}$. If now the inert mass of the entire pendulum $= M$ be reduced to the distance (r), and the angular velocity with which it passes from the position AB into that of $PQ = \omega$, we shall then have $\frac{M\omega^2}{2} = \frac{K(\alpha^2 - \phi^2)}{2}$; and, hence, $\omega = \sqrt{\frac{K}{M}(\alpha^2 - \phi^2)}$; and finally, the time of oscillation $t = \pi \sqrt{\frac{M}{K}}$.

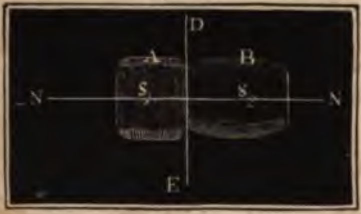
CHAPTER IV.

THE DOCTRINE OF IMPACT.

§ 253. *Impact in General.*—In virtue of the impenetrability of matter, two bodies cannot simultaneously occupy one and the same position. But when two bodies in motion come into contact with one another, so that the one strives to penetrate the space occupied by the other, a reciprocal action takes place, producing a consequent change in the conditions of motion of the two bodies. This reciprocal action is what is called *impact* or *collision.*

The relations of impact depend upon *the law of equality of action and reaction* (§ 62); during impact, the one body presses exactly as forcibly on the other as does this latter in an opposite direction on the former. The straight line, perpendicular to the surfaces in which the two bodies touch, and passing through the point of contact, is the direction of the impact. If the centres of gravity of the two bodies lie within this line, the impact is then called a *centric*, but if without, an *excentric, impact*. The bodies A and B, in Fig. 314, give a centric impact, because their centres of gravity S_1 and S_2 lie in the normal $\overline{NN}$ to the plane of contact DE; of the bodies A and B, Fig. 315, A thrusts centrically, and B excentrically, because S_1 lies within, and S_2 without, the normal line $\overline{NN}$.

Fig. 314.



With respect to the direction of motion, we distinguish between *direct impact* and *oblique impact*. The direction of motion, in the case of direct impact, lies in the line of impact; but in that of oblique impact, there is a deviation between the two directions. If, for example, the bodies A and B, Fig. 316, move in the directions $S_1\,C_1$

Fig. 315.

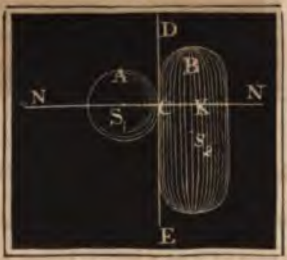
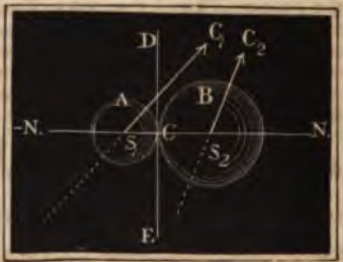
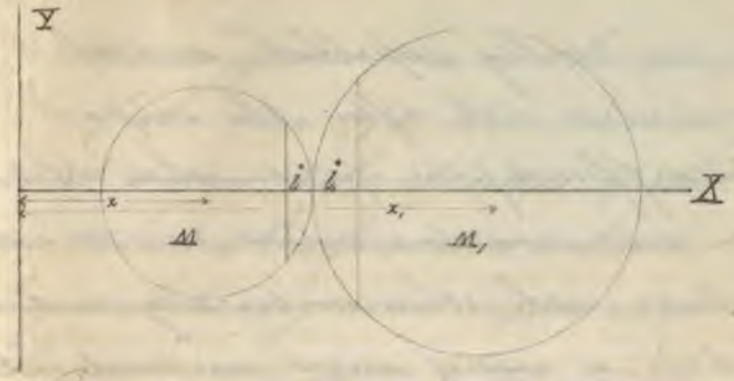


Fig. 316.





From the period when a body first receives the impact of another, until that period of the impact when for an instant they both move with the same velocity it is evident that the surfaces must have been in a state of continually increasing velocity compression; the instant when they acquire therefore their gre common velocity is that of their greatest compression. When this common velocity shall have been attained their mutual pressures will have ceased when they are inelastic bodies and they will move with a common motion. if they be elastic, their surfaces will in the act of recovering their forms perfectly elastic impact substitute for v' & v_1 the already found values

$$v = \frac{(M - M_1)V + 2M_1V_1}{M + M_1} \quad \& \quad v_1 = \frac{(M_1 - M)V_1 + 2MV}{M + M_1}$$

performing this substitution we will have

$$MV^2 + M_1V_1^2 - M\left[\frac{(M - M_1)V + 2M_1V_1}{M + M_1}\right]^2 - M_1\left[\frac{(M_1 - M)V_1 + 2MV}{M + M_1}\right]^2 = MV^2 + M_1V_1^2 - M\left[\frac{(M - M_1)^2V^2 + 4M_1(M - M_1)VV_1 + 4M_1^2V_1^2}{(M + M_1)^2}\right] - M_1\left[\frac{(M_1 - M)^2V_1^2 + 4M(M_1 - M)VV_1 + 4M^2V^2}{(M + M_1)^2}\right] = MV^2 + M_1V_1^2 - M\left[\frac{(M^2 - 2MM_1 + M_1^2)V^2 + 4M_1(M - M_1)VV_1 + 4M_1^2V_1^2}{(M + M_1)^2}\right] - M_1\left[\frac{(M_1^2 - 2MM_1 + M^2)V_1^2 + 4M(M_1 - M)VV_1 + 4M^2V^2}{(M + M_1)^2}\right]$$

$$= \frac{M^3V^2 + 2M^2M_1V^2 + MM_1^2V^2 + M_1M^2V_1^2 + 2MM_1^2V_1^2 + M_1^3V_1^2 - M^3V^2 + 2M^2M_1V^2 - MM_1^2V^2 - 4M_1M^2VV_1 + 4MM_1^2VV_1 - 4MM_1^2V_1^2 - M_1^3V_1^2 + 2MM_1^2V_1^2 - M_1M^2V_1^2 - 4MM_1^2V_1V + 4M^2M_1VV_1 - 4M_1M^2V^2}{(M + M_1)^2} = \frac{0}{(M + M_1)^2}$$

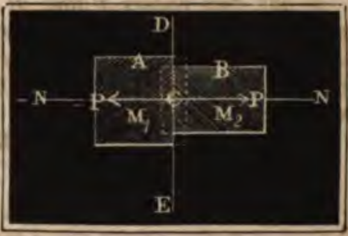
or the loss of vis viva in perfectly elastic impact is zero. This we might have known for the body returns perfectly to its original form, no work is done & hence there can be no loss of living force. We might have known algebraically that our last equation would be true already, for we obtained the values of v & v_1 from equations 1 and then substituted them immediately in that equation and S_2C_2, which deviate from the normal or line of impact NN, an oblique impact will take place; whilst, if the directions coincided with the normal, it would be direct.

We make the further distinction of *the impact of free bodies* and *the impact of bodies entirely or partially supported.*

§ 254. The time occupied in the communication or change of motion by impact is indeed very small, but by no means indefinitely small; it depends, as well as the impact itself, upon the mass, velocity, and elasticity of the impinging bodies. We may regard this time as consisting of two periods. In the first period, the bodies become mutually compressed, and in the second, they again partially or entirely extend themselves. Elasticity is brought into action by this compression, and puts itself into equilibrium with the inertia, and thereby alters the state of motion of the impinging bodies. If the limit of elasticity is not exceeded by the compression, the body at the end of the impact perfectly recovers its former figure, and we then call it a *perfectly elastic* body; but if, at the end of the impact, a disfigurement takes place, we then call it an *imperfectly elastic body;* and lastly, if the body retains its original form, at a maximum pressure, and therefore has no tendency to expansion, we call it an *inelastic body.* At any rate, however, the distinction must only be taken as correct relatively to a certain strength of impact, for it is possible that one and the same body may show itself elastic to a weak, and inelastic to a stronger, impact. Strictly speaking, no body is perfectly elastic or perfectly inelastic; yet we shall, in the sequel, call bodies elastic which nearly recover their form after impact, and those inelastic which undergo a considerable and permanent disfigurement by impact (compare § 181).

In practical mechanics, impinging bodies, such as wood, iron, &c., are generally considered inelastic bodies, because they possess but little elasticity, and by repetition of the blows, lose still further that elasticity. It is a most important rule, moreover, to avoid, as far as possible, in machines and constructions, all jars or impacts, or so to moderate their effects as to convert them into elastic ones; because shocks and abrasions would be thereby produced, and a part of the mechanical effect consumed.

Fig. 317.



§ 255. *Inelastic Impact.* — Let us in the first place develop the laws of the direct central impact of freely moving bodies. Let us suppose the time of impact to be made up of equal parts τ, and let us assume that the pressure during the first instant is P_1, during the second P_2, during the third P_3, and so on. Let, now, the mass of the one body be $A = M$, Fig. 317, we shall then have the corresponding accelerating force:

$$p_1 = \frac{P_1}{M_1},\ p_2 = \frac{P_2}{M_1},\ p_3 = \frac{P_3}{M},\ \text{\&c.};$$

but from § 19, the change of velocity due to the accelerating force p and particle of time τ is $\kappa = p\,\tau$; hence, for the ensuing fall, we shall have the elementary increment or decrement: $\kappa_1 = \frac{P_1\tau}{M_1}$, $\kappa_2 = \frac{P_2\tau}{M_1}$, $\kappa_3 = \frac{P_3\tau}{M_1}$, &c., and the consequent increment or decrement of velocity of the mass M_1 in a given finite time $\kappa_1 + \kappa_2 + \kappa_3 + \ldots = (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_1}$, as also the consequent change of velocity of the mass B of the magnitude M_2: $= (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_2}$.

In the following or impinging body A, the pressure acts opposite to the velocity c_1, consequently here a decrement of velocity takes place; and after a certain time, the residuary velocity of the body is: $v_1 = c_1 - (P_1 + P_2 + \ldots)\frac{\tau}{M_1}$; in the preceding or impinged body B, on the other hand, the pressure acts in the direction of motion; hence there is an increment of velocity c_2, and it is converted into

$$v_2 = c_2 + (P_1 + P_2 + \ldots)\frac{\tau}{M_2}.$$

If we eliminate from both equations $(P_1 + P_2 + \ldots)\tau$, there will then remain the general formula:

I. $M_1(c_1 - v_1) = M_2(v_2 - c_2)$, or $M_1v_1 + M_2v_2 = M_1c_1 + M_2c_2$.

The product of the mass and velocity of a body is called the *momentum of the body*, and it may therefore be enunciated, *that for each instant of the time of impact, the aggregate of the momenta of the two bodies is as great as before impact.*

At the instant of maximum compression, both bodies have an equal velocity; hence, instead of v_1 and v_2, we may put this value into the equation found; then $M_1v + M_2v$ will remain $= M_1c_1 + M_2c_2$, and the velocity of the two bodies at the instant of maximum compression will be:

$$v = \frac{M_1c_1 + M_2c_2}{M_1 + M_2}.$$

If the two bodies A and B are inelastic, they exert therefore no power after compression to re-expand themselves, and the communication of a change of motion will then cease, if both bodies are compressed to a maximum; and hence the two will go on after impact with a common velocity:

$$v = \frac{M_1c_1 + M_2c_2}{M_1 + M_2}.$$

Examples.—1. An inelastic body B of 30 lbs. weight, moves with a 3 feet velocity, and is struck by another inelastic body A having a 7 feet velocity, the two will then proceed, after the blow, with the velocity $v = \frac{50\,.\,7 + 30\,.\,3}{50 + 30} = \frac{350 + 90}{80} = \frac{44}{8} = \frac{11}{2} = 5\tfrac{1}{2}$ feet.—2. To cause a body of 120 lbs.

weight to pass from a velocity $c_2 = 1\frac{1}{2}$ feet into a 2 feet velocity v, it is struck by a body of 50 lbs. weight, what velocity will the body acquire? Here

$$c_1 = v + \frac{(v - c_2)\, M_2}{M_1} = 2 + \frac{(2 - 1{,}5)\, .\, 120}{50} = 2 + \frac{6}{5} = 3{,}2 \text{ feet.}$$

§ 256. *Elastic Impact.*—If the impinging bodies are perfectly elastic, they will then expand themselves after compression in the first period, gradually again in the second period of the time of impact, and when they have resumed the former shape, they will proceed in their motions with different velocities. But, since the mechanical effect which is expended on the compression of an elastic body is equal to the effect which the same gives out again by its expansion, no loss in *vis viva* will take place from the collision of elastic bodies, and hence the second following equation will be also true for this case:

$$\text{II. } M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2, \text{ or}$$
$$M_1 (c_1^2 - v_1^2) = M_2 (v_2^2 - c_2^2).$$

From the equations I. and II., the velocities v_1 and v_2 of the bodies after impact may be found. First, it follows by division that

$\frac{c_1^2 - v_1^2}{c_1 - v_1} = \frac{v_2^2 - c_2^2}{v_2 - c_2}$, i. e. $c_1 + v_1 = v_2 + c_2$, or $v_2 - v_1 = c_1 - c_2$; if now we put the resulting value of $v_2 = c_1 + v_1 - c_2$, into the equation I:

$$M_1 v_1 + M_2 v_1 + M_2 (c_1 - c_2) = M_1 c_1 + M_2 c_2, \text{ or,}$$

$(M_1 + M_2)\, v_1 = (M_1 + M_2)\, c_1 - 2\, M_2 (c_1 - c_2)$, from which we have the value:

$$v_1 = c_1 - \frac{2\, M_2}{M_1 + M_2} (c_1 - c_2), \text{ and}$$
$$v_2 = c_1 - c_2 + c_1 - \frac{2\, M_2 (c_1 - c_2)}{M_1 + M_2} = c_2 - \frac{2\, M_1 (c_1 - c_2)}{M_1 + M_2}.$$

Whilst for inelastic bodies the loss in velocity of the one body is

$$c_1 - v = c_1 - \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2} = \frac{M_2 (c_1 - c_2)}{M_1 + M_2},$$

for elastic bodies it comes out twice as great, namely:

$$c_1 - v_1 = \frac{2 M_2 (c_1 - c_2)}{M_1 + M_2},$$

and while the gain in velocity of the other body for inelastic bodies is:

$$v - c_2 = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2} - c_2 = \frac{M_1 (c_1 - c_2)}{M_1 + M_2},$$

for elastic bodies it is

$$v_2 - c_2 = \frac{2 M_1 (c_1 - c_2)}{M_1 + M_2}, \text{ likewise twice as great.}$$

Example. Two perfectly elastic spheres, the one of 10 lbs. the other of 16 lbs. weight, impinge with the velocities 12 and 6 feet against each other, what will be their velocities after impact? Here $M_1 = 10$ and $c_1 = 12$ feet, but $M_2 = 16$ and $c_2 = -6$ feet, hence the loss of velocity of the first body will be

$$c_1 - v_1 = \frac{2\, .\, 16\, (12 + 6)}{10 + 16} = \frac{2\, .\, 16\, .\, 18}{26} = 22{,}154 \text{ feet,}$$

and the gain in velocity of the other: $v_2 - c_2 = \frac{2\, .\, 10\, .\, 18}{26} = 13{,}846$ feet. From this the first body after impact will recoil with the velocity $v_1 = 12 - 22{,}154 = -10{,}154$ feet; and the other with that of $-6 + 13{,}846 = 7{,}846$ feet. Moreover, the measure of *vis viva* of the two bodies after impact $= M_1 v_1^2 + M_2 v_2^2 = 10 \,.\, 10{,}154^2 + 16 \,.\, 7{,}846^2 = 1031 + 985 = 2016$, as likewise of that before impact, namely: $M_1 c_1^2 + M_2 c_2^2 = 10 \,.\, 12^2 + 16 \,.\, 6^2 = 1440 + 576 = 2016$. Were these bodies inelastic, the first would only lose in velocity $\frac{c_1 - v_1}{2} = 11{,}077$ feet, and the other gain $\frac{v_2 - c_2}{2} = 6{,}923$ feet; the first would still retain, after impact, the velocity $12 - 11{,}077 = 0{,}923$ feet, and the second acquire the velocity $-6 + 6{,}923 = 0{,}923$, and the loss of mechanical effect would be $(2016 - (10 + 16)\, 0{,}923^2) \div 2g = (2016 - 22{,}2) \,.\, 0{,}0155 = 29{,}35$ ft. lbs.

§ 257. *Particular Cases.*—The formula developed in the foregoing paragraphs, for the final velocities of impact, hold good also in the case where the one body is at rest, or where both bodies move opposed to each other, or where the mass of the one is indefinitely great compared with the other. If the mass M_2 be at rest, we then have $c_2 = 0$, hence for the inelastic body

$$(1)\quad v = \frac{M_1 c_1}{M_1 + M_2}, \text{ and for the elastic:}$$

$$(2)\quad v_1 = c_1 - \frac{2\, M_2 c_1}{M_1 + M_2} = \frac{M_1 - M_2}{M_1 + M_2}\, c_1, \text{ and}$$

$$(3)\quad v_2 = 0 + \frac{2\, M_1 c_1}{M_1 + M_2} = \frac{2\, M_1}{M_1 + M_2}\, c_1.$$

If the bodies meet, c_2 is therefore negative, and for an inelastic body it will follow that (4) $v = \frac{M_1\, c_1 - M_2\, c_2}{M_1 + M_2}$, and for an elastic one:

$$(5)\quad v_1 = c_1 - \frac{2\, M_2\, (c_1 + c_2)}{M_1 + M_2}, \text{ and } v_2 = -\, c_2 + \frac{2\, M_1\, (c_1 + c_2)}{M_1 + M_2}. \quad (6)$$

If in this case the momenta be equal, $M_1 c_1 = M_2 c_2$, for the inelastic body then $v = 0$, *i. e.* the bodies bring each other to rest, but for elastic bodies:

$$v_1 = c_1 - \frac{2\, (M_2 c_1 + M_1 c_1)}{M_1 + M_2} = c_1 - 2\, c_1 = -\, c_1, \text{ and} \quad (7)$$

$$v_2 = -\, c_2 + \frac{2\, (M_2 c_2 - M_1 c_2)}{M_1 + M_2} = -\, c_2 + 2\, c_2 = +\, c_2; \quad (8)$$

then the bodies rebound after impact with opposite velocities. If on the other hand the masses are equal, we have then for inelastic bodies $v = \frac{c_1 - c_2}{2}$, and for elastic $v_1 = -\, c$, and $v_2 = c_1$, *i. e.* the masses rebound with their velocities interchanged.

If the masses again meet in the same direction, and if the preceding mass M_2 be indefinitely great, we shall then have for inelastic bodies $v = \frac{M_2 c_2}{M_2} = c_2$, and for elastic $v_1 = c_1 - 2\,(c_1 - c_2) = 2\, c_2 - c_1$, (9) $v_2 = c_2 + 0 = c_2$; the velocity therefore of the indefinitely great mass will not be altered by the collision of the finite mass. If, now, the indefinitely great mass be at rest, therefore, $c_2 = 0$, we shall then have for inelastic bodies $v = 0$, and for elastic $v_1 = -\, c_1$, $v_2 = 0$; the indefinitely great mass will then remain at rest, but in the first case, the impinging body will entirely lose its velocity, and in the second case this will be converted into an opposite one.

Particular Cases (§ 57)

Take general equations for velocity

$v_c = \frac{mV + m_1V_1}{m + m_1}$ [& [illegible] of page 8 of note

for Inelastic Impact ------

for Elastic impact [page 9 of note]

$v = \frac{(m - m_1)V + 2m_1V_1}{m + m_1}$, $v_1 = \frac{(m_1 - m)V_1 + 2mV}{m + m_1}$

I. If impinged body m be at rest $V = 0$ and

for inelastic impact $v_c = \frac{m_1V_1}{m + m_1}$ -(1)

for elastic " $v_1 = \frac{(m_1 - m)V_1}{m + m_1}$ -(2)

" " " $v = \frac{2m_1V_1}{m + m_1}$ ---(3)

II. If the bodies meet V is negative and hence

for inelastic impact $v_c = \frac{m_1V_1 - mV}{m + m_1}$ --(4)

" elastic impact $v_1 = \frac{(m_1 - m)V_1 - 2mV}{m + m_1}$ (5)

" " " $v = \frac{(m_1 - m)V + 2m_1V_1}{m + m_1}$ -(6)

In this case the momenta are equal i.e. $m_1V_1 = mV$ then $v_c = 0$, bodies bring each other to rest. and for elastic bodies

$v_1 = \frac{m_1V_1 - mV_1 - 2m_1V_1}{m + m_1} = \frac{-(m + m_1)V_1}{m + m_1} = -V_1$ -(7)

$v = \frac{m_1V - mV + 2mV}{m + m_1} = \frac{(m + m_1)V}{m + m_1} = V$ ---8

hence the bodies rebound after impact with opposite velocities; if the masses are equal, then $v_c = \frac{V_1 - V}{2}$ .. $v_1 = -V$.

$v = V_1$ i.e. masses rebound with interchanged velocities ——

III. Bodies moving in the same direction and M indefinitely great compared with

$v_c = \frac{MV + M_1V_1}{M} = V + 0.V_1 = V$ = initial velocity of M.

$v_1 = \frac{-MV_1 + 2MV}{M} = 2V - V_1$ (9).

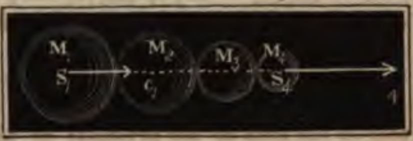
$v = \frac{MV + 2M_1V_1}{M} = V + 0.V_1 = V$. hence velocity of M is not changed by impact with an infinitely smaller mass M_1 ——

If under same conditions M should be at rest, $V = 0$ hence $v_c = 0$ .. $v_1 = -V_1$ .. $v = 0$

i.e. M will remain at rest and for in-elastic impact M_1 will lose its velocity. for elastic it will of course rebound with its initial velocity. Chas. J.

Examples.—1. With what velocity must a body of 8 lbs. impinge against another at rest of 25 lbs., in order that the last may have a velocity of 2 feet? Were the bodies inelastic, we should then have to put: $v = \frac{M_1 c_1}{M_1 + M_2}$, i. e. $2 = \frac{8 \cdot c_1}{8 + 25}$, hence $c_1 = \frac{33}{4} = 8\frac{1}{4}$ feet, the required velocity; but were they elastic, we should have $v_2 = \frac{2 M_1 c_1}{M_1 + M_2}$; hence, $c_1 = \frac{33}{8} = 4\frac{1}{8}$ feet.—2. If a sphere M_1, Fig. 318, strike against a mass at rest $M_2 = nM_1$ with the velocity c_1, the second, a third mass $M_3 = n\, M_2 = n^2 M_1$ with the velocity communicated by the impact, this again another mass $M_4 = nM_3 = n^3 M_1$, &c., we shall have from the perfect elasticity of these masses, the velocity

Fig. 318.



$v_2 = \frac{2M_1}{M_1 + nM_1} c_1 = \frac{2}{1+n} \cdot c_1, v_3 = \frac{2\,M_2}{M_2 + n\,M_2}\, v_2 = \frac{2}{1+n} \cdot v_2 = \left(\frac{2}{1+n}\right)^2 c_1,$
$v_4 = \left(\frac{2}{1+n}\right)^3 c_1$, &c. If, for example, the weight of each mass be half as great as that of the succeeding one, and we have therefore the exponents of the geometrical series formed by the masses: $n = \frac{1}{2}$, it will follow that

$$v_2 = \frac{4}{3} c_1, v_3 = \left(\frac{4}{3}\right)^2 c_1, v_4 = \left(\frac{4}{3}\right)^3 c_1 \ldots, v_{10} = \left(\frac{4}{3}\right)^9 c_1 = 13{,}32 \,.\, c_1.$$

§ 258. *Loss of Mechanical Effect.*—In the collision of inelastic masses, a loss of *vis viva* constantly ensues, whence the masses after impact have not the power of producing so much mechanical effect, as before impact. Before impact the masses M_1 and M_2 proceeding with the velocities c_1 and c_2, contain the *vis viva*, $M_1 c_1^2 + M_2 c_2^2$, but after impact the masses proceeding with the velocity $v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}$ have the *vis viva* $M_1 v^2 + M_2 v^2$; hence the subtraction of these forces will give the loss in *vis viva* by the collision: $K = M_1 (c_1^2 - v^2) + M_2 (c_2^2 - v^2) = M_1 (c_1 + v)(c_1 - v) - M_2 (c_2 + v)(v - c_2)$, but $M_1) c_1 - v) = M_2 (v - c_2) = \frac{M_1 M_2 (c_1 - c_2)}{M_1 + M_2}$, hence

$$K = (c_1 + v - c_2 - v) \cdot \frac{M_1 M_2 (c_1 - c_2)}{M_1 + M_2} = \frac{(c_1 - c_2)^2 M_1 M_2}{M_1 + M_2} = \frac{(c_1 - c_2)^2}{\frac{1}{M_1} + \frac{1}{M_2}}.$$

If the weight of the masses are G_1 and G_2, M_1 is, therefore, $= \frac{G_1}{g}$, and $M_2 = \frac{G_2}{g}$, we shall from this have the loss in mechanical effect: $L = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1\, G_2}{G_1 + G_2}$. We call $\frac{G_1\, G_2}{G_1 + G_2}$ *the harmonic mean* of G_1 and G_2, and we may from this assert that *the loss in mechanical effect* which is produced by the impact of two inelastic masses, and which is expended upon the disfigurement of these, *is equivalent to the product of the harmonic mean of both masses, and of the height of fall which is due to the difference of the velocities of these masses.*

If one of the masses, for example M_2, be at rest, we shall have the loss in mechanical effect $L = \frac{c_1^2}{2g} \cdot \frac{G_1\, G_2}{G_1 + G_2}$, and if the mass moved M_1 be very great in comparison with the one at rest, G_1 will vanish as compared with G_2, and there will remain $L = \frac{c_1^2}{2g} \cdot G_2$.

For the rest we may put

$$K = M_1(c_1^2 - v^2) + M_2(c_2^2 - v^2) = M_1(c_1^2 - 2\,c_1v + v^2 + 2\,c_1v - 2\,v^2) + M_2(c_2^2 - 2\,c_2v + v^2 + 2\,c_2v - 2v_2^2)$$

$$= M_1(c_1 - v)^2 + 2\,M_1v\,(c_1 - v) + M_2(c_2 - v)^2 + 2\,M_2v\,(c_2 - v)$$

$$= M_1(c_1 - v)^2 + M_2(c_2 - v)^2, \text{ because } M_1(c_1 - v) = M_2(v - c_2).$$

From this, therefore, *the vis viva lost by inelastic impacts is equivalent to the sum of the products of the masses and the squares of their loss or gain in velocity.*

Examples.—1. If in a machine, 16 blows per minute take place between two inelastic bodies $M_1 = \frac{1000}{g}$ lbs. and $M_2 = \frac{1200}{g}$ lbs., with the velocities $c_1 = 5$ feet, and $c_2 = 2$ feet, then the loss in mechanical effect from these blows will be:

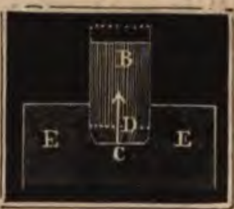
$$L = \frac{16}{60} \cdot \frac{(5-2)^2}{2g} \cdot \frac{1000\,.\,1200}{2200} = \frac{4}{15} . 9\,.\,0{,}0155\,\frac{6000}{11} = 0{,}558\,.\,\frac{400}{11} = 20{,}29$$

ft. lbs. per second.—2. If two trains upon a railroad of 120000 lbs. and 160000 lbs. weight, come into collision with the velocities $c_1 = 20$, and $c_2 = 15$ feet, there will ensue a loss of mechanical effect expended upon the destruction of the locomotives and carriages, which, in the case of perfect inelasticity of the impinging parts, will amount to

$$= \frac{(20 + 15)^2}{2g} \cdot \frac{120000\,.\,160000}{280000} = 35^2 . 0{,}0155\,\frac{1920000}{28} = 1302000 \text{ ft. lbs.}$$

§ 259. *Pile Driving.*—The effects of impact are very often applied to ram or drive one body B, Fig. 319, into another E, a soft mass, for instance. If the resistance which the latter mass opposes to the penetration of the former be constant and $= P$, and the depth of penetration by one blow $= s$, a mechanical effect Ps will be then expended. If, on the other hand, this resistance at the commencement be $= 0$, and if it increase simultaneously with the depth of penetration, so that at the end, after the body has penetrated the second mass a depth s, it be $= P$, the mechanical effect expended will be then only $\frac{(0+P)}{2}\, s = \frac{1}{2}\, Ps$. If, lastly, the initial resistance be $= P_1$, and increase simultaneously with the space, so that, after describing a space s, it becomes P_2, we shall then have to put the mechanical effect $= \frac{(P_1 + P_2)}{2}$.

Fig. 319.



If the body B, whose mass may be M, begins with the velocity v to penetrate a mass, and if this velocity of penetration ~~increase~~, it will, in virtue of its *vis viva*, have produced the mechanical effect $\frac{Mv^2}{2} = \frac{v^2}{2g}\, G$, if $G = Mg$ represent its weight.

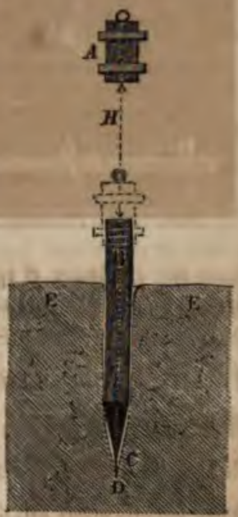
is gradually lost

When the resistance is constant, we must put: $Ps = \frac{v^2}{2g} G$; on the other hand, when the resistance beginning from nought gradually increases: $Ps = \frac{v^2}{2g} . 2\,G$. ; and when it increases gradually from P_1 to P_2: $(P_1 + P_2)\,s = \frac{v^2}{2g} . 2\,G$.

The initial velocity v is generated if a third mass A, whose magnitude may be $= M_1$ and weight $= G_1$, be allowed to impinge upon the second mass B, with a certain velocity c. If, now, these masses are inelastic, we then have the velocity with which the two proceed after impact, and begin to penetrate the mass E:

$$v = \frac{M_1 c}{M + M_1} = \frac{G_1 c}{G + G_1}.$$

Fig. 320.



In the driving of a *pile* or *post*, Fig. 320, B consists of a pile shod with iron, and A of a heavy body falling from a certain height, which is called a *ram*, or block of iron. If the height of fall $= H$, we shall have:

$$\frac{v^2}{2g} = \left(\frac{G_1}{G + G_1}\right)^2 \cdot \frac{c^2}{2g} = \left(\frac{G_1}{G + G}\right)^2 . H,$$

hence the mechanical effect of the pile due to the velocity v

$$= \left(\frac{G_1}{G + G_1}\right)^2 GH,$$

and that of the pile and ram together

$$= \left(\frac{G_1}{G + G_1}\right)^2 (G + G_1)\,H = \frac{G_1^2 H}{G + G_1}.$$

But if the resistance of the bed of earth be constant, the mechanical effect expended in the penetration of the pile will be $= Ps$, hence we shall have to put:

$$1.\ Ps = \left(\frac{G_1}{G + G_1}\right)^2 GH;\ \text{or, } 2.\ Ps = \frac{G_1^2 H}{G + G_1},$$

the first if the ram does not, during penetration, remain upon the pile, and the second if both go down together.

The weight $G + G_1$ produces, in penetrating, the mechanical effect $(G + G_1)\,s$, we may then more correctly put: $(P - G - G_1)\,s = \frac{G_1^2 H}{G + G_1}$; but $G + G_1$ is small compared with P, and may generally be neglected.

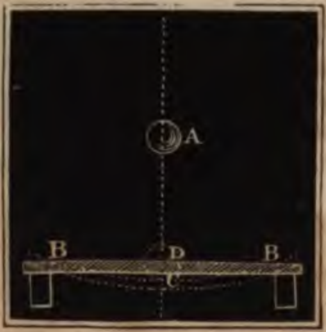
Hence, were the impact perfectly elastic, we should have to put: $Ps = \left(\frac{2\,G_1}{G + G_1}\right)^2 . GH$. Were G small, compared with G_1, as for instance, in the driving of a nail, we should have either $Ps = G_1 H$, or $Ps = 4\,GH$.

Example. A pile of 400 lbs. weight is driven by the last round of 20 blows of a 700 lbs. ram, falling from a height of 5 feet, 6 inches deeper, what resistance will the ground offer, or what load will the pile sustain without penetrating deeper? Here $G = 400$, $G_1 = 700$ lbs., $H = 5$, and $s = \frac{0,5}{20} = 0,025$ feet, whereby it is supposed that the pile penetrates equally far for each blow. From the first formula $P = \left(\frac{700}{700 + 400}\right)^2 \frac{400 \,.\, 5}{0,025} = \left(\frac{7}{11}\right)^2 . \, 80000 = 32400$ lbs.; and from the second: $P = \frac{700^2 \,.\, 5}{1100 \,.\, 0,025} = \frac{4900}{11} \,.\, 200 = 89100$ lbs.

For duration, with security, such piles are only loaded from $\frac{1}{100}$ to $\frac{1}{10}$ of their strength.

§ 260. The formulæ found above are applicable to the breaking of bodies by descending weights or balls. Let *BB*, Fig. 321, be a prismatic body of the mass *M*, or weight $G = Mg$, supported at its extremities, which is bent a depth $CD = s$, by a weight G_1 falling from a height $AD = H$ upon its middle, and in this manner broken; the conditions under which this is possible are to be determined.

Fig. 321.



From § 190, the deflexion, or the height of the arc, is given $s = \frac{P l^3}{48 \, WE}$, from the pressure *P* in the middle of the beam, and from its length $BB = l$, and if further its moment of flexure *WE* is known; therefore, inversely, the pressure corresponding to a certain deflexion *s* is: $P = \frac{48 \, WEs}{l^3}$. This pressure however is not constant, but increases simultaneously with *s*, hence the mechanical effect expended in the deflexion by a depth *s*, not $= Ps$, but only $\frac{1}{2} Ps$, *i. e.* $\frac{1}{2} \,.\, \frac{P^2 \, l^3}{48 \, WE} = \frac{P^2 \, l^3}{96 \, WE}$. This mechanical effect may now be equated to that which the falling body communicates to the beam. Since the beam rests on its extremities, we must (from § 219) consider only the third part of its mass as inert, and hence put for this mechanical effect:

$$\left(\frac{G_1}{\frac{1}{3} G + G_1}\right)^2 . \, \frac{1}{3} GH, \text{ or } \frac{G_1^{\,2} H}{\frac{1}{3} G + G_1}.$$

The first, if the weight G_1 ~~flies back~~ [illegible] after its descent, and the second if it remains on the beam during the fracture.

If we suppose a rectangular beam of the depth *h*, and breadth *b*, we shall then have to put: $W = \frac{1}{12} bh^3$, and $P = \frac{4}{6} \frac{bh^2}{l} K$, hence $\frac{P^2}{W} = \frac{16}{3} \,.\, \frac{bhK^2}{l^2}$; accordingly, the mechanical effect for the rupture of the beam will be $= \frac{1}{96} \cdot \frac{P^2 l^3}{WE} = \frac{bhlK^2}{18\,E}$, and we may now put:

$$1.\ GH\left(\frac{G_1}{\frac{1}{3}\,G + G_1}\right)^2 = \frac{bhlK^2}{6\,E}, \text{ or } 2.\ \frac{G_1{}^2 H}{\frac{1}{3}\,G + G_1} = \frac{bhlK^2}{18\,E}.$$

Example.—From what height must an iron weight G_1 of 100 lbs. be allowed to fall to break a cast-iron plate, 36 inches long, 12 inches broad, and 3 inches thick, in its middle? The modulus of elasticity of cast-iron $E = 17000000$, and the modulus of strength $K = 19000$, hence it follows that:

$$\frac{bhlK^2}{6\,E} = \frac{12\,.\,3\,.\,36\,.\,19000^2}{6\,.\,17000000} = \frac{216\,.\,19^2}{17} = \frac{216\,.\,361}{17} = 4587.$$

If now a cubic inch of cast iron weighs 0.275* lbs., the weight of a plate G will then be $= 12\,.\,3\,.\,36\,.\,0{,}275 = 1296\,.\,0{,}275 = 356{,}4$ lbs.; hence:

$$G\left(\frac{G_1}{\frac{1}{3}\,G + G_1}\right)^2 = 356{,}4\,.\left(\frac{100}{218{,}8}\right)^2 = 74{,}44; \text{ on the other hand,}$$

$$\frac{G_1{}^2}{\frac{1}{3}\,G + G_1} = \frac{10000}{218{,}8} = 45{,}70.$$ Hence the height of fall required is:

$$H = \frac{4587}{74{,}44} = 61{,}6 \text{ inches, or } H = \frac{4587}{3\,.\,45{,}7} = 33{,}5 \text{ inches.}$$

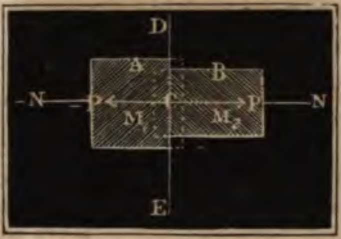
§ 261. *Hardness.*—When the modulus of elasticity of the impinging bodies is known, we may then find the force of compression and its amount. Let the transverse sections of the bodies A and B, Fig. 322, be F_1 and F_2, the lengths l_1 and l_2, and the moduli of elasticity E_1 and E_2. If both impinge against each other with a force P, the compressions effected will be from § 183:

$$\lambda_1 = \frac{Pl_1}{F_1 E_1}, \text{ and } \lambda_2 = \frac{Pl_2}{F_2 E_2},$$

and their ratio:

$$\frac{\lambda_1}{\lambda_2} = \frac{F_2 E_2}{F_1 E_1} \cdot \frac{l_1}{l_2}.$$

Fig. 322.



If for simplicity we represent $\frac{FE}{l}$ by H, we obtain $\lambda_1 = \frac{P}{H_1}$, and $\lambda_2 = \frac{P}{H_2}$, as well as $\frac{\lambda_1}{\lambda_2} = \frac{H_2}{H_1}$. If, after the example of Whewell,† we call the quantity $\frac{FE}{l}$ the hardness of a body, it follows that *the depth of compression is inversely proportional to the hardness.*

If a mass $M = \frac{G}{g}$ impinges with the velocity c upon an immovable or indefinitely great mass, it then expends its whole *vis viva* upon the compression, hence $\frac{1}{2}\,Ps = \frac{Mc^2}{2} = \frac{c^2}{2g}\,G$. But now the space s is equal to the aggregate of the compressions λ_1 and λ_2, and λ_1

---

* A nearer statement of its weight is, 0,2604 lbs. avoirdupois.—Am. Ed.

† The Mechanics of Engineering, § 207.

$= \frac{P}{H_1}$, and $\lambda_2 = \frac{P}{H_2}$; hence it follows that

$$s = \lambda_1 + \lambda_2 = P\left(\frac{1}{H_1} + \frac{1}{H_2}\right) = \frac{H_1 + H_2}{H_1 H_2} \cdot P,$$

as, inversely, $P = \frac{H_1 H_2}{H_1 + H_2}\, s$, and the equation of condition $\frac{1}{2} \cdot \frac{H_1 H_2}{H_1 + H_2} \cdot s^2 = \frac{c^2}{2g}\, G$, therefore,

$$s = c\sqrt{\frac{H_1 + H_2}{H_1 H_2} \cdot \frac{G}{g}}.$$

from which P, λ_1 and λ_2 may be calculated.

Example. If a wrought-iron hammer, of 4 square inches base and 6 inches high, strikes with a velocity of 50 feet upon a plate of lead, of 2 square inches base and 1 inch thick, the following relations present themselves. The modulus of elasticity of wrought iron is $E_1 = 29000000$, and that of lead $E_2 = 700000$; hence, the hardness of these bodies is: $H_1 = \frac{F_1 E_1}{l_1} = \frac{4 \,.\, 29000000}{6} = 19333333$, and $H_2 = \frac{F_2 E_2}{l_2} = \frac{2 \,.\, 700000}{1} = 1400000$. If we put these values into the formula $s = c\sqrt{\frac{H_1 + H_2}{H_1 H_2} \cdot \frac{G}{g}}$, and substitute for the weight of the hammer $= 4 \,.\, 6 \,.\, 0{,}29 = 7$ lbs.; therefore, $\frac{G}{g} = 7 \,.\, 0{,}031 = 0{,}217$, we shall then obtain the space of the hammer in the compression:

$s = 50\sqrt{\frac{20733333 \,.\, 0{,}224}{19333333 \,.\, 1400000}} = 60\sqrt{\frac{0{,}46443}{2706666}} = 0{,}0207$ inches $= 0{,}249$ lines. From this the force of impact or pressure follows:

$P = \frac{H_1 H_2}{H_1 + H_2} \cdot s = \frac{19333333 \,.\, 1400000}{20733333} \,.\, 0{,}0207 = 27037$ lbs.; further, the compression of the hammer is: $\lambda_1 = \frac{P}{H_1} = \frac{27037}{19333333} = 0{,}0014$ inches $= 0{,}016$ lines, and that of the leaden plate: $\lambda_2 = \frac{P}{H_2} = \frac{27037}{1400000} = 0{,}0193$ inches $= 0{,}233$ lines.

§ 262. *Elastic and Inelastic Impact.*—If two masses M_1 and M_2 move with the velocities c_1 and c_2, the common velocity of the two at the moment of maximum compression will then be from § 256 $v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}$, and the mechanical effect expended on the compression from § 259:

$$L = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2}.$$

This mechanical effect may be also put:

$$= \tfrac{1}{2} Ps = \tfrac{1}{2} P(\lambda_1 + \lambda_2) = \tfrac{1}{2} \cdot \frac{H_1 H_2}{H_1 + H_2}\, s^2,$$

consequently the sum of the compressions of both masses will be:

$$s = (c_1 - c_2)\sqrt{\frac{G_1 G_2}{g(G_1 + G_2)} \cdot \frac{H_1 + H_2}{H_1 H_2}},$$

from which the compressing force P, and the compressions of the separate masses λ_1 and λ_2, may be found.

If the masses are inelastic, these compressions will remain after impact, but if only one of the two bodies be inelastic, the other will again recover its form in the second period, and produce a mechanical effect which will generate a new change of velocity. If, for example, $M_1 = \frac{G_1}{g}$ be elastic, the mechanical effect in this second period of impact $\frac{1}{2} Ph_1 = \frac{1}{2} \cdot \frac{P^2}{H_1} = \frac{1}{2H_1}\left(\frac{H_1 H_2}{H_1 + H_2}\right)^2 s^2 = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2} \cdot \frac{H_2}{H_1 + H_2}$ will be given out; hence we shall have in this

to be perfectly elastic and the plate of lead [illegible], we [illegible] velocity of the 7 lbs. hammer, descending with a 50 feet velocity, since c_2 [illegible] $M_2 = \infty$.

$$c_1 - v_1 = c_1 \left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right) = 50 \left(1 + \sqrt{\frac{1400000}{20733333}}\right)$$

$= \frac{P}{H_1}$, and $\lambda_2 = \frac{P}{H_2}$; hence it follows that

$$s = \lambda_1 + \lambda_2 = P\left(\frac{1}{H_1} + \frac{1}{H_2}\right) = \frac{H_1 + H_2}{H_1 H_2} \cdot P,$$

as, inversely, $P = \frac{H_1 H_2}{H_1 + H_2}\, s$, and the equation of condition $\frac{1}{2}$
$\frac{H_1 H_2}{H_1 + H_2} \cdot s^2 = \frac{c^2}{2g}\, G$, therefore,

Equation (1) translated means
the vis viva after impact = the Vis
the period of greatest compression + the
Viva given out during the expansion.
Equation (2) gives us the relation that
the Vis Viva after impact = the Vis
-fore impact – the Vis Viva lost by th
-pression + the Vis Viva given out by th
-pansion.
Equation (3) which is but a reduction
the vis viva after impact – the Vis
-fore impact – the total Vis Viva lost

from which the compressing force P, and the compressions of the separate masses λ_1 and λ_2, may be found.

If the masses are inelastic, these compressions will remain after impact, but if only one of the two bodies be inelastic, the other will again recover its form in the second period, and produce a mechanical effect which will generate a new change of velocity. If, for example, $M_1 = \frac{G_1}{g}$ be elastic, the mechanical effect in this second period of impact $\frac{1}{2}\, P\lambda_1 = \frac{1}{2} \cdot \frac{P^2}{H_1} = \frac{1}{2 H_1} \left(\frac{H_1 H_2}{H_1 + H_2}\right)^2 s^2 = \frac{(c_1 - c_2)^2}{2 g} \cdot \frac{G_1 G_2}{G_1 + G_2} \cdot \frac{H_2}{H_1 + H_2}$ will be given out; hence we shall have in this case for the velocities v_1 and v_2 after impact the formula:

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2, \text{ and}$$

$$M_1 v_1^2 + M_2 v_2^2 = M_1 v^2 + M_2 v^2 + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2} \quad (1)$$

$$= M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2}, \quad (2)$$

$$\text{i. e. } M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2}. \quad (3)$$

If the loss of velocity $c_1 - v_1$ be put $= x$, we shall then have the gain of velocity $v_2 - c_2 = \frac{M_1 x}{M_2}$, and the last equation will assume the form:

$$x\,(2\,c_1 - x) - x\left(2\,c_2 + \frac{M_1 x}{M_2}\right) - (c_1 - c_2)^2 \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2} = 0, \text{ or,}$$

$$\frac{M_1 + M_2}{M_2}\, x^2 - 2\,(c_1 - c_2)\, x + (c_1 - c_2)^2 \cdot \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2} = 0.$$

If this be multiplied by $\frac{M_2}{M_1 + M_2}$ and $\frac{H_1}{H_1 + H_2}$ be put $= 1 - \frac{H_2}{H_1 + H_2}$, we shall then obtain the quadratic equation:

$$x^2 - 2\,(c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\, x + (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2$$

$$= (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \cdot \frac{H_2}{H_1 + H_2}, \text{ or,}$$

$$\left(x - (c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\right)^2 = (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \cdot \frac{H_2}{H_1 + H_2},$$

whose solution will give x, or the loss in velocity of the first body:

$$c_1 - v_1 = (c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right),$$

and the gain in velocity of the second:

$$(v_2 - c_2 = c_1 - c_2)\,\frac{M_1}{M_1 + M_2}\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right).$$

Example. If we assume the iron hammer in the example of the preceding paragraph to be perfectly elastic and the plate of lead inelastic, we shall then obtain the loss in velocity of the 7 lbs. hammer, descending with a 50 feet velocity, since $c_2 = 0$ and $M_2 = \infty$.

$$c_1 - v_1 = c_1\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right) = 50\left(1 + \sqrt{\frac{1400000}{20733333}}\right)$$

$$= 50\,(1 + 0{,}26) = 63 \text{ feet};$$

hence, the velocity of the hammer after the blow is : $v_1 = c_1 - 63 = 50 - 63 = -13$ feet. The velocity of the supported plate of lead $= 0$.

§ 263. *Imperfectly Elastic Impact.*—If the bodies impinging against each other are imperfectly elastic, they only partially recover their figure in the second period of the time of impact, the *vis viva* expended in compression during the first period, will not, therefore, again be completely given out. If, again, λ_1 and λ_2 are the depths of penetration, and P the pressure, we shall then have the loss of mechanical effect by the compression $= \frac{1}{2} P\lambda_1$, and $\frac{1}{2} P\lambda_2$, and if during the expansion the μth part of this, or generally during the expansion of the one body the μ_1th, and during that of the second the μ_2th be given back, there will remain the aggregate loss of mechanical effect after impact:

$$L = \tfrac{1}{2} P\,[(1 - \mu_1)\,\lambda_1 + (1 - \mu_2)\,\lambda_2], \text{ or } \lambda_1 = \frac{P}{H_1}, \text{ and } \lambda_2 = \frac{P}{H_2},$$

$$L = \tfrac{1}{2} P^2 \left[\frac{1 - \mu_1}{H_1} + \frac{1 - \mu_2}{H}\right].$$ But from the former paragraph:

$$P = \frac{H_1 H_2 s}{H_1 + H_2}, \text{ and } s = (c_1 - c_2)\sqrt{\frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1 + H_2}{H_1 H_2}},$$

hence the loss of the mechanical effect in question is known:

$$L = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1 H_2}{H_1 + H_2}\left(\frac{1 - \mu_1}{H_1} + \frac{1 - \mu_2}{H_2}\right).$$

Now, in order to find the velocities v_1 and v_2 after impact, we have to combine them with each other and to solve the equations:

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2, \text{ and}$$
$$M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2$$
$$- (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{(1 - \mu_1)\,H_2 + (1 - \mu_2)\,H_1}{H_1 + H_2}.$$

In the same manner as in the former §, the *loss of velocity of the first body is given:*

$$c_1 - v_1 = (c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right),$$

and the *gain of velocity of the body preceding:*

$$v_2 - c_2 = (c_1 - c_2)\,\frac{M_1}{M_1 + M_2}\left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right).$$

These two general formulæ also embrace the laws of perfectly elastic and perfectly inelastic impact. If in them we put $\mu_1 = \mu_2 = 1$, we then obtain the formula already found above for the impact of perfectly elastic bodies, but if we assume $\mu_1 = \mu_2 = 0$, we then obtain the formula for inelastic impact, &c. If both bodies have the same degree of elasticity, therefore, $\mu_1 = \mu_2$, we have more simply:

$$c_1 - v_1 = (c_1 - c_2)\,\frac{M_2}{M_1 + M_2}\,(1 + \sqrt{\mu}), \text{ and}$$
$$v_2 - c_2 = (c_1 - c_2)\,\frac{M_1}{M_1 + M_2}\,(1 + \sqrt{\mu}).$$

The general equations given previously for the equality of the momenta and for the Vis Viva of impinging bodies may by a slight change be brought to a form which is of great use in the simpler reduction of the particular cases given in page 29 et seq. The first two equations are (A) $Mv + M_1v_1 = MV + M_1V_1$ & (B) $Mv^2 + M_1v_1^2 = MV^2 + M_1V_1^2 - 2\int N(di + di_1)$; from the first by transposition $M(V - v) = M_1(v_1 - V_1)$. (1) in the second consider only for a moment the case of perfectly elastic impact then the second term of the second member equals zero and ∴ by transposition $M(V^2 - v^2) = M_1(v_1^2 - V_1^2)$ Divide this by equation (1) member by member $V + v = V_1 + v_1$ and from this $v - v_1 = V_1 - V$ taking now the general case for imperfectly elastic impact the only difference that would be introduced in this last formula would be that the second member would be multiplied by some coefficient which we term e; the equation now is $v - v_1 = (V_1 - V)e$ (C)

$m = \infty$ this [illegible] [illegible]

or the velocity of the impinged body after impact is in this case = 0

$v = 0$ Substituting in equation (6) for $(V_1 -$ [illegible] for V, 0, & for v, 0 it reduces to $v_1 = -V_1e$ which as will be seen by inspection are the same as those given in [illegible]

These two formulae are easily applicable to the case of the inelastic bodies treated of in § 267; keeping the same notation that we have throughout, we have from (C) $v = v_1 + (V_1 - v_1)e$ ∴ substituting in (B)

$$M_1(V_1 - v_1) = M(v_1 + (V_1 - v_1)e - V)$$

$$M_1V_1 - M_1v_1 = Mv_1 + MV_1e - Mv_1e - MV \text{ transposing}$$

$$M_1v_1 - Mv_1 = -M_1V_1 + MV_1e \;[illegible]\; - MV$$

$$v_1 = \frac{M_1V_1 - Mv_1e + MV_1e + MV}{M + M_1} \text{ [illegible]}$$

$$V_1 - v_1 = V_1 - \frac{M_1V_1 - MV_1e + MVe + MV}{M + M_1}$$

$$= \frac{MV_1 + [illegible] + MV_1e - MVe - MV}{M + M_1}$$

$$= \frac{MV_1 - MV + MV_1e - MVe}{M + M_1} = \frac{MV_1(1+e) - MV(1+e)}{M + M_1}$$

$$= \frac{M(V_1 - V)(1+e)}{M + M_1} \;[illegible]$$

[illegible]

[illegible] mp.

If, further, the mass M_2 is at rest, and infinitely great, it then follows that:

$$c_1 - v_1 = c_1 (1 + \sqrt{\mu}), \text{ i. e. } v_1 = - c_1 \sqrt{\mu},$$

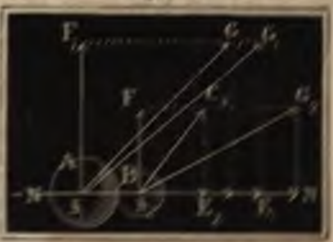
as inversely, $\mu = \left(\frac{v_1}{c_1}\right)^2$. If now M_1 be allowed to fall from a height h upon a similar mass M_2, and if it reascend to a height h_1, we may then find from the two the co-efficient of imperfect elasticity, by the formula $\mu = \frac{h_1}{h}$. Newton has already found in this manner for ivory $\mu = \left(\frac{8}{9}\right)^2 = \frac{64}{81} = 0{,}79$; for glass $\mu = \left(\frac{15}{16}\right)^2 = 0{,}9375^2 = 0{,}879$; for cork, steel, and wool, $\mu = \left(\frac{5}{9}\right)^2 = 0{,}555^2 = 0{,}309$. It must be here supposed that the impinging, or striking body, is spherical, and the body impinged upon, or the support, flat.

Example. What velocities will two steel plates acquire after impact, if they possessed before impact the velocities $c_1 = 10$ and $c_2 = - 6$ feet, the one weighs 30 the other 40 lbs.? Here

$c_1 - v_1 = (10 + 6)\frac{40}{70}\left(1 + \frac{5}{9}\right) = 16 \cdot \frac{4}{7} \cdot \frac{14}{9} = \frac{16 \cdot 8}{9} = 14{,}22$ feet; hence, the velocities sought are $v_1 = c_1 - 14{,}22 = 10 - 14{,}22 = - 4{,}22$ feet, and $v_2 = c_2 + 10{,}66 = - 6 + 10{,}66 = 4{,}66$ feet.

§ 264. *Oblique Impact.*—If the directions of motion $\overline{S_1C_1}$ and $\overline{S_2C_2}$ of two bodies A and B, Fig. 323, deviate from the normal $\overline{NN}$ to the plane of contact, the impact is then *oblique*. We may reduce the theory of this to that of direct impact if we resolve the velocities $S_1C_1 = c_1$, and $S_2C_2 = c_2$, in a normal and tangential direction; the lateral velocities in the direction of the normal $\overline{NN}$ communicate a certain impact, and hence are altered to the same amount as for centric impact; the velocities, on the other hand, parallel to the plane of contact, communicate no impact, and hence remain unaltered. If we join the normal velocity of a body changed in accordance with the laws of centric impact to the remaining unchanged tangential velocity, we shall obtain the resultant velocities of these bodies after impact. If we represent the angles which the directions of motion make with the normal by α_1 and α_2, then $C_1S_1N = \alpha_1$, and $C_2S_2N = \alpha_2$, we shall obtain for the normal velocities S_1E_1 and S_2E_2 the values c_1 *cos.* α_1 and c_2 *cos.* α_2, for the tangential velocities on the other hand S_1F_1 and S_2F_2: c_1 *sin.* α_1 and c_2 *sin.* α_2. The first velocities suffer alteration from the effect of the impact, and the one passes into

Fig. 323.



$$v_1 = c_1 \text{ cos. } \alpha_1 - (c_1 \text{ cos. } \alpha_1 - c_2 \text{ cos. } \alpha_2) \frac{M_2}{M_1 + M_2}(1 + \sqrt{\mu}),$$

and the second into:

$$v_2 = c_2 \cos.\alpha_2 + (c_1 \cos.\alpha_1 - c_2 \cos.\alpha_2)\frac{M_1}{M_1 + M_2}(1+\sqrt{\mu}),$$

M_1 and M_2 representing the masses of the bodies.

The resultant velocity $S_1 G_1$ of the first body is given by v_1 and c $\sin.\alpha_1$, $w_1 = \sqrt{v_1^2 + c_1^2 \sin.\alpha_1^2}$, and the velocity $S_2 G_2$ of the second body by v_2 and $c_2 \sin.\alpha_2$; $w_2 = \sqrt{v_2^2 + c_2^2 \sin.\alpha_2^2}$; the deviations from the normal are also given by the formula:

$$tang.\phi_1 = \frac{c_1 \sin.\alpha_1}{v_1}, \text{ and } tang.\phi_2 = \frac{c_2 \sin.\alpha_2}{v_2},$$

ϕ_1 representing the angle $G_1 S_1 N$ and ϕ_2 the angle $G_2 S_2 N$.

Example. Two spheres of 30 and 50 lbs. impinge against each other with the velocities $c_1 = 20$ and $c_2 = 25$ feet, which deviate from the normal by the angle $\alpha_1 = 21°$ $35'$ and $\alpha_2 = 65° 20'$, in what directions and with what velocities will the two bodies proceed after impact? The uniform component velocities are: $c_1 \sin.\alpha_1 = 20 . \sin. 21° 35' = 7,357$ feet, and $c_2 \sin.\alpha_2 = 25 . \sin. 65° 20' = 22,719$ feet; the variable, on the other hand, $c_1 \cos.\alpha_1 = 20 . \cos. 21° 35' = 18,598$ feet, and $c_2 \cos.\alpha_2 = 25 . \cos. 65° 20', = 10,433$ feet. If the bodies are inelastic, then $\mu = 0$; hence, the altered normal velocities are: $v_1 = 18,598 - (18,598 - 10,433)\frac{50}{80} = 18,598 - 5,103 = 13,495$ feet, and $v_2 = 10,433 + 8,165 . \frac{3}{8} = 10,433 + 3,062 = 13,495$ feet. The resultant velocities are now:

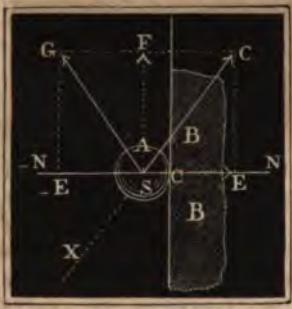
$$w_1 = \sqrt{13,495^2 + 7,357^2} = \sqrt{236,24} = 15,37 \text{ feet, and}$$
$$w_2 = \sqrt{13,495^2 + 22,719^2} = \sqrt{698,27} = 26,42 \text{ feet; and}$$

we have for their directions the $tang.\phi_1 = \frac{7,357}{13,495}$, $\log. tang.\phi_1 = 0,73653 - 1$, $\phi_1 = 28° 36'$ and $tang.\phi_2 = \frac{22,719}{13,495}$, $\log. tang.\phi_2 = 0,22622$, $\phi_2 = 59° 17'$.

5

§ 266. If a mass A, Fig. 324, strikes against another mass, indefinitely great, or against an immovable resistance BB, we have $c_2 = 0$ and $M_2 = \infty$, it then follows that

Fig. 324.



$$v_1 = c_1 \cos.\alpha_1 - c_1 \cos.\alpha_1 (1 + \sqrt{\mu})$$
$$= - c_1 \cos.\alpha_1 \sqrt{\mu} \text{ and}$$
$$v_2 = 0 + c_1 \cos.\alpha_1 . \frac{M_1(1 + \sqrt{\mu})}{\infty}$$
$$= 0 + 0 = 0;$$

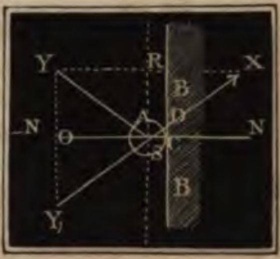
if now, further, $\mu = 0$, v_1 will also $= 0$; but if $\mu = 1$, v_1 will $= - c_1 \cos.\alpha_1$; *i. e. in inelastic impact, the normal velocity is entirely lost; in elastic, on the other hand, it is changed in the opposite direction.* For the angle by which the direction of motion after impact deviates from the normal $tang.\phi_2$ is

$$= \frac{c_1 \sin.\alpha_1}{v_1} = - \frac{c_1 \sin.\alpha_1}{c_1 \cos.\alpha_1 \sqrt{\mu}} = - tang.\alpha_1 \sqrt{\frac{1}{\mu}};$$

for inelastic bodies the $tang.\phi_1$ is therefore $= - \frac{tang.\alpha_1}{0} = \infty$, *i. e.* $\phi_1 = 90°$, and for elastic $tang.\phi_1 = - tang.\alpha_1$, *i. e.* $\phi_1 = -\alpha_1$. After the impact of an inelastic body against an inelastic resistance, the first proceeds with a tangential velocity c_1 *sin.* α_1 in the direction *SF* of the plane of contact; after the impact of an elastic body against an elastic resistance, the body proceeds with a uniform velocity in the direction *SG*, which lies in a plane with the normal $\overline{NN}$ and the initial direction *XS*, and makes with the normal the same angle $GS\overline{N}$ as does the direction of motion with it before impact, but on the opposite side. The angle $XS\overline{N}$ which the direction of motion before impact makes with the normal or the vertical, is called the *angle of incidence*, and the angle *GSN*, which the direction of motion after impact makes with the same, the *angle of reflexion;* and it may therefore be enunciated, *that in perfectly elastic impact, the angles of reflexion and incidence lie in the same plane with the vertical, and are equal to each other.*

In imperfectly elastic impact, the ratio $\sqrt{\mu}$ of the tangents of these angles, is equal to the ratio of the velocity given back by the expansion to that of the velocity lost by the compression. By the aid of this law, the direction may be easily found in which the body *A*, Fig. 325, must impinge against the immovable resistance *BB*, that after impact it may pursue a certain direction *SY*. If the impact be elastic we must let fall from a point *Y* of the given direction the perpendicular *YO* on the incident perpendicular $\overline{NN}$, prolong the same until the prolongation OY_1 is equal to the perpendicular: SY_1 is then the direction of the impact in question, for from this construction, the angle $NSY_1 = NSY$. If the impact be inelastic, we may make $OY_1 = \sqrt{\mu} \, . \, OY$, then Y_1S will be likewise the initial direction sought, since $\frac{tang. \, \alpha_1}{tang. \, \phi_1} = \frac{OY_1}{OY}$, and also $= \sqrt{\mu}$.

Fig. 325.



If a perpendicular *YR* be let fall upon the line *SR* parallel to the plane of contact, and its prolongation *RX* be made $= \sqrt{\frac{1}{\mu}} \, RY$, we shall then also obtain, for evident reasons, in *SX* the direction of the incidence sought.

Remark. The theory of oblique impact has especial application to the game of billiards. (*See* Théorie mathématique des effets du jeu de billard," par Coriolis.) According to Coriolis, in the striking of a billiard-ball against the cushion, the ratio of the reflected velocity to that of the incident $= 0{,}5$ to $0{,}6$; therefore $\mu = 0{,}5^2 = 0{,}25$ to $0{,}6^2 = 0{,}36$. With the assistance of this value, the direction may now be known in which a ball *A* must strike against the cushion *BB* that it may be reflected from this towards a given point *Y*. We may let fall from the given point *Y* the perpendicular *YR* to the line of gravity of the ball parallel to the cushion, prolong it by $RX = \sqrt{\frac{1}{\mu}} = \frac{10}{6}$ to $\frac{10}{5}$ of its value, and draw the straight line Y_1X: the point of intersection *D* is the place where the

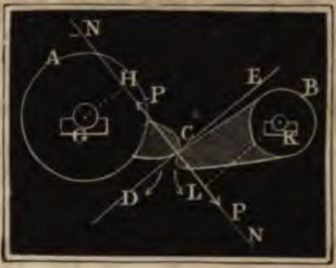
velocity of rotation generated by the impact against the plane is:

$$w_1 = 2f(1+\sqrt{\mu})\, c \cos. \alpha \text{ and } = \frac{5}{2} f(1+\sqrt{\mu})\, c \cos. \alpha.$$

Example. If a billiard-ball, with a 15 feet velocity and at an angle of incidence $\alpha = 45^{\circ}$, strike against the cushion, what motion will it take after impact? If we put for the $\sqrt{\mu}$ the mean value 0,55, we shall have the lateral normal velocity after impact $= -\sqrt{\mu}\, c \cos. \alpha = -0{,}55 \,.\, 15 \,.\, \cos. 45^{\circ} = -8{,}25 \,.\, \sqrt{\frac{1}{2}} = -5{,}833$ ft. and if with Coriolis we take $f = 0{,}20$, we shall then obtain the lateral velocity parallel to the cushion $= c \sin. \alpha - f(1+\sqrt{\mu})\, c \cos. \alpha = (1 - 0{,}20 \,.\, 1{,}55)\, 10{,}607 = 0{,}69 \,.\, 10{,}607 = 7{,}319$ feet, and for the angle of reflexion ϕ:

$tang. \phi = \frac{7{,}319}{5{,}833} = 1{,}2548$; therefore, $\phi = 51^{\circ}\, 27'$, and the velocity after impact will remain $= \frac{5{,}833}{\cos. 51^{\circ}\, 27'} = 9{,}360$ feet. Besides, the ball will have further a velocity of rotation $\frac{5}{2} f \,.\, 1{,}55 \,.\, 10{,}607 = 8{,}220$ feet about its vertical line of gravity. Since the ball moves with a rolling and not a sliding motion, we must assume that besides its progressive velocity $c = 15$ feet, it possesses an equal amount of velocity of rotation, and this may likewise be resolved into the components $c \cos. \alpha = 10{,}607$ and $c \sin. \alpha = 10{,}606$. The first component answers to a rotation about an axis parallel to the cushion, and passes into $c \cos. \alpha - \frac{5}{2} f(1+\sqrt{\mu})\, c \cos. \alpha = 10{,}607 - 8{,}220 = 2{,}387$ feet, the other component $c \sin. \alpha = 10{,}607$ feet answers to a rotation about an axis normal to the cushion, and remains uniform.

Fig. 327.



§ 267. *Rotary Bodies.*—If two bodies, *A* and *B*, capable of rotating about two fixed axes *G* and *K*, Fig. 327, strike against each other, changes of velocity ensue, which may be determined from the moments of inertia $M_1 y_1^2$ and $M_2 y_2^2$ of the masses of these bodies about the fixed axes, with the assistance of the formulæ already found. If the perpendiculars *GH* and *KL*, let fall from the axis of rotation upon the line of impact, are a_1 and a_2, we then have the inert masses reduced to the points *H* and *L* where the perpendiculars meet the line of impact $= \frac{M_1 y_1^2}{a_1^2}$ and $\frac{M_2 y_2^2}{a_2^2}$, and if these values be substituted for M_1 and M_2 in the formula for free centric impact, we obtain the changes of the velocity of the points *H* and *L* (§ 264)

$$= (c_1 - c_2) \frac{M_2 y_2^2 \div a_2^2}{M_1 y_1^2 \div a_1^2 + M_2 y_2^2 \div a_2^2} (1+\sqrt{\mu})$$

$$= (c_1 - c_2) \frac{M_2 y_2^2 a_1^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} (1+\sqrt{\mu}) \text{ and}$$

$$(c_1 - c_2) \frac{M_1 y_1^2 \div a_1^2}{M_1 y_1^2 \div a_1^2 + M_2 y_2^2 \div a_2^2} (1+\sqrt{\mu})$$

$$= (c_1 - c_2) \frac{M_1 y_1^2 a_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} (1+\sqrt{\mu}),$$

c_1 and c_2 representing the velocity of these points before impact.

But if we introduce the angular velocities, and represent these before impact by ε_1 and ε_2, and after impact by ω_1 and ω_2, we shall have to put $c_1 = a_1\,\varepsilon_1$, $c_2 = a_2\,\varepsilon_2$, and shall obtain for the loss in angular velocity of the impinging body:

$$= a_1\,(a_1\,\varepsilon_1 - a_2\,\varepsilon_2)\,\frac{M_2 y_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2}\,(1 + \sqrt{\mu}),$$

and for the body impinged upon, the gain of the same:

$$= a_2\,(a_1\,\varepsilon_1 - a_2\,\varepsilon_2)\,\frac{M_1 y_1^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2}\,(1 + \sqrt{\mu}),$$

consequently the angular velocities themselves, after impact, will be:

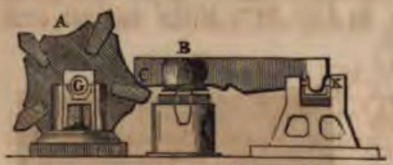
$$\omega_1 = \varepsilon_1 - a_1\,(a_1\,\varepsilon_1 - a_2\,\varepsilon_2)\,(1 + \sqrt{\mu})\,\frac{M_2 y_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} \text{ and}$$

$$\omega_2 = \varepsilon_2 + a_2\,(a_1\,\varepsilon_1 - a_2\,\varepsilon_2)\,(1 + \sqrt{\mu})\,\frac{M_1 y_1^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2}.$$

If both bodies are perfectly elastic, we shall have $\mu = 1$, therefore $1 + \sqrt{\mu} = 2$, and if inelastic, $\mu = 0$, therefore $1 + \sqrt{\mu} = 1$. In the latter case, the loss of *vis viva* produced by impact

$$= (a_1\,\varepsilon_1 - a_2\,\varepsilon_2)^2 \cdot \frac{M_1 y_1^2 \,.\, M_2 y_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2}.$$

Fig. 328.



Example. The armed axle AG, Fig. 328, has the moment of inertia about its axis of rotation G, $= M_1\,y_1^2 = 40000 \div g$, and the tilt hammer BK one about its axis K, $= 150000 \div g$; the arm GC of the axle is 2 feet, and the arm KC of the hammer 6 feet, and the angular velocity of the axle at the moment of impact on the hammer $= 1{,}05$ feet; what is this velocity after impact, and what effect is lost at each blow, if there is an entire absence of elasticity? The angular velocity of the axle sought is:

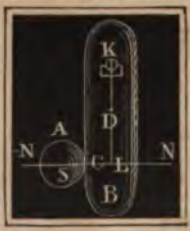
$$\omega_1 = 1{,}05 - \frac{4 \,.\, 1{,}05 \,.\, 150000}{40000 \,.\, 36 + 150000 \,.\, 4} = 1{,}105\left(1 - \frac{60}{204}\right) = 1{,}05 \,.\, 0{,}706$$

$= 0{,}741$ feet, and that of the hammer $= \frac{2 \,.\, 6 \,.\, 1{,}05 \,.\, 4}{204} = 0{,}247$ feet, *i. e.*, only one-third that of the axle. The loss of mechanical effect by each blow is:

$$L = \frac{(2 \,.\, 1{,}05)^2}{2g} \cdot \frac{4 \,.\, 1{,}05 \,.\, 150000}{40000 \,.\, 36 + 150000 \,.\, 4} = 0{,}0155 \,.\, (2{,}1)^2 \,.\, \frac{600000}{144 + 60}$$

$$= 0{,}0155 \,.\, 4{,}41 \,.\, \frac{150000}{51} = 201 \text{ ft. lbs.}$$

Fig. 329.



§ 268. A body A, in a state of free and progressive motion, Fig. 329, impinges against a body BCK, capable of rotating about a fixed axis K, the velocities after impact may be found, if, in place of $a_1\,\varepsilon_1$ and $a_1\,\omega_1$ in the formula of the preceding paragraph, we put the progressive velocities c_1 and v_1, and instead of $\frac{M_1 y_1^2}{a_1^2}$ the inert mass M_1 of the first body, the other denominations remaining the same. Hence, the velocity of the first mass after impact is:

$$v_1 = c_1 - (c_1 - a_2 \varepsilon_2)(1 + \sqrt{\mu}) \cdot \frac{M_2 y_2^2}{M_1 a_2^2 + M_2 y_2^2},$$

and the angular velocity of the second:

$$\omega_2 = \varepsilon_2 + a_2 (c_1 - a_2 \varepsilon_2)(1 + \sqrt{\mu}) \cdot \frac{M_1}{M_1 a_2^2 + M_2 y_2^2}.$$

If the mass M_2 be at rest, therefore $\varepsilon_2 = 0$, we have:

$$v_1 = c_1 - c_1 (1 + \sqrt{\mu}) \cdot \frac{M_2 y_2^2}{M_1 a_2^2 + M_2 y_2^2} \text{ and}$$

$$\omega_2 = a_2 c_1 (1 + \sqrt{\mu}) \cdot \frac{M_1}{M_1 a_2^2 + M_2 y_2^2}.$$

If, on the other hand, M_1 is at rest, that is, the oscillating body the impinging one, we shall have $c_1 = 0$, and hence

$$v_1 = a_2 \varepsilon_2 (1 + \sqrt{\mu}) \cdot \frac{M_2 y_2^2}{M_1 a_2^2 + M_2 y_2^2} \text{ and}$$

$$\omega_2 = \varepsilon_2 \left(1 - (1 + \sqrt{\mu}) \frac{M_1 a_2^2}{M_1 a_2^2 + M_2 y_2^2}\right).$$

The velocity communicated by impact to another at rest, depends not only on the velocity of impact and of the masses of the bodies, but also on the distance $KL = a_2$ at which the direction of impact NN is distant from the axis K of the rotary body. If the free mass be the impinging one, the rotary mass will assume the angular velocity

$$\omega_2 = c_1 (1 + \sqrt{\mu}) \frac{M_1 a_2}{M_1 a_2^2 + M_2 y_2^2},$$

and if the oscillating mass strike against the free, this will acquire the velocity

$$v_1 = \varepsilon_2 (1 + \sqrt{\mu}) \frac{M_2 y_2^2 \cdot a_2}{M_1 a_2^2 + M_2 y_2^2},$$

but both velocities will be so much the greater, the greater

$\frac{a_2}{M_1 a_2^2 + M_2 y_2^2}$ or $\frac{1}{M_1 a_2 + \frac{M_2 y_2^2}{a_2}}$ is, and therefore the less

$M_1 a_2 + M_2 \frac{y_2^2}{a_2}$ is. See page Note*

If for a_2 we put $a \pm x$, when x is very small, we shall obtain the value of the last impression:

$$M_1 (a \pm x) + \frac{M_2 y_2^2}{a \pm x} = M_1 a \pm M_1 x + \frac{M_2 y_2^2}{a} \left(1 \mp \frac{x}{a} + \frac{x^2}{a^2} \pm \ldots\right)$$

or in consequence of the smallness of the powers of x,

$$= M_1 a + \frac{M_2 y_2^2}{a} \pm \left(M_1 - \frac{M_2 y_2^2}{a^2}\right) x + \ldots$$

If now a correspond to the least of all the values of $M_1 a_2 + \frac{M_2 y_2^2}{a_2^2}$, the member $\pm \left(M_1 - \frac{M_2 y_2^2}{a^2}\right) x$ will disappear, because the addition of the quantity (x) will give to it a different sign to that of a diminutive $(- x.)$ Therefore:

$\left(M_1 - \frac{M_2 y_2^2}{a^2}\right) x$ must be $= 0$, *i. e.* $\frac{M_2 y_2^2}{a^2} = M_1$, consequently:

$$a = \sqrt{\frac{M_2 y_2^2}{M_1}} = y_2 \sqrt{\frac{M^2}{M_1}}.$$

If at this distance one body impinge against the other, then will the latter take the greatest velocity, and be in fact

$$\omega = c_1 (1 + \sqrt{\mu}) \frac{1}{2\, y_2} \sqrt{\frac{M_1}{M_2}},$$

in the case where the rotary body is impinged upon; and

$$v = \tfrac{1}{2}\, y_2 c_2 (1 + \sqrt{\mu}) \sqrt{\frac{M_2}{M_1}},$$

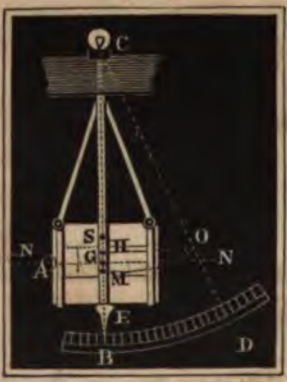
when the free body receives the blow.

The point D in the line of impact of the distance corresponding to the greatest velocity, or of the arm a, is sometimes improperly called the centre of impact, but more properly the *point of impact.*

Example. What position has the point of impact if the free mass consist of an iron sphere of 16 lbs. weight, and the rotary mass have a moment of inertia of $1000 \div g$? The distance of this point from the fixed axis of the last body: $a = \sqrt{\frac{1000}{16}} = \sqrt{62{,}5} = 7{,}906$ feet. If the impact be inelastic, and the block strike against the sphere with the velocity $c = 3$ feet, the latter will receive the velocity $v = \frac{3}{2} \cdot 7{,}906 = 11{,}86$ feet.

§ 269. *Ballistic Pendulum.*—An application of the laws laid down, is found in the theory of the *ballistic pendulum*, or the *pendulum of Robins.* It consists of a mass MH, turning about a horizontal axis C, Fig. 330, which is set into oscillating motion by a ball projected against it, which serves for the measurement of its velocity. That as inelastic a blow as possible may ensue, there is an opening made on the further side, which from time to time is filled by fresh wood or clay, &c. The ball remains after each projection sticking in this mass, and oscillating in common with the whole body. For the measurement of the velocity of the ball, it is requisite to know the angle of elongation of this pendulum, on which account there is further a graduated arc BD applied, and an index E fixed to the centre of gravity of the pendulum, which slides along with the former.

Fig. 330.



From the foregoing paragraph, the angular velocity of the ballistic pendulum after the impact of the ball is: $\omega = \frac{M_1 a_2 c_1}{M_1 a_2^2 + M_2 y_2^2}$, if M_1 is the mass of the ball, $M_2 y_2^2$ the moment of inertia of the pendulum,

c_1 the velocity of the ball, and a_2 the arm CG of the impact, or the distance of the line of impact NN from the axis of revolution of the pendulum. If the distance CM of the centre of oscillation M of the entire mass, together with the ball from the centre of suspension C, *i. e.* the length of the simple pendulum, which oscillates in equal times with the ballistic, $= l$, and the angle of elongation $BCD = \alpha$, we have the height of the isochronously oscillating simple pendulum: ?

$$h = CM - CH = l - l \cos. \alpha = l(1 - \cos. \alpha) = 2l \cancel{=} \left(\sin. \frac{\alpha}{2}\right)^2;$$

and hence the velocity at the lowest point of its path:

$$v = \sqrt{2gh} = 2\sqrt{gl} \sin. \frac{\alpha}{2}.$$

or the corresponding angular velocity:

$$\omega = \frac{v}{l} = 2\sqrt{\frac{g}{l}} . \sin. \frac{\alpha}{2} .$$

By equating these two values it will follow that ~~the angular~~ velocity:

$$c_1 = \frac{M_1 a_2^2 + M_2 y_2^2}{M_1 a_2} . 2 \sqrt{\frac{g}{l}} . \sin. \frac{\alpha}{2}.$$

compound

But now, according to the theory of the ~~simple~~ pendulum,

$$l = \frac{\text{moment of inertia}}{\text{statical moment}} = \frac{M_1 a_2^2 + M_2 y_2^2}{(M_1 + M_2)\, s},$$

s being the distance of the centre of gravity S from the axis of revolution, hence

$$M_1 a_2^2 + M_2 y_2^2 = (M_1 + M_2)\, sl, \text{ and}$$

$$c_1 = 2\left(\frac{M_1 + M_2}{M_1} .\right) \frac{s}{a_2} \sqrt{gl} . \sin. \frac{\alpha}{2}.$$

If the pendulum makes n oscillations per minute, the time of oscillation

$$\pi \sqrt{\frac{l}{g}} = \frac{60''}{n}, \text{ hence } \sqrt{gl} = \frac{60'' . g}{n\pi},$$

and the required velocity of the ball

$$c_1 = \frac{M_1 + M_2}{M_1} . \frac{120\, gs}{n\, \pi\, a_2} . \sin. \frac{\alpha}{2}.$$

Example. If a ballistic pendulum, of 3000 lbs. weight, whose angle of elongation amounts to 15°, is set into oscillation by the projection of a 6 lbs. ball, if, further, the distance s of the centre of gravity from the axis $= 5$ feet, and that of the line of projection from this axis $= 5\frac{1}{2}$ feet, and lastly, the number of oscillations per minute $n = 40$, from the above formula the velocity of the ball at the moment of the impact will be:

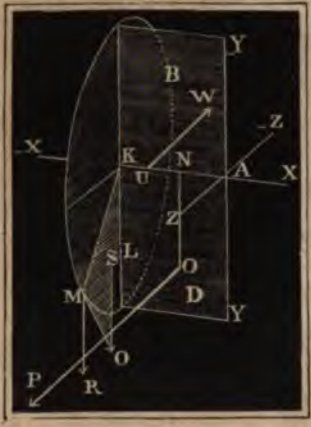
$$c = \frac{3006}{6} . \frac{120 . 31{,}25 . 5}{40 . 3{,}1416 . 5{,}5} \sin. 7\tfrac{1}{2}^\circ = \frac{501 . 3750 . \sin. 7^\circ 30'}{44 . 3{,}1416} = 1774 \text{ feet.}$$

§ 270. *Centre of Percussion.*—If a body turning about a fixed axis C is impinged upon by another, a reaction from the blow will generally take place upon the axis of the body, which is dependent principally upon the distance between the direction of the impact and that of the axis. Let us determine this reaction or this axial pressure in the simple case, when the direction of the blow is perpendicular to the plane passing through the axis of revolution and the centre of gravity of the body.

Let BD be a plane of gravity passing through the axis of revolution $X\overline{X}$ of the body in Fig. 331, $Y\overline{Y}$ a second perpendicular axis in this plane, and $Z\overline{Z}$ a third axis, at right angles to this plane of gravity. An element M_1 of the body is determined by the co-ordinates $AK = x_1$, $KL = y_1$, and $LM = z_1$, in this system of axes intersecting in the point A, and another element by the co-ordinates x_2, y_2, z_2, &c. If $\varkappa$ be the angular acceleration, we shall have the force of inertia of the element $M_1 : Q_1 = M_1 . \varkappa . \overline{KM}$, and if these be resolved into the component forces R parallel, and S at right angles to the imaginary plane of gravity, the similarity of the triangles KML and QMR or MQS, gives $R = M_1 . \varkappa z_1 = \varkappa M_1 z_1$, and $S = \varkappa M_1 y_1$. From this the aggregate of the component forces parallel to the plane $= \varkappa (M_1 z_1 + M_2 z_2 + ..)$ and that of the component forces at right angles to this plane:

$$\varkappa (M_1 y_1 + M_2 y_2 + ..).$$

Fig. 331.



Since the plane BD passes through the centre of gravity, the sum of the moments $M_1 z_1 + M_2 z_2 + .. = 0$, hence there remains only the sum of the forces $\varkappa (M_1 y_1 + M_2 y_2 + ..)$. If, now, P be a blow or impact, and W the resistance, or the reaction of the axis, we shall, in the first place, have to put:

$$P = W + \varkappa (M_1 y_1 + M_2 y_2 . . .).$$

The statical moment of the force:

$$Q_1 = M_1 \varkappa . \overline{KM} = M_1 \varkappa . \overline{KM} . \overline{KM} = \varkappa . M_1 \overline{KM}^2,$$

or the distance KM represented by $r_1, = \varkappa M_1 r_1^2$, the moment of the force of another particle of the mass $= \varkappa M_2 r_2^2$, &c., hence the statical moment of the entire inertia $= \varkappa (M_1 r_1^2 + M_2 r_2^2 + . . .)$. If now we put the distance NO of the direction of the impact from the direction of the axis $= b$, we shall have the moment of percussion P about $X\overline{X} = Pb$, whilst that of $W = 0$; we may hence also put:

$$Pb = \varkappa (M_1 r_1^2 + M_2 r_2^2 + ..),$$

and obtain by elimination of $\varkappa$ from both equations:

$$P = W + \frac{Pb (M_1 y_1 + M_2 y_2 + ..)}{M_1 r_1^2 + M_2 r_2^2 + ..},$$ *i. e.* the reaction sought:

$$W = P \left(1 - \frac{b (M_1 y_1 + M_2 y_2 + ..)}{M_1 r_1^2 + M_2 r_2^2 + ..}\right).$$

If, lastly, we represent the distance AN of the direction of the impact from the axis $Y\overline{Y}$ by a, and the distance AU of the point of application U of the reaction W from the initial point by u, we shall have further the moment $Pa =$ mom. $Wu +$ mom. $\varkappa (M_1 x_1 y_1 + M_2 x_2 y_2 + ..)$, and the distance of the point of application sought will be:

$$u = \frac{Pa - \varkappa (M_1 x_1 y_1 + M_2 x_2 y_2 + \ldots)}{W}, \text{ i. e.}$$

$$u = \frac{a (M_1 r_1^2 + M_2 r_2^2 + \ldots) - b (M_1 x_1 y_1 + M_2 x_2 y_2 + \ldots)}{M_1 r_1^2 + M_2 r_2^2 + \ldots - b (M_1 y_1 + M_2 y_2 + \ldots)}$$

The reaction $W = 0$, if $b (M_1 y_1 + M_2 y_2 + \ldots) = M_1 r_1^2 + M_2 r_2^2 + \ldots$,

$$\text{i. e. } 1.\ b = \frac{M_1 r_1^2 + M_2 r_2^2 + \ldots}{M_1 y_1 + M_2 y_2 + \ldots} = \frac{\text{moment of inertia}}{\text{statical moment}},$$

and also its moment $= 0$, if

$$Pa = \varkappa (M_1 x_1 y_1 + M_2 x_2 y_2 + \ldots), \text{ i. e.}$$

$$2.\ a = \frac{M_1 x_1 y_1 + M_2 x_2 y_2 + \ldots}{M_1 y_1 + M_2 y_2 + \ldots}.$$

The point O determined by these co-ordinates a and b, in the plane of gravity containing the fixed axis, is called the *centre of percussion*. Every blow passing through this point, and at right angles to the plane of gravity, is completely taken up by the mass, without leaving any residuary effect upon the axis, or producing any pressure. The formula (1) shows that the centre of percussion is at the same distance from the axis of revolution (compare § 251) as the centre of suspension. oscillation and coincides with the centre percuss § 27

That a hammer may not jar by its blow the hand which holds it, or react upon the wrist about which it turns, it is requisite that the blow pass through the centre of percussion.

Examples.—1. In a prismatic bar CA, Fig. 332, which turns about one of its extreme points, the centre of percussion lies about $CO = b = \frac{\frac{1}{3} l^2}{\frac{1}{2} l} = \frac{2}{3} l = \frac{2}{3} CA$ from the axis. If, therefore, the bar be fixed at one extremity, and be struck at a point O at the distance $CO = \frac{2}{3} CA$, then no jar will be felt.—2. In a parallelopiped BDE, Fig. 333, which

Fig. 332.

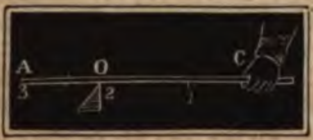
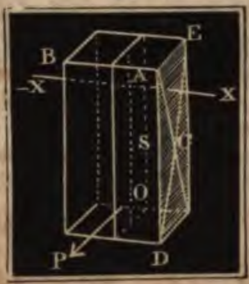


Fig. 333.

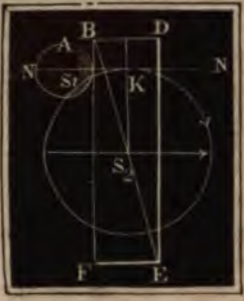


turns about an axis $\overline{XX}$ running parallel to its four sides, and distant about $SA = s$ from the centre of gravity, the distance AO of the centre of percussion O from the axis $b = \frac{s^2 + \frac{1}{3} d^2}{s}$, where d is the semi-diagonal of the lateral surfaces through which the axis $\overline{XX}$ passes (§ 220). If the force of the blow P were to pass through the centre of gravity, the reaction would be:

$$W = P \left(1 - s \cdot \frac{s}{s^2 + \frac{1}{3} d^2}\right) = P \left(1 - \frac{s^2}{s^2 + \frac{1}{3} d^2}\right) = \frac{1}{3} \cdot \frac{Pd^2}{s^2 + \frac{1}{3} d^2} = \frac{Pd^2}{3 s^2 + d^2}.$$

§ 271. *Excentric Impact.*—Lastly, let us further investigate a simple case of excentric impact, when both masses are perfectly free. When two bodies A and BE, Fig. 334, impinge upon each other so that the direction of impact NN passes through the centre of gravity S_1 of the one body, and beyond the centre of gravity S_2 of the other body, the impact with respect to the first body is centric, and with respect to the other, excentric. The effects of this excentric impact may be found from the proposition, § 214, if we assume first, that the second body is free, and that the direction of impact passes through the centre of gravity S_2, and secondly, that this body is fixed at its centre of gravity, and that the percutient force acts as a force of revolution. If now c_1 be the initial velocity of A, c_2 that of the centre of gravity of BE, and if the two velocities pass into v_1 and v_2 from the effect of the blow, there will remain as in § 256,

Fig. 334.



$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2.$$

If further ε be the initial angular velocity of the body BE, and in its revolution about the axis passing through the centre of gravity, and perpendicular to the plane $N\overline{N}S_2$, and if this velocity pass from the effect of the blow into ω, and $M_2 y_2^2$ represent the moment of inertia of this body about S_2, or s the distance S_2K of the centre of gravity S_2 from the direction of impact, we shall then also have

$$M_1 v_1 + \frac{M_2 y_2^2}{s^2} \cdot s\omega = M_1 c_1 + \frac{M_2 y_2^2}{s^2} s\varepsilon.$$

If both bodies are inelastic, then the points of contact of the two will have at the end equal velocities, therefore, further v_1 will $= v_2 + s\omega$. If we determine from the above equations v_2 and ω by v_1, and put the values obtained into the last equation, we shall then obtain:

$$v_1 = \frac{M_1 (c_1 - v_1)}{M_2} + c_2 + \frac{M_1 s^2 (c_1 - v_1)}{M_2 y_2^2} + s\varepsilon,$$

and from this may be determined the loss in velocity of the first body:

$$c_1 - v_1 = \frac{M_2 y_2^2 (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2},$$

the gain in progressive velocity of the second:

$$v_2 - c_2 = \frac{M_1 y^2 (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2},$$

and the gain in angular velocity:

$$\omega - \varepsilon = \frac{M_1 s (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2}.$$

For the case of perfectly elastic impact, these values are double; and for that of imperfectly elastic impact, they are $(1 + \sqrt{\mu})$ times as great.

Example. An iron ball A, of 65 lbs. weight, strikes a parallelopiped BE of fir, originally at rest, with a 36 feet velocity; the length of this body is 5 feet, its breadth 3 feet, and thickness 2 feet, and the direction of the impact $N\overline{N}$ deviates by $S_2 K = s = 1\frac{3}{4}$ feet from the centre of gravity S_2, then the following velocities after impact are given. The specific gravity of fir may be taken $= 0,45$, the weight of the parallelopiped is therefore $= 5 \times 3 \times 2 \times 62,5 \times 0,45 = 843,75$ lbs. The square of the semi-diagonal of the lateral surface parallel to the direction of impact is:

$$y^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{2}{2}\right)^2 = 7,25,$$

hence the velocity of the ball after impact is:

$$v_1 = c_1 - \frac{M_2 y^2 c_1}{(M_1 + M_2) y^2 + M_1 s^2} = 36 \left(1 - \frac{843,75 . 7,25}{956 . 7,25 + 65 . 1,75^2}\right)$$

$$= 36 \left(1 - \frac{843,75 . 7,25}{7130,06}\right) = 36 \, (1 - 0,958) = 1,512 \text{ feet}:$$

further, the velocity of the centre of gravity of the parallelopiped:

$$v_2 = \frac{M_1 y^2 c_1}{(M_1 + M_2) y^2 + M_1 s^2} = \frac{65 . 7,25 . 36}{7130,06} = 2,379 \text{ feet}:$$

lastly, the angular velocity of this body is:

$$\omega = \frac{M_1 s c}{(M_1 + M_2) y^2 + M_1 s^2} = \frac{65 . 1,75 . 36}{7130,06} = 0,574 \text{ feet.}$$

SECTION V.

STATICS OF FLUID BODIES.

CHAPTER I.

ON THE EQUILIBRIUM AND PRESSURE OF WATER IN VESSELS.

§ 272. *Fluidity.*—We regard *fluid bodies* as systems of material points, whose cohesion is so feeble, that the smallest forces are sufficient to effect a separation, and to move them amongst each other (§ 59). Many bodies in nature, such as air, water, &c., possess this property of fluidity in a high degree; other bodies, on the contrary, such as oil, fat, soft earth, &c., are fluid in a low degree. The former are called *perfectly fluid*, the latter *imperfectly fluid bodies*. Certain bodies, as, for instance, paste, are intermediate between solid and fluid bodies.

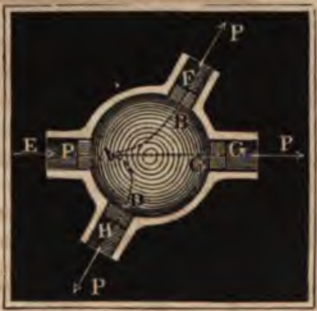
Perfectly fluid bodies, of which only we shall subsequently speak, are at the same time perfectly elastic, *i. e.*, they may be compressed by external forces, and will perfectly resume their former volume after the withdrawal of these forces. The amount of the change of volume corresponding to a certain pressure, is different for different fluids; in *liquid bodies* this is scarcely perceptible, while in *aëriform bodies*, which, on this account, are also called elastic fluids, it is very great. This slight degree of compressibility of liquid bodies is the reason why in most investigations in hydrostatics (§ 63) they are considered and treated as incompressible or inelastic. As water, of all liquids, is the one most generally diffused, and the most useful for the purposes of life, it is taken as the representant of all these fluids, and in the investigations of the mechanics of fluids, water only is spoken of, whilst it is tacitly understood that the mechanical properties of other liquids are the same as those of water.

From a similar reason in the mechanics of the elastic fluid bodies ordinary atmospheric air is only spoken of.

Remark.—A column of water of one square inch transverse section is compressed by a weight of 15 lbs., which corresponds to the atmospheric pressure, by about 0,00005 or 50 millionths of its volume, while the same column of air under this pressure would be compressed to one half of its original volume.

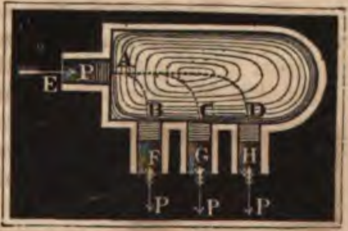
§ 273. *Principle of Equality of Pressures.*—The characteristic property of fluids, which essentially distinguishes them from solid bodies, and which serves as a basis of the laws of the equilibrium of fluid bodies, *is the capability of transmitting the pressure which is exerted upon a part of the surface of the fluid in all directions unchanged.* The pressure on solids is transmitted only in its proper direction (§ 83); while, on the other hand, when water is pressed on one side, a tension takes place in the entire mass, which exerts itself on all sides, and may be observed at all parts of the surface. To satisfy ourselves of the correctness of this law, we may make use of an apparatus filled with water, as is shown in the horizontal section in Fig. 335. The tubes *AE* and *BF*, &c., equally distant, and at an equal height above the horizontal base, are closed by perfectly movable and accurately fitting pistons; the water presses, therefore, by its weight, as strongly against the one piston as against the other. Let us do away with this pressure, and regard the water as devoid of weight. Let us press the one piston with a certain pressure *P* against the water, this pressure will then be transmitted by the water to the other pistons *B*, *C*, *D*, and for the restoration of equilibrium, or to prevent the pushing back of these pistons, it is requisite that an equal and opposite pressure *P* act against each of these pistons. We are, therefore, justified in assuming, that the pressure *P*, acting upon a point *A* of the surface of the mass of water, produces in it a tension, and not only transmits this in the straight line *AC*, but also in every other direction *BF*, *DH*, &c., to every equal area of the surface *C*, *B*, *D*.

Fig. 335.



If the axes of the tubes *BF*, *CG*, &c., Fig. 336, are parallel to each other, the pressures which act upon their pistons may be united by addition into a single pressure; if n be the number of the pistons, then the aggregate pressure upon these amounts to $P_1 = nP$, and in the case represented in the figure $P_1 = 3P$. But now the areas F_1 of the pressed surfaces *B*, *C*, *D*, are equal to n times the pressed surface F of the one piston, hence n may not only be put $= \frac{P_1}{P}$, but also $= \frac{F_1}{F}$, therefore $\frac{P_1}{P} = \frac{F_1}{F}$.

Fig. 336.



If the tubes *B*, *C*, *D*, form a single one, as in Fig. 337, and if we close it by a single piston, F_1 then becomes a single surface, and P_1

is the pressure acting upon it, hence there follows this general law, *the pressure which a fluid body exerts upon different parts of the sides of a vessel, is proportional to the area of these parts.*

Fig. 337.

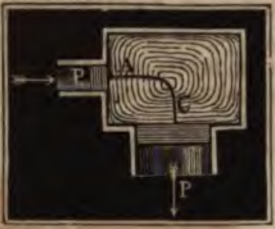
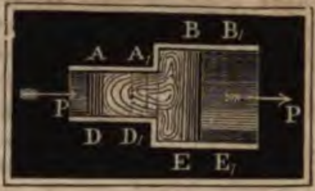


Fig. 338.



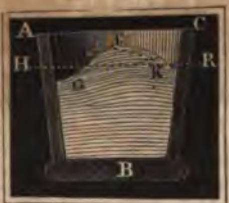
This law corresponds also to the principle of virtual velocities. If the piston $AD = F$, Fig. 338, moves inwards through a space $AA_1 = s$, it then presses the column of water Fs from its tube, and if the piston $BE = F_1$, it passes outwards through the space $BB_1 = s_1$, it then leaves a space $F_1 s_1$ behind. But since we have supposed that mass of water neither allows of expansion nor compression, its volume then by this motion of the piston must remain unaltered, that is, the increase Fs must be equal to the decrease $F_1\, s_1$. But the equation $F_1\, s_1 = Fs$ gives $\frac{F_1}{F} = \frac{s}{s_1}$, and by combining this proportion with the proportion $\frac{P_1}{P} = \frac{F_1}{F}$, it follows that $\frac{P_1}{P} = \frac{s}{s_1}$, hence, therefore, the mechanical effect $P_1 s_1 =$ mechanical effect Ps (§ 80).

Example. If the piston AD has a diameter of 1½ inches, and the piston BE one of 10 inches, and each is pressed by a force P of 36 lbs. upon the water, this piston exerts a pressure $P_1 = \frac{F}{F_1} P = \frac{10^2}{1{,}5^2} \,.\, 36 = 1600$ lbs. If the first piston is pushed forwards 6 inches, the second will only go back by $s_1 = \frac{F}{F_1} s = \frac{9 \,.\, 6}{400} = \frac{27}{200} = 0{,}135$ in.

Remark. Numerous applications of this law will come before us in the hydraulic press, or water column machines, in pumps, &c.

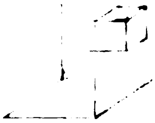
§ 274. *The Fluid Surface.*—The gravity inherent in water causes all its particles to tend downwards, and they would actually so move unless this motion were prevented. In order to obtain a coherent mass of water, it is necessary to enclose it in vessels. The water in the vessel ABC, Fig. 339, is then only in equilibrium if its free surface HR is perpendicular to the direction of gravity, and therefore horizontal, for so long as this surface is curved or inclined to the horizon: then there are elementary portions E, F, &c., lying higher, which, from their extreme mobility in virtue of their gravity, slide down on those below them, as if it were on an inclined plane GK.

Fig. 339.



Conditions of equilibrium of any fluid, homogeneous or otherwise, compressible or incompressible, when under the action of known forces.

Let $x\,y\,z$ be the ordinates



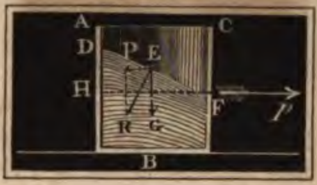
of one of its elements. Divide fluid into cubes by planes parallel to each of the coordinate planes respectively. The edges are dx, dy, dz, and the volume will be $dx\,dy\,dz$. We know that $m = \frac{G}{g} = \frac{V\gamma}{g}$, let us now call $\frac{\gamma}{g} = \rho$ so that our expression for the mass is $m = V\rho$ then $dm = dV\rho = dx\,dy\,dz\,\rho$. Let P be the total pressure upon a unit of mass and resolve it in the direction of three coördinate axes into components X. Y. Z. but the force upon elementary volume = the force upon a unit of mass multiplied by the mass so that the force upon our element will be $P\,dm$ and its three components will now also become $X\,dm + Y\,dm + Z\,dm$.

Since the directions of gravity for great distances can no longer be regarded as parallel, we must, therefore, consider the free surface, or the level of water in a large vessel, as for example, in a great lake, no longer as a plane, but as part of a spherical surface.

If any other force than that of gravity act upon the particles of water, the fluid surface in the state of equilibrium, will be perpendicular to the direction of the resultant arising from gravity and the concurrent force.

If a vessel ABC, Fig. 340, is moved forward horizontally by a uniformly accelerating force p, the free surface of the water in it will form an inclined plane DF, for in this case every element E of this surface will be impelled downwards by its weight G, and horizontally by its inertia $P = \frac{p}{g} G$, there will then be a resultant R, which will make with the direction of gravity a uniform angle $REG = \alpha$. This angle is at the same time the angle DFH which the surface of the water makes with the horizon. It is determined by $tang.\, \alpha = \frac{P}{G} = \frac{p}{g}$.

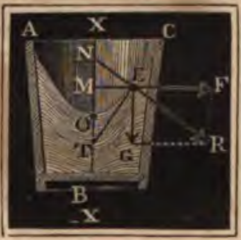
Fig. 340.



If, on the other hand, a vessel ABC, Fig. 341, rotates uniformly about its vertical axis $\overline{XX}$, the surface of the water then forms a hollow surface AOC, whose sections through the axis are parabolic. If ω be the angular velocity of the vessel and the water in it, G the weight of an element of water E, and y its distance ME from the vertical axis, we shall then have for the centrifugal force of this element $F = \omega^2 \frac{G y}{g}$ (§ 231), and hence for the angle $REG = TEM = \phi$, which the resultant R makes with the vertical or the tangent to the water profile with the horizon:

$$tang.\, \phi = \frac{F}{G} = \frac{\omega^2 y}{g}.$$

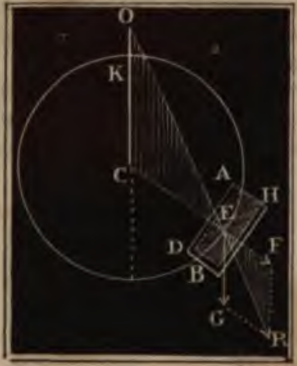
Fig. 341.



From this, therefore, the tangent of the angle which the line of contact makes with this ordinate, is proportional to the ordinate. As this property belongs to the common parabola (§ 144), the vertical section AOC of the surface of water is also a parabola whose axis coincides with the axis of revolution $\overline{XX}$.

If a vessel ABH be moved in a vertical circle, Fig. 342, uniformly about a horizontal parallel axis C, the surface of the water will form in it a cylindrical surface with circular sections DEH. If we prolong the direction of the resultant R of the gravity G, and the centrifugal force F of an element E to the intersection O with the vertical CK passing through the centre of revolution, we shall then

Fig. 342.



obtain the similar triangles ECO and EFR, for which

$$\frac{OC}{EC} = \frac{FR}{EF} = \frac{G}{F},$$

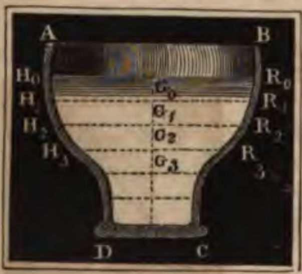
but now, if we put the radius of [redacted] $EC = y$, and retain the last notation, $F = \frac{\omega^2 Gy}{g}$, it follows that the line

$$CO = \frac{g}{\omega^2} = \frac{32,2}{(3,1416)^2}\left(\frac{30}{u}\right)^2 = \frac{2936}{u^2}$$

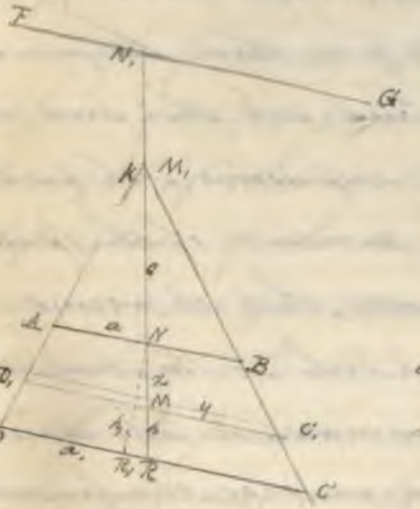
if u represents the number of revolutions per minute. As this value of CO is one and the same for all the particles of water, it follows that the components of all the particles forming the section DEH are directed towards O, and hence the section perpendicular to the directions of these forces, is a circle described from O as a centre. According to this, the surfaces of water in the buckets of an overshot wheel form perfect cylindrical surfaces, corresponding to one and the same horizontal axis.

§ 275. *Pressure on the Bottom.*—The pressure of water in a vessel $ABCD$, Fig. 343, immediately under the water level is the least, but becomes greater and greater in proportion to the depth, and is greatest immediately above the bottom. To prove this generally, let us assume that the level of the water H_0R_0, whose area may be F_0, is uniformly pressed by a force P_0, for example, by the superincumbent atmosphere, or by a piston, and let us suppose the whole mass of water divided by horizontal planes, as H_1R_1, H_2R_2, into equally thick strata of water. If now λ be the thickness or the height of such a stratum, and γ the density of water, we shall then have the weight of the first stratum $G_1 = F_0\lambda\gamma$, and hence the entire pressure on the subjacent water: $P_1 = P_0 + F_0\lambda\gamma$. If we divide this pressure by the area F_1 of the following horizontal section H_1R_1, we shall obtain the pressure for each unit of this surface: $p_1 = \frac{P_0}{F_1} + \frac{F_0}{F_1},\lambda\gamma$, or, since F_1, on account of the infinitely small distance between H_0R_0 and H_1R_1, is infinitely little different from F_0, and may be substituted for this: $p_1 = p_0 + \lambda\gamma$, where p_0 represents the external pressure on the unit of surface. The pressure of the succeeding horizontal section H_2R_2 may be determined as exactly as the pressure of the stratum H_1R_1, if we take into consideration that the initial pressure upon the unit is now $p_1 = p_0 + \lambda\gamma$, whilst it was then only p_0. The pressure in the horizontal stratum H_2R_2 then follows: $p_2 = p_1 + \lambda\gamma = p_0 + \lambda\gamma + \lambda\gamma = p_0 + 2\lambda\gamma$; likewise the pressure in

Fig. 343.



Centre of Pressure



Let us consider the case of a submerged trapezium; let FNG be the free fluid surface; to make this discussion perfectly general we do not take the free fluid surface horizontal but inclined to it at any angle, we may however always take the two parallel bases of trapezium parallel to it draw RNN, bisecting the two parallel sides of the ~~parallelogram~~ the trapezium and for the more general case suppose it inclined to the free fluid surface at an angle RN,G = α; this line is not necessarily vertical although it has fallen very nearly so in our figure; let us the a forces applied at all the points of the surface which is exactly what we have to determine the centre of Gravity.

the third stratum $H_3R_3 = p + 3\lambda\gamma$, in the fourth $= p_0 + 4\lambda\gamma$, and in the nth $= p_0 + n\lambda\gamma$. But now $n\lambda$ is the depth $G_0Gn = h$ of the nth stratum below the level of the water, hence the pressure upon each unit of surface in the nth horizontal stratum may be put: $p = p_0 + h\gamma$.

The depth h of an element of surface below the water level, is called the *head of water*, and the pressure of water upon any unit of surface may from this be found, if the externally acting pressure be increased by the weight of a column of water whose base is this unit, and whose height is the head of water.

The head of water h on a horizontal surface, for instance, on the bottom CD, is at all places one and the same; hence the area of this surface $= F$, and the pressure of water against it is: $P = (p_0 + h\gamma) F = Fp_0 + Fh\gamma = P_0 + Fh\gamma$, or if we abstract the outer pressure: $P = Fh\gamma$. *The pressure of water against a horizontal surface is therefore equivalent to the weight of the superincumbent column of water Fh.*

This pressure of water against a horizontal surface—a horizontal bottom, for instance—or against a horizontal part of a lateral wall, is independent of the form of the vessel; whether, therefore, the vessel AC, Fig. 344, be prismatic as a, or wider above than below as b, or wider below than above, as c, or inclined as d, or bulging out as e, &c., the pressure on the bottom will be always equal to the weight of a column of water whose base is the bottom and whose height is the depth of the bottom below the level of the water. As the pressure of water transmits itself on all sides, this law is therefore applicable when the surface, as BC, Fig. 345, is pressed upon from below upwards. Every unit of surface in the stratum lying in BC is pressed by a column of water of the height $HB = RK = h$; consequently, the pressure against $CB = Fh\gamma$, F being the area of the surface.

Fig. 344.

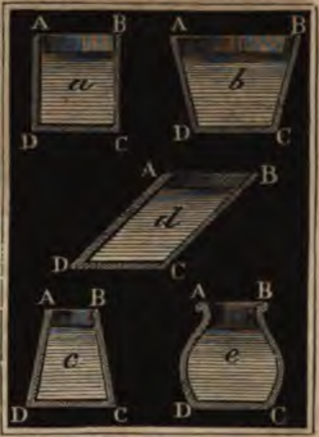


Fig. 345.

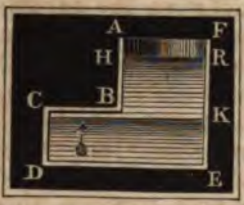
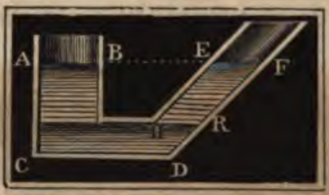


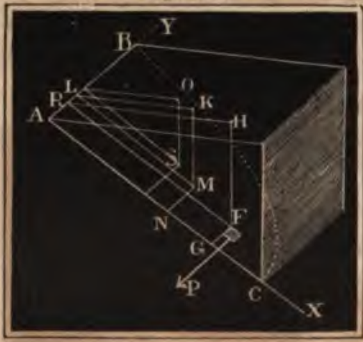
Fig. 346.



It further follows from this, that the water in tubes communicating with each other ABC and DEF, Fig. 346, when equilibrium subsists, stands equally high, or that the two levels AB and EF are in one and the same horizontal plane. For the subsistence of equilibrium, it is requisite that the stratum of water *HR* be as forcibly pressed downwards by the superincumbent column of water *ER*, as pressed upwards by the mass of water lying below it. But as in both cases the surface pressed is one and the same, so must the head of water in both cases be one and the same, therefore the level *AB* must stand as high above *HR* as the level *EF*.

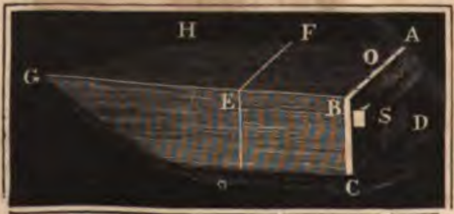
§ 276. *Lateral Pressure.*—The laws found above for the pressure of water against a horizontal surface, are not directly applicable to a plane surface inclined to the horizon; for in this case the heads of water at different places are different. The pressure $p = h\gamma$ on each unit of surface within the horizontal stratum of water, which lies a depth h below the level, acts in all directions (§ 273), and consequently also perpendicular to the fixed lateral walls of the vessel, which (from § 128) perfectly counteract it. If now F_1 be the area of an element of a lateral surface *ABC*, Fig. 347, and h_1 its head of water *FH*, we shall then have the normal pressure of the water against it: $P_1 = F_1 . h_1\gamma$; if F_2 be a second element of the surface, and h_2 its head of water, we shall then have the normal pressure on it: $P_2 = F_2h_2\gamma$; and for a third element $P_3 = F_3h_3\gamma$, &c. These normal pressures form a system of parallel forces, whose resultant P is the sum of these pressures; therefore $P = (F_1h_1 + F_2h_2 + \ldots)\gamma$. But now, further, $F_1h_1 + F_2h_2 + \ldots$ is the sum of the statical moments of F_1, F_2, &c., with respect to the surface *OHR* of the water, and $= Fh$, F representing the area of the whole surface, and h the depth *SO* of its centre of gravity below the level; hence, the aggregate normal pressure against the plane surface is $P = Fh\gamma$. We mean here, by the head of water of a surface, the depth *SO* of its centre of gravity below the level of the water; the general rule, therefore, is true that: *the pressure of water against a plane surface is equivalent to the weight of a column of water whose base is the surface and whose height is the head of water of the surface.*

Fig. 347.



It must further be stated, that this pressure of the water is not dependent on the quantity of water which is before or below the pressed surface, that therefore, for example, a flood-gate, *AC*, Fig. 348, under otherwise similar circumstances, has to sustain the same pressure, whether the water to be dammed up be that of a small sluice

Fig. 348.



ACEF, or that of a larger dam *ACGH*, or that of a great reservoir. From the breadth $AB = CD = b$ and height $AD = BC = a$ of a rectangular flood-gate, $F = ab$, and the head of water $SO = \frac{a}{2}$; hence the pressure of water

$$P = ab \,.\, \frac{a}{2} \gamma = \tfrac{1}{2} a^2 b\gamma.$$

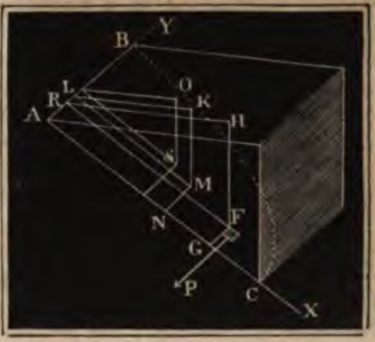
Therefore the pressure increases as the breadth, or as the square of the height of the pressed surface.

Example. If the water stand $3\frac{1}{2}$ feet high before a board of oak 4 feet broad, 5 feet high, and $2\frac{1}{2}$ inches thick, what will be the force required to draw it up? The volume of the board is $4 \,.\, 5 \,.\, \frac{5}{24} = \frac{25}{6}$ cubic feet. If now we take the density of oak saturated with water from § 58 at $62{,}5 \times 1{,}11 = 67{,}3$ lbs., the weight of this board will be: $G = \frac{25}{6} \,.\, 67{,}3 = 280{,}5$ lbs. The pressure of the water against the board, and also the pressure of this last against the guides will be:

$P = \frac{1}{2} \cdot \left(\frac{7}{2}\right)^2 \,.\, 4 \,.\, 62{,}5 = 49 \,.\, 30{,}25 = 1531{,}25$ lbs.; if now we take the co-efficient of friction for wet wood from § 161, $f = 0{,}68$, the friction of this board against its guides will be $F = f\,P = 0{,}68 \,.\, 1531{,}25 = 1041{,}25$ lbs. If to this be added the weight of the board, we shall obtain the force required to pull it up $= 1041{,}25 + 67{,}3 = 1108{,}55$ lbs.

§ 277. *Centre of Pressure.*—The resultant $P = Fh\gamma$ of the collective elementary pressures $F_1 h_1 \gamma$, $F_2 h_2 \gamma$, &c., has, like every other system of parallel forces, a definite point of application, which is called the *centre of pressure*. Equilibrium will subsist for the whole pressure of the surface, if this point be supported. The statical moments of the elementary pressures $F_1 h_1 \gamma$, $F_2 h_2 \gamma$, &c., with respect to the plane of the level *OHR*, Fig. 349, are: $F_1 h_1 \gamma \,.\, h_1 = F_1 h_1^2 \gamma$, $F_2 h_2^2 \gamma$, &c.; therefore, the statical moment of the whole pressure with respect to this plane is: $(F_1 h_1^2 + F_2 h_2^2 + \ldots)\,\gamma$. If we put the distance *KM* of the centre *M* of this pressure from the level of the water $= z$, we shall then have the moment of pressure $= Pz = (F_1 h_1 + F_2 h_2 + \ldots)\,z\gamma$, and by equating both moments, the depth in question of the centre *M* below the surface:

Fig. 349.



$$1.\; z = \frac{F_1 h_1^2 + F_2 h_2^2 + \ldots}{F_1 h_1 + F_2 h_2 + \ldots}, \text{ or } = \frac{F_1 h_1^2 + F_2 h_2^2 + \ldots}{Fh},$$

if, as before, F represent the area of the whole surface, and h the depth of its centre of gravity below the surface. To determine this pressure completely we must know further its distance from another plane or line. If we put the distances $F_1 G_1$, $F_2 G_2$, &c., of the elements of the surface $F_1 F_2$, &c., from the line *AC* which determines the angle of inclination of the plane $= y_1 y_2$, &c., we shall then have the moments of the elementary pressures with respect to this line $= F_1 h_1 y_1 \gamma$, $F_2 h_2 y_2 \gamma$, &c., therefore, the moment of the whole surface $= (F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots)\,\gamma$; and if we represent the distance *MN* of the centre *N* from this line by v, we shall then have the moment also $= (F_1 h_1 + F_2 h_2 + \ldots)\,v\gamma$; if, lastly, we make both moments equal, we shall obtain the second ordinate:

$$2.\ v = \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{F_1 h_1 + F_2 h_2 + \ldots},\ \text{or} = \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{Fh}.$$

If α be the angle of inclination of the plane *ABC* to the horizon, and x_1, x_2, &c., the distances $F_1 R_1$, $F_2 R_2$, &c., of the elements F_1, F_2, &c., as likewise u the distance of the centre of pressure *M* from the line of intersection *AB* of the plane with the level of the water, we shall then have:

$$h_1 = x_1\, sin.\, \alpha,\ h_2 = x_2\, sin.\, \alpha,\ \text{\&c., as well as } z = u\, sin.\, \alpha;$$

and if these values be put into the expressions for z and v, we shall then obtain:

$$u = \frac{F_1 x_1^2 + F_2 x_2^2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{moment of inertia}}{\text{statical moment}},\ \text{and}$$

$$v = \frac{F_1 x_1 y_1 + F_2 x_2 y_2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{centrifugal moment}}{\text{statical moment}}.$$

We may, therefore, find the distances u and v of the centre of pressure from the horizontal axis *AY*, and from the axis *AX* formed by the line of fall, if we divide the statical moment of the surface with respect to the first axis, once by its moment of inertia with respect to the same axis, and a second time by its centrifugal moment with respect to both axes. The first distance is at once the distance of the centre of suspension from the line of intersection with the line of the water (§ 251). It is easy to see that the centre of pressure coincides perfectly with the centre of percussion, determined in § 270, if the line of intersection *AY* of the surface with the level, be regarded as the axis of revolution.

If the pressed surface is a rectangle *AC*, Fig. 350, with horizontal base *CD*, the centre of pressure *M* will be found in the line *LK* let fall upon *CD* bisecting the basis, and will be distant $\frac{2}{3}$ of this line from the side *AB* in the surface of water. If this rectangle does not reach the surface as in Fig. 351, if further the distance *KL* of the lower base *CD* from the surface be l_1, and that of the upper base

Fig 350

$$x_1 = \frac{\int_0^h b\,dx\,x^2}{\int_0^h b\,dx\,x} = \frac{2}{3}h$$

Fig 351

$$x_1 = \frac{\int_{h_2}^{h_1} b\,dx\,x^2}{\int_{h_2}^{h_1} b\,dx} = \frac{2}{3}\,\frac{h_1^3 - h_2^3}{h_1^2 - h_2^2}$$

Fig. 350.

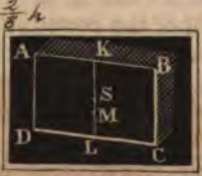


Fig. 351.

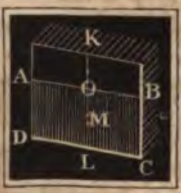
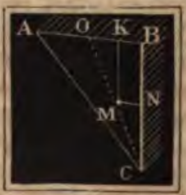


Fig. 352.



$$x_1 = \frac{\int_0^l y\,dx\,x^2}{\int_0^l y\,dx\,x} = \frac{\int \frac{b(l-x)}{l}x^2dx}{\int \frac{b(l-x)}{l}x\,dx} = \frac{\int (l-x)x^2dx}{\int (l-x)x\,dx} = \frac{\frac{l^4}{3}-\frac{l^4}{4}}{\frac{l^3}{2}-\frac{l^3}{3}} = \frac{\frac{l^4}{12}}{\frac{l^3}{6}} = \frac{1}{2}l$$

$$x_1 = \frac{\int_0^l y\,dx\,x^2}{\int_0^l y\,dx\,x} = \int \frac{\frac{bx}{l}x^2dx}{\frac{bx}{l}x\,dx} = \int \frac{x^3dx}{x^2dx} = \frac{\frac{l^4}{4}}{\frac{l^3}{3}} = \frac{3}{4}l$$

$AB = l_2$, we then have the distance KM of the centre of pressure from the fluid surface:

$$u = \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2}.$$

For the case of a right-angled triangle ABC, Fig. 352, whose base AB lies in the fluid surface, the distance KM of the centre of pressure M from AB (§ 223), $u = \frac{\frac{1}{6} F \cdot l^2}{\frac{1}{3} F \cdot l} = \frac{1}{2} l$, if l represent the height BC of the triangle, and the distance of the same point from the other leg, as this point in every case lies in the line CO bisecting the triangle, which passes from the point O to the middle point of the base, $NM = v = \frac{1}{4} b$, where b represents the base AB.

If the point C lies in the surface, as in Fig. 353, therefore, the base AB below this point, we have

$$KM = u = \frac{\frac{1}{2} Fl^2}{\frac{2}{3} Fl} = \frac{3}{4} l \text{ and } NM = v = \frac{3}{4} \cdot \frac{b}{2} = \frac{3}{8} b.$$

[illegible]

passing to the limits [illegible]

$$= \frac{l_1^2 + 2 l_1 l_2 + 3 l_2^2}{2(l_1 + 2 l_2)}$$

Fig. 353.

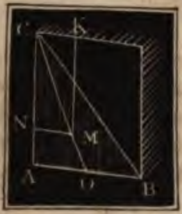
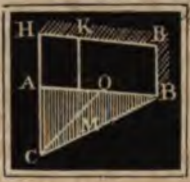


Fig. 354.



If the whole triangle ABC, Fig. 354, be under water, if the base AB is at a distance $AH = l_2$, and the point a distance $CH = l_1$ from the surface HR, we then have the distance MK from the surface HR:

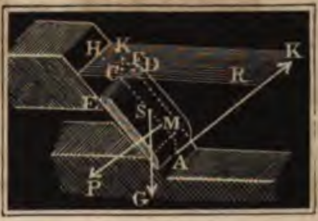
$$u = \frac{\frac{1}{18} F (l_1 - l_2)^2 + F \left(l_2 + \frac{l_1 - l_2}{3}\right)^2}{F \left(l_2 + \frac{l_1 - l_2}{3}\right)}$$

$$= \frac{\frac{1}{18} (l_1 - l_2)^2 + \frac{1}{9} (2 l_2 + l_1)^2}{\frac{1}{3} (2 l_2 + l_1)}$$

$$= \frac{l_1^2 + 2 l_1 l_2 + 3 l_2^2}{2 (l_1 + 2 l_2)}.$$

In a similar manner the centres of pressure may be determined for other figures.

Fig. 355.



Example. What force K must be expended to draw up a trap-door AC turning about an axis EF, Fig. 355? Let its length $CA = 1\frac{1}{4}$ feet, its breadth $EF = 1\frac{1}{2}$ feet, its weight $= 35$ lbs.; further, the distance CK of the axis of revolution C from the surface HR, measured in the plane of the door, $= 1$ foot, and the angle of inclination of this plane to the horizon $= 68°$.

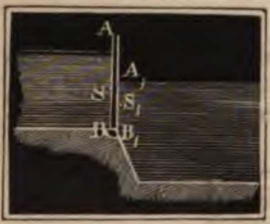
The pressed surface is $F = \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8}$ square feet, and the head of water or the depth of its centre of gravity below the surface, $h = HS \, sin. \, a = (HC + CS) \, sin. \, a = (HC + \frac{1}{2} AC) \, sin. \, a = \left(1 + \frac{1}{2} \cdot \frac{5}{4}\right) sin. \, 68° = \frac{13}{8} \, sin. \, 68° = \frac{13 \,.\, 0{,}92718}{8} = 1{,}5067$ feet; hence, the pressure of water on the surface is: $P = Fh\gamma = \frac{15}{8} \,.\, 1{,}5067 \,.\, 66 = 186{,}45$ lbs. The arm of this force about the axis of revolution is the distance CM of the centre of pressure M from this axis; therefore $= HM - HC$

$$= \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} - l_2 = \frac{2}{3} \cdot \frac{\left(\frac{9}{4}\right)^3 - \left(\frac{4}{4}\right)^3}{\left(\frac{9}{4}\right)^4 - \left(\frac{4}{4}\right)^2} - 1 = \frac{1}{6} \cdot \frac{729 - 64}{81 - 16} - 1$$

$= 0{,}705$ ft.; hence the statical moment of the pressure of water $= 186{,}45 \,.\, 0{,}705 = 131{,}46$ ft. lbs. If the centre of gravity S of the trap-door lies about half the length $CS = \frac{1}{2} \cdot \frac{5}{4} = \frac{5}{8}$ feet from the axes of revolution, the arm CD of the weight of the revolving door will be $= CS \, cos. \, a = \frac{5}{8} \,.\, cos. \, 68° = \frac{5}{8} \,.\, 0{,}3746 = 0{,}2341$ ft., and hence the statical moment of this weight $= 35 \,.\, 0{,}2341 = 8{,}19$ ft. lbs. By the addition of both moments, we obtain the whole moment for drawing up the trap-door $= 131{,}46 + 8{,}19 = 139{,}65$ ft. lbs.; and if the force K for this effect act at the arm $CA = 1{,}25$ feet, its amount will be $= \frac{139{,}65}{1{,}25} = 112$ lbs.

§ 279. If water presses against both sides of a plane surface AB, Fig. 356, there arises from the resultant forces corresponding to the two sides a new resultant, which is obtained by the subtraction of the former, because these two act oppositely to each other.

Fig. 356.

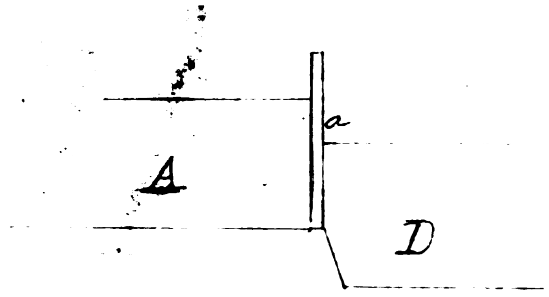


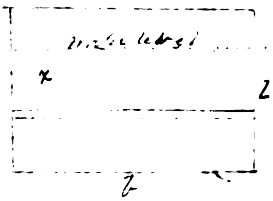
If F is the area of the pressed portion on the one side of the surface AB, and h the depth AS of its centre of gravity below the level of the water; further, F_1 the area of the portion $A_1 B_1$ on the other side of the surface, and h_1 the depth $A_1 S_1$ of its centre of gravity below the corresponding level of the water, we then have for the resultant sought, $P = Fh\gamma - F_1 h_1 \gamma = (Fh - F_1 h_1) \gamma$.

If the moment of inertia of the first portion of the fluid surface with respect to the line in which the plane of the surface intersects that of the water, $= Fx^2$, the statical moment of the pressure of water of the one side is, therefore, $= Fx^2 \,.\, \gamma$; if, further, the moment of inertia of the second portion with respect to the line of intersection with the second surface of water $= F_1 x_1^2$, the statical moment of the pressure of water of the other side about the axis lying on the second surface is then $= F_1 x_1^2 \gamma$. Further, if the distance AA_1 of the axes $= a$, we then obtain the augmentation of the last moment in its transit from the axis A_1 to the axis A, $= F_1 h_1 \, a \gamma$, and hence the statical moment of the pressure of water with respect to the axis in the first surface

$$= F_1 x_1^2 \gamma + F_1 h_1 \,.\, a \,.\, \gamma = (F_1 x_1^2 + F_1 a h_1) \gamma.$$

From this, then, it follows that the statical moment of the difference



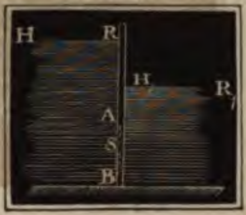


Let us consider the case where we have a plane rectangular area submerged by equal depths on either side; we wish to find the point of application of the resultant pressure in this case; suppose our two figures to represent a side and an end elevation of the partially submerged rectangle; call b its breadth and l the vertical distance between the water level and the lower base on the A side of the submerged figure; let x represent the distance of ~~the~~ an elementary strip below the water level; we have the elementary surface = bdx then the elementary pressure, the pressure upon

of both mean pressures $= (Fx^2 - F_1 x_1^2 - a\, F_1 h_1)\, \gamma$, and the arm of this latter force, or the distance of the centre of pressure from the axis in the first surface of water is:

$$u = \frac{Fx^2 - F_1 x_1^2 - a\, F_1 h_1}{Fh - F_1 h_1}$$

Fig. 357.



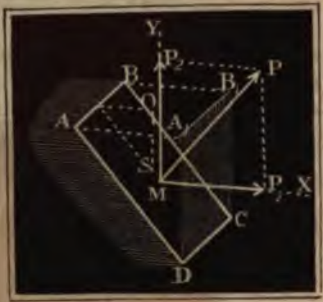
If the portions of surface pressed are equal to one another, which takes place when, as Fig. 357 represents, the entire surface *AB* is below the water, we have then more simply $P = F\,(h - h_1)\,\gamma$ and $u = h$; the last, because $h - h_1 = a$, and $x_1^2 = x^2 - 2\,a\,h + a^2$ (§ 217). In the last case, therefore, the pressure is equivalent to the weight of a column of water, whose base is the surface pressed, and whose height is the difference of altitude $R H_1$ of both surfaces of water, and the centre of pressure coincides with the centre of gravity S of the surface. This law is also further correct if both surfaces of water are besides further pressed by equal forces, for example, by a piston or by the atmosphere. For this pressure upon each unit of surface $= p$, and therefore the corresponding height of a column of water $x = \frac{p}{\gamma}$ (§ 275), we have then to substitute for h, $h + x$, and for h_1, $h_1 + x$; and by subtraction, we have the residuary force $P = (h + x - [h_1 + x])\, F\gamma = (h - h_1)\, F\gamma$. For this reason, the pressure of the atmosphere in hydrostatic investigations is generally left out of consideration.

Example. The height *AB* of the upper surface of water in a canal, Fig. 356, amounts to 7 feet, the water in the lock stands 4 feet high at the sluice-gate, and the breadth of the canal and of the lock measure 7,5 feet, what mean pressure has the sluice-gate to sustain? It is $F = 7\,.\,7{,}5 = 52{,}5$, and $F_1 = 4\,.\,7{,}5 = 30$ square feet. Further, $h = \frac{1}{2}\,.\,7 = \frac{7}{2}$ and $h_1 = \frac{4}{2} = 2$ feet, $a = 7 - 4 = 3$ feet, $x^2 = \frac{1}{3}\,.\,7^2 = \frac{49}{3}$ and $x_1^2 = \frac{1}{3}\,.\,4^2 = \frac{16}{3}$; hence it follows, that the mean pressure sought is: $P = (Fh - F_1 h_1)\,\gamma$ $= (52{,}5\,.\,\frac{7}{2} - 30\,.\,2)\,.\,62{,}5 = 123{,}75\,.\,62{,}5 = 7734{,}375$ lbs.; and the depth of its point of application below the surface of the water is:

$$u = \frac{52{,}5\,.\,\frac{49}{3} - 30\,.\,\frac{16}{3} - 3\,.\,60}{52{,}5\,.\,\frac{7}{2} - 60} = \frac{517{,}5}{123{,}75} = 4{,}182 \text{ feet.}$$

§ 280. *Pressure in a Definite Direction.*—In many cases it is of importance to know only one part of the pressure acting in a definite direction upon a surface. In order to find this component, we resolve the normal pressure $MP = P$ of the surface $AC = F$, Fig. 358, in the given direction MX, and in the direction MY perpendicular to it into two component pressures $MP_1 = P_1$, and $MP_2 = P_2$. Let a be the angle PMX, which the normal in the given direction MX makes with the component, we shall then obtain for the components; $P_1 = P$

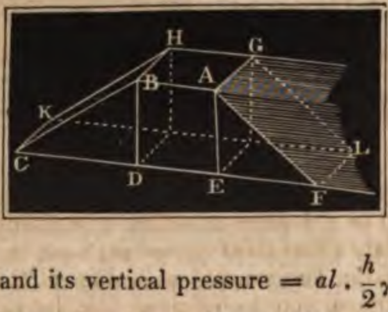
Fig. 358.



$\cos. \alpha$ and $P_2 = P \sin. \alpha$. Let a projection $A_1 B_1 CD$ of the surface AB be made on a plane at right angles to the given direction MX, we shall then have for its area F_1, the formula $F_1 = F \,.\, \cos. ADA_1$, or since the angle of inclination ADA_1 of the surface from its projection is equal to the angle $PMX = \alpha$ between the normal pressure P and its component P_1, we then have $F_1 = F \cos. \alpha$, or inversely: $\cos. \alpha = \frac{F_1}{F}$, and hence $P_1 = P \,.\, \frac{F_1}{F}$. But as the normal pressure $P = Fh\gamma$, it follows finally that $P_1 = F_1 h\gamma$, *i. e. the pressure with which water presses against a surface in a given direction, is equal to the weight of a column of water which has for base the projection of the surface perpendicular to the given direction, and for height, the depth of the centre of gravity of the surface below that of the water.*

It is important, in most cases of application, to know only the vertical or the horizontal component of the pressure of water against a surface. Since the projection at right angles to the vertical direction is the horizontal, and the projection at right angles to the horizontal direction, a vertical projection, the vertical pressure of water against a surface may be found, if the horizontal projection or its trace be considered as the surface pressed, and on the other hand the horizontal pressure of the water in any direction may be also found, if the vertical projection or the elevation of the surface at right angles to the given direction be considered as the surface pressed, but in both cases the depth of the centre of gravity of the surface below that of the water taken as the head of water.

Fig. 359.



For a prismatic dam ACH, Fig. 359, the longitudinal profile EG for the horizontal pressure of the water and the horizontal projection EL of the surface of water for the vertical pressure must be regarded as the surfaces pressed. Hence, if the length AG of the dam $= l$, the height $AE = h$, and the front slope $EF = a$, we have then the horizontal pressure of the water $= lh \,.\, \frac{h}{2}\gamma = \frac{1}{2} h^2 l\gamma$, and its vertical pressure $= al \,.\, \frac{h}{2}\gamma = \frac{1}{2}\, alh\gamma$. If now, further, the upper breadth of the top of the dam $= b$, the slope at the back $CD = a_1$, and the density of the mass of the dam $= \gamma_1$, we then have the weight of the dam $= \left(b + \frac{a + a_1}{2}\right) hl\gamma_1$, and the whole vertical pressure of this against the horizontal bottom

$$= \frac{1}{2} alh\gamma + \left(b + \frac{a + a_1}{2}\right) hl\gamma_1 = \left[\frac{1}{2} a\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] hl.$$

If we put the co-efficient of friction $= f$, then the friction or force to push the dam forward is:

$$F = \left[\frac{1}{2} a\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] fhl.$$

In the case where the horizontal pressure of the water is to effect this displacement, we have:

$$\frac{1}{2} h^2 l\gamma = \left[\frac{1}{2} a\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] fhl, \text{ or more simply:}$$

$$h = f\left[a + \left(2b + a + a_1\right)\frac{\gamma_1}{\gamma}\right].$$

Therefore, in order that the dam may not be pushed away by the water, we must have:

$$h < f\left[a + \left(2b + a + a_1\right)\frac{\gamma_1}{\gamma}\right], \text{ or,}$$

$$b > \frac{1}{2}\left[\left(\frac{h}{f} - a\right)\frac{\gamma}{\gamma_1} - \left(a + a_1\right)\right].$$

For safety we assume that the base of the dam is quite permeable, on which account there is further a counter pressure from below upwards $= (b + a + a_1)\, lh\gamma$ to abstract, and we may put

$$h < f\left[\left(2b + a + a_1\right)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right].$$

//?

Example. The density of the mass of a clay dam is nearly twice as great as that of water, therefore, $\frac{\gamma_1}{\gamma} = 2$ and $\frac{\gamma_1}{\gamma} - 1 = 1$; hence, for such a dam we may put simply $h > f(2b + a)$. According to experience, a dam will resist a long time if its height, slope and breadth at the top are equal to one another; if in the last formula we put $h = b = a$, then $f = \frac{1}{3}$, whence we must in other cases put:—
$h = \frac{1}{3}\left[(2b + a + a_1)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right]$, and for clay dams especially, $h = \frac{1}{3}(2b + a)$, and inversely, $b = \frac{3h - a}{2}$. If the height of the dam be 20 feet, and the angle of slope $\alpha = 36°$, the slope a will be
$= h \operatorname{cotg.} \alpha = 20 \,.\, \operatorname{cotg.} 36° = 20 \,.\, 1{,}3764 = 27{,}53$ feet,
and hence the upper breadth of the dam $b = \frac{60 - 27{,}53}{2} = 16{,}24$ feet.

§ 281. *Pressure on Curved Surfaces.*—The law found in the last paragraph on the pressure of water in a definite direction is true only for plane surfaces, or for the separate elements of curved surfaces, but not for curved surfaces in general. The normal pressures on the separate elements of a curved surface may be resolved into lateral components parallel to a given direction, and into others acting in the plane normal to it; these components form a system of parallel forces, whose resultant gives the pressure in the given direction, and these components may be reduced to a resultant, but the two resultants admit of no further composition when their directions do not intersect. It is not possible in general to reduce the aggregate pressures against the elements of a curved surface to a single force, but particular cases present themselves where this composition is possible.

Let G_1, G_2, G_3, &c., be the projections, and h_1, h_2, h_3, &c., the heads of water of the elements F_1, F_2, F_3, &c., of a curved surface, we then have the pressure of water in the direction perpendicular to the plane of projection:

$$P_1 = (G_1 h_1 + G_2 h_2 + G_3 h_3 + \ldots)\gamma,$$

and its moment with respect to the plane of the surface of water.

$$P_1 u = (G_1 h_1^2 + G_2 h_2^2 + G_3 h_3^2 + \ldots)\gamma.$$

If the curved surface pressed upon can be decomposed into elements which have a uniform ratio to their projections, we may then put

$$\frac{F_1}{G_1} = \frac{F_2}{G_2} = \frac{F_3}{G_3}, \text{ \&c.}, = n, \text{ we then have:}$$

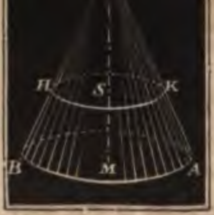
$$P_1 = \left(\frac{F_1 h_1}{n} + \frac{F_2 h_2}{n} + \ldots\right)\gamma = \left(\frac{F_1 h_1 + F_2 h_2 + \ldots}{n}\right)\gamma,$$

or, since the ratio of the entire curved surface F to its projection G, *i. e.* $\frac{F}{G}$ is $= n$,

$P_1 = \frac{Fh}{n}\gamma = Gh\gamma$; in this case we have, as for every plane surface, the pressure in any direction equivalent to the weight of a prism of water, whose basic surface is at right angles to the projection of he curved surface in the given direction, and whose height is equal to the depth of the centre of gravity of the curved surface below the surface of water.

So, for example, the vertical pressure of water against the envelope of a conical vessel ACB, filled with water, Fig. 360, is equal to the weight of a column of water which has the bottom for its base, and two-thirds of the length of the axis CM for height, because the horizontal projection of the envelope of a right cone upon its base, as likewise the envelope, may be resolved into exactly similar triangular elements, and because the centre of gravity S of the surface of the cone is distant two-thirds of the height of the cone from the vertex (§ 110). If r be the radius of the base, and h the height of the cone, we shall then have the pressure against the bottom $= \pi r^2 h\gamma$, and the vertical pressure against the envelope $= \frac{2}{3}\pi r^2 h\gamma$, but as the bottom is rigidly connected with the sides, and both pressures act opposed to each other, the force with which the vessel is pressed downwards by the water is:

Fig. 360.

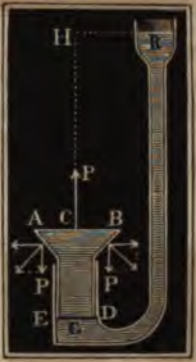


$$= \left(1 - \frac{2}{3}\right) \pi r^2 h \gamma = \frac{1}{3} \pi r^2 h \gamma =$$

to the weight of the whole mass of water. If the bottom be separated by a fine cut from the envelope, this will then press with its full force $\pi r^2 h \gamma$ downwards, or on its support, and on the other hand it would be necessary to hold down the envelope with a force $\frac{2}{3} \pi r^2 h \gamma$ to prevent its being raised off.

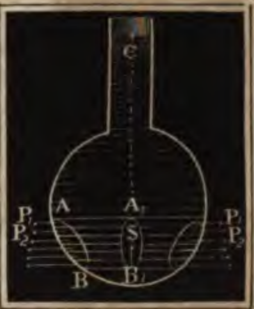
Remark. From this the force which the steam of a steam-engine or the water of a water-column machine exerts on the piston, is independent of the form of the piston. Whether the surface of pressure be augmented by being hollowed out or rounded, the pressure with which the steam or water pushes forward the piston is equivalent to the product of the cross section or horizontal projection of the piston and the pressure on a unit of surface. The pressure on the larger surface of a funnel-shaped piston AB, Fig. 361, whose greater radius $CA = CB = r$ and lesser radius $GD = GE = r_1$, is $= \pi r^2 p$, and the reaction upon the envelope $= \pi (r^2 - r_1^2) p$; hence, the residuary effective pressure is $= \pi r^2 p - \pi (r^2 - r_1^2) p = \pi r_1^2 p$ = the cross section of the cylinder multiplied by the pressure on a unit of surface.

Fig. 361.



§ 282. *Horizontal and Vertical Pressure.*—Whatever may be the form of a curved surface, AB, Fig. 362, the horizontal pressure of the water against it is always equivalent to the weight of a column of water, whose base is the vertical projection A_1B_1 of the surface perpendicular to the given direction of pressure, and whose height of pressure is the depth CS of the centre of gravity S of the projection below the surface of water. The correctness of this follows directly from the formula $P_1 = (G_1h_1 + G_2h_2 + \ldots) \gamma$, when we consider that the height of pressure h_1, h_2, &c., of the elements of the surface are also the heights of pressure of their projections, that, therefore, $G_1h_1 + G_2h_2 + \ldots$ is the statical moment of the whole projection, *i. e.* the product Gh of the vertical projection G and the depth h of its centre of gravity below the surface of water. We have here, therefore, again to put $P_1 = Gh\gamma$, and to consider h as the height of pressure of the vertical projection.

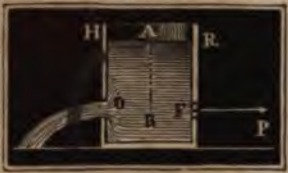
Fig. 362.



The vertical section which divides a vessel containing water into two equal or unequal parts, is at once the vertical projection of the two parts, but the horizontal pressure on one part of the wall of the vessel is proportional to the product of its vertical projection and to the depth of its centre of gravity below the surface of the water, consequently the horizontal pressure on a part of the wall of the vessel is exactly equal in amount to the oppositely acting horizontal pressure on the part opposite, and consequently the two forces balance each other in the vessel; the whole vessel is therefore equally pressed by the enclosed water in all horizontal directions. ?

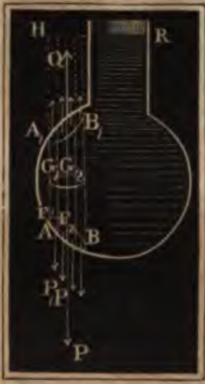
If an opening O be made in the side of a vessel HBR, Fig. 363, the part of the pressure corresponding to the section of this opening disappears, and the pressure on the oppositely situated portion of the surface F now comes into action. Whilst, therefore, the water flows out at the lateral aperture, an equal distribution of the horizontal pressure no longer takes place over the whole extent, and there ensues a reaction opposite to the motion of the flowing water: $P = Fh\gamma$, F being the projection of the aperture, and h the height of pressure of its projection. By this reaction the vessel may be set into motion.

Fig. 363.



The vertical pressure of water is $P_1 = G_1 h_1 \gamma$ against an element of surface F_1, Fig. 364, of the side of the vessel, since the horizontal projection G_1 may be regarded as the transverse section, and the height of pressure h_1 as the height, and therefore $G_1 h_1$ as the volume of a prism, equivalent to the weight of a column of water HF_1 incumbent on the element, and reaching the surface of the water. The elements of the surface which make up a finite part of the bottom, or side of the vessel, hence suffer a vertical pressure which is equivalent to the weight of all the incumbent columns of water, *i. e.* to the weight of a column of water incumbent on the whole portion. Let this volume $= V_1$, we then obtain for the vertical pressure $P = V_1 \gamma$. For another portion $A_1 B_1$, which lies vertically above the former, we have the vertical pressure opposed to it $Q = V_2 \gamma$; but if both portions are rigidly connected with each other, there results from the two forces the force acting vertically downwards $R = (V_1 - V_2)\gamma = V\gamma =$ *to the weight of the columns of water contained between the two portions of the surface.* If, lastly, we apply this law to the whole vessel, it follows that *the aggregate vertical pressure of the water against the vessel is equivalent to the weight of the enclosed mass of water.*

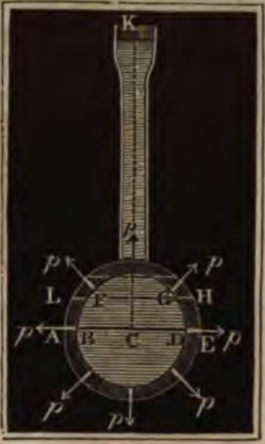
Fig. 364.



§ 283. *Thickness of Pipes.*—The application of the laws of the pressure of water to pipes, boilers, &c., is of particular importance. That these vessels may adequately resist the pressure, and be prevented bursting from its effect, we must give a certain thickness to their sides, corresponding to the head of water and the internal width. The bursting of a pipe may take place in various ways, viz., transversely or longitudinally; the latter happens more frequently than the former, as will be soon understood from what follows.

The width of the pipe $BD = 2r$, Fig. 365, and the head of water

Fig. 365.



$CK = h$, therefore the pressure on a unit of surface $p = h\gamma$, we then have the whole pressure in the direction of the axis of the pipe $= \pi r^2 p = \pi r^2 h\gamma$; if the thickness of the side $AB = DE = e$, we then have the transverse section of the mass of the pipe $= \pi\,(r+e)^2 - \pi r^2 = 2\,\pi r e + \pi e^2 = 2\,\pi r e \left(1 + \frac{e}{2r}\right)$, and if lastly we put the modulus of elasticity $= K$, we then have the pressure for rupture over the whole section of the pipe

$$= 2\,\pi r e \left(1 + \frac{e}{2r}\right) K,$$

for this reason we have now to put:

$$2\,\pi r e \left(1 + \frac{e}{2r}\right) K = \pi r^2 p,$$

or approximately and more simply $2\,e\,K = rp$, and hence the thickness of the pipe $e = \frac{r\,p}{2\,K} = \frac{ph\gamma}{2\,K}$. In order, therefore, to avoid any transverse rent in the pipe or in the boiler, the thickness of the sides must be made $e > \frac{r\,p}{2\,K}$. Of all longitudinal rents, AE, LH, &c., those running diametrically, such as AE, take place the most easily, because they have the smallest area, whence we must only take these into account. Let us consider a portion of a pipe of the length l, and let us have regard to the occurrence of a rent of the length l, we then obtain a transverse section of the surface of separation $= le$, and hence the force for rupture in this surface leK. For two oppositely situated rents this force is $= 2\,leK$, whilst the pressure of water for each half of the pipe is proportional to the transverse section $2\,rl$, and hence is $= 2\,rlp$. By equating the two expressions, it follows that $2\,leK = 2\,rlp$, i. e. $e\,K = rp$, therefore the thickness $e = \frac{rp}{K}$. To provide against longitudinal rents, the sides must be made as thick again, as to provide against transverse rents.

From the formula $e = \frac{rp}{K} = \frac{rh\gamma}{K}$, it follows that *the strength of similar pipes is as the widths and as the heads of water or pressures upon a unit of surface.* A pipe three times the width of another, which has five times the pressure to sustain on each unit of surface that the other has, must have its sides fifteen times as thick.

Hollow spheres, which have to sustain a pressure p from within on each unit of surface, require a thickness $e = \frac{rp}{2\,K}$, because here the projection of the surface of pressure is the greatest circle πr^2, and the surface of separation the ring $2\,\pi r e \left(1 + \frac{e}{2r}\right)$, or approximately for a smaller thickness $= 2\,\pi r e$.

The formula found give for $p = o$, also $e = o$, for this reason, therefore, pipes which have no internal pressure to sustain, may be made indefinitely thin; but as each pipe must sustain a certain pressure from its own weight, we must still give to it a certain thickness e_1, to obtain the strength of a tube which will resist under all circumstances. Hence, for cylindrical pipes or boilers we must put $e = e_1 + \frac{rh\gamma}{K}$, or more simply, if d represents the interior diameter of the pipe, n the pressure in atmospheres, each corresponding to a column of water 33 ft. high, and μ a number from experiment $e = e_1 + \mu nd$.

From experiments made we must take for pipes of

Iron plate	$e = 0{,}00086\, nd + 0{,}12$ inches
Cast iron	$e = 0{,}00238\, nd + 0{,}33$ "
Copper	$e = 0{,}00148\, nd + 0{,}16$ "
Lead	$e = 0{,}00242\, nd + 0{,}20$ "
Zinc	$e = 0{,}00507\, nd + 0{,}16$ "
Wood	$e = 0{,}0323\, nd + 1{,}04$ "
Natural stones	$e = 0{,}0369\, nd + 1{,}15$ "
Artificial stones	$e = 0{,}0538\, nd + 1{,}53$ "

Example. If a perpendicular water-column machine has cast-iron pipes of 10 inches inner width, how thick must these be at 100, 200, and 300 feet depths? From the formula, for 100 feet pressure, this thickness is:

$$= 0{,}00238 \,.\, \frac{100}{33} \,.\, 10 + 0{,}33 = 0{,}07 + 0{,}33 = 0{,}40 \text{ inches};$$

for 200 feet, $= 0{,}14 + 0{,}33 = 0{,}47$ inches; and for 300 feet pressure, $= 0{,}22 + 0{,}33 = 0{,}55$ inches. Cast iron conducting pipes are commonly proved at 10 atmospheres, for which reason, $e = 0{,}0238 \,.\, d + 0{,}33$ inches; therefore, for pipes of 10 inches width, the thickness $e = 0{,}24 + 0{,}33 = 0{,}57$ inches must be given.

Remarks. The thickness of the sides of steam-boilers will be considered in the Second Part. Concerning the theory of the strength of pipes, a treatise by Brix, in the "Verhandlungen des Vereins zur Beförderung des Gewerbfleiszes in Preuszen," Jahrgang, 1834, may be consulted. The technical relations and the proving of pipes are fully treated of in Hagen's "Handbuch der Wasserbaukunst," vol. i., and in Genieys' "Essai sur les moyens de conduire, &c., les eaux."

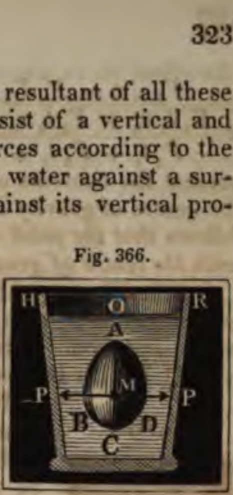
[For a view of the general principles governing the construction and strength of cylindrical steam-boilers, the editor may refer to his paper on that subject read before the Franklin Institute, July 26, 1832, and published in its Journal, in which the relation stated in the text, between the strength required in the direction of the curvature and that in the direction of the length of the tube or boiler, was pointed out, accompanied by a table of diameters and thicknesses of boilers, with the tenacities per inch of iron required in each direction for a given pressure. See, likewise, American Journal of Science and Arts, vol. xxiii. No. 1.]

CHAPTER II.

ON THE EQUILIBRIUM OF WATER WITH OTHER BODIES.

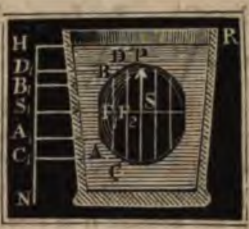
§ 284. *Buoyancy.*—A body immersed under water is pressed upon by the water on all sides, and now the question arises as to the amount, direction and point of application of the resultant of all these pressures. Let us imagine this resultant to consist of a vertical and a horizontal component, and determine these forces according to the rules of § 282. The horizontal pressure of the water against a surface is equivalent to the horizontal pressure against its vertical projection, but now every projection of a body, AC, Fig. 366, is at the same time the projection of the fore part ADC and the back part ABC of its surface; hence, also, the horizontal pressure of water against the back portion of the surface of a body is equal in amount to that of the front portion, and as both pressures are exactly opposite, their resultant $= 0$. As this relation takes place for every arbitrary horizontal direction, and the vertical projection corresponding to this, it follows that the resultant of all the horizontal pressures is nothing; that, therefore, the body AC below the water is equally pressed in all horizontal directions, and for this reason exerts no effort to move forward in a horizontal direction.

Fig. 366.



To find the vertical pressure of the water against the body BCS, Fig. 367, let us suppose it made up of vertical elementary prisms, AB, CD, &c., and determine the vertical pressures on their terminating surfaces A and B, C and D. If the lenghts of these prisms are l_1, l_2, &c., the depths of their upper extremities B, D, &c., below the surface of water HR : h_1, h_2, &c., and the horizontal transverse sections $F_1 F_2$, &c., we then have for the vertical pressures acting from above downwards against the extremities, B, D, &c., $= F_1 h_1 \gamma$, $F_2 h_2 \gamma$, &c.; on the other hand, the pressures acting from below upwards and against the extremities A, C, &c., $= F_1 (h_1 + l_1)\gamma$, $F_2 (h_2 + l_2)\gamma$, &c.; and it now follows, from a composition of these parallel forces, that the resultant P

Fig. 367.



$$= F_1 (h_1 + l_1)\gamma + F_2 (h_2 + l_2)\gamma + \ldots - F_1 h_1 \gamma - F_2 h_2 \gamma - \ldots$$
$$= (F_1 l_1 + F_2 l_2 + \ldots)\gamma = V\gamma,$$

if V represents the volume of the immersed body or the water displaced.

Therefore the buoyancy or the force with which the water strives to push a body immersed from below upwards, is equivalent to the weight of water displaced, or to a quantity of water which has the same volume as the submerged body.

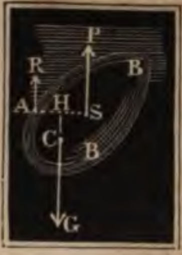
Further, to find the point of application of this resultant, let us put the distances AA_1, CC_1, &c., of the elementary columns AB, CD, &c., from a vertical plane HN : a_1, a_2, &c., and determine the moments of the forces with respect to this plane. If S is the point of application of the upward pressure, and $SS_1 = x$ its distance from that principal plane, we shall then have:

$$V\gamma \, . \, x = F_1 \, l_{1}\gamma \, . \, a_1 + F_2 \, l_{2}\gamma \, . \, a_2 + \, . \, . \, ;$$ and hence,

$$x \, \frac{F_1 \, l_1 \, a_1 + F_2 \, l_2 \, a_2 + \, . \, .}{F_1 \, l_1 + F_2 \, l_2 + \, . \, .} = \frac{V_1 \, a_1 + V_2 \, a_2 + \, . \, .}{V_1 + V_2 + \, . \, .},$$ if V_1, V_2, &c., represent the contents of the elementary columns. Since (from § 100) the centre of gravity is accurately determined by the same formula, it follows *that the point of application S of the upward pressure coincides with the centre of gravity of the water displaced.*

§ 285. The weight G of the body acting in an opposite direction associates itself with the buoyancy of the body immersed or under water, and from the two there arises a resultant $R = G - V\gamma$ or $= (\varepsilon - 1) \, V\gamma$, if ε be the specific gravity of the body.

Fig. 368.

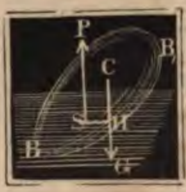


If the mass of the body be homogeneous, the centre of gravity of the displaced water will coincide with that of the body, and hence this point will be the point of application of the resultant R; but if there be not homogeneity, then these centres of gravity do not coincide, and the point of application of the resultant R deviates from both centres of gravity. Let us put the horizontal distance SH, Fig. 368, of both centres of gravity from each other, $= b$, and the horizontal distance SA of the point of application A sought from the centre of gravity S of the displaced water $= a$, we shall have the equation $Gb = Ra$, from which is given:

$$a = \frac{G \, b}{R} = \frac{G \, b}{G - P}.$$

If the immersed body be left to its own gravity, the three following cases may present themselves. Either the specific gravity of the body is equal to that of the water, or it is greater, or it is less than the specific gravity of the water. In the first case the buoyancy is equal, in the second it is less, and in the third it is greater than the weight of the water. Whilst, in the first case, equilibrium subsists between the weight and the buoyancy, the body must in the second case sink with the force $G - V\gamma = (\varepsilon - 1) \, V\gamma$, and, in the third case, rise with the force $V\gamma - G = (1 - \varepsilon) \, V\gamma$. The rising goes on only as long as the mass of water V_1, cut off from the plane of the surface and displaced by the body, has the same weight as the entire body. The weight $G = V \varepsilon \gamma$ of the body BB_1, Fig. 369, and the buoyancy $P = V_1\gamma$ now constitute a couple, by which the body is made to revolve until the directions of both coincide, or until the centre of gravity of the body lies in one and the same vertical line with the centre of gravity of the displaced water.

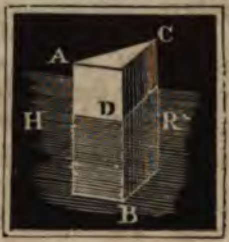
Fig. 369.



The line passing through the centre of gravity of the floating body and through that of the displaced water, is called *the axis of floatation;* and on the other hand, the section of the body formed by the plane of the surface of the water, *the plane of floatation.* Every plane which divides a body, so that one part is to the whole as the specific gravity of the body to that of the fluid, and that the centres of gravity of the two parts lie in a line normal to this plane, is a plane of floatation of the body.

§ 286. *Depth of Floatation.* If the figure and weight of a floating body be known, the depth of immersion may be calculated beforehand, with the help of the previous rule. If G be the weight of the body, we may then put the volume of the displaced water $V = \frac{G}{\gamma}$; if we combine with it the ? stereometrical formula for the volume V, we shall obtain the equation of condition. Hence, for the prism ABC, Fig. 370, with vertical axis, for example, $V = Fy$, if F represent the section and y the depth BD of immersion, $Fy = \frac{G}{\gamma}$ and $y = \frac{G}{F\gamma}$. For a pyramid ABC, Fig. 371, whose vertex floats under the water, $V = \frac{1}{3} f y^3$, if f represents the section at the distance of unity from the vertex; hence it follows, that:

Fig. 370.



$$\tfrac{1}{3} f y^3 = \frac{G}{\gamma}, \text{ and hence the depth } CE = y = \sqrt[3]{\frac{3\,G}{f\,\gamma}}.$$

Fig. 371.

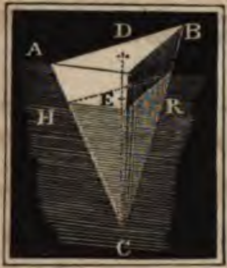
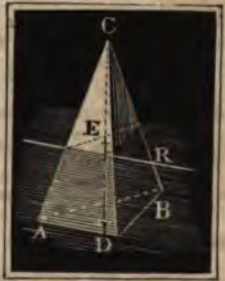


Fig. 372.



For a pyramid ABC, Fig. 372, floating with its base below the water, the distance is given $CE = y_1$ of the vertex from the surface, from the height h of the entire pyramid, if we put:

$$V = \tfrac{1}{3} f (h^3 - y_1^3),\; y_1 = \sqrt{h^3 - \frac{3\,G}{f\gamma}}.$$

For a sphere AB, Fig. 373, with the radius $CA = r$,

Fig. 373.

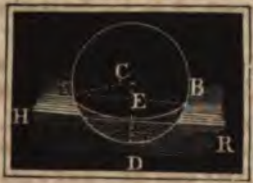
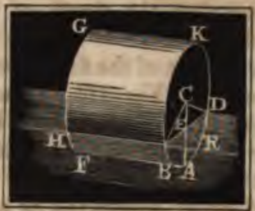


Fig. 374.



$V = \pi\, y^2\left(r - \frac{y}{3}\right)$, hence we shall have to solve the cubic equation $y^3 - 3\, r\, y^2 + \frac{3\, G}{\pi\gamma} = 0$, to find the depth of immersion *DE* of the sphere.

For a floating cylinder *AG*, with horizontal axis, Fig. 374, of a radius $BC = DC = r$, if α^0 be the angle *BCD* subtended at the centre by the arc immersed, the depth of immersion $AE = y = r\,(1 - cos.\ \frac{1}{2}\,\alpha)$, but to find the arc immersed, we must put the volume of the water displaced = to the segment $\frac{r^2\alpha}{2}$ less the triangle $\frac{r^2\, sin.\ \alpha}{2}$, multiplied by the length $GK = l$ of the cylinder; therefore, $(\alpha - sin.\ \alpha)\,\frac{lr^2}{2} = \frac{G}{\gamma}$, and solve the equation $\alpha - sin.\ \alpha = \frac{2\,G}{lr^2\gamma}$, by approximation, with respect to α.

Examples.—1. A wooden sphere, of 10 inches diameter, floats 4½ inches deep, the volume of water displaced by it is then:

$$V = \pi\left(\frac{9}{2}\right)^2\left(5 - \frac{9}{6}\right) = \frac{\pi\,.\,81\,.\,7}{8} = \frac{567\,.\,\pi}{8} = 222{,}66 \text{ cubic inches,}$$

whilst the solid contents of the sphere are $\frac{\pi\, d^3}{6} = \frac{\pi\,.\,10^3}{6} = 523{,}6$ cubic inches.

From this, 523,6 cubic inches of the mass of the sphere weigh as much as 222,66 cubic inches of water, and it follows that the specific gravity of the former is:

$$s = \frac{222{,}66}{523{,}6} = 0{,}425\,..$$

2. How deep will a wooden cylinder of 10 inches diameter and specific gravity $s = 0{,}425$ sink? $\frac{\alpha - sin.\ \alpha}{2} = \frac{\pi\, r^2\, l\,.\,s\gamma}{l\, r^2\,\gamma} = \pi\, s = 0{,}425\,.\,\pi = 1{,}3352$; now a table of segments gives for the area $\frac{\alpha - sin.\ \alpha}{2} = 1{,}32766$ of a circular segment, the angle subtended at the centre by the arc $\alpha^0 = 166^\circ$, and for $\frac{\alpha - sin.\ \alpha}{2} = 1{,}34487$, the same angle $= 167^\circ$; hence, simply, the angle subtended at the centre corresponding to the slice 1,3352 is:

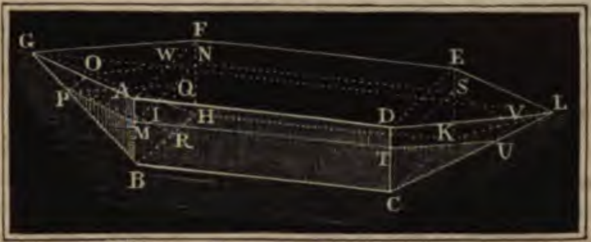
$$\alpha^0 = 166^\circ + \frac{1{,}3352 - 1{,}32766}{1{,}34487 - 1{,}32766}\,.\,1^\circ = 166^\circ + \frac{754^\circ}{1721} = 166^\circ\, 26'$$; therefore the depth of immersion:

$$y = r\,(1 - cos.\ \tfrac{1}{2}\,\alpha) = 5\,(1 - cos.\ 83^\circ\, 13') = 5\,.\,0{,}8819 = 4{,}41 \text{ inches.}$$

§ 287. The determination of the depth of immersion occurs chiefly in the case of ships, boats, &c. If these have a regular form, the depth may be calculated from geometrical formulæ; but if this regularity fails, or the law of configuration is not known, or if the form is very complex, the depth of immersion must then be determined by experiment.

An example of the first case is in the body *ACLEG* (a pointed scow), represented in Fig. 375, bounded by plane surfaces. It consists of a parallelopiped *ACE*, and of two four-sided pyramids *BFG* and *CEL*, forming the head and the stern, and its plane of floatation is composed of a parallelogram *MS*, and two trapeziums *MO* and *SU*, and cuts off a bulk of water, consisting of a parallelopiped *MCS*, and two triangular prisms PNR, and two quadrilateral pyramids BQP. If we put the length AD of the middle portion $= l$, the breadth $AF = b$, and the

Fig. 375.



depth $AB = h$; further, the length GW of each of the two ends $= c$, and the depth of immersion, *i. e.*, $BM = CT = y$, the immersed part MCS of the middle portion will be: $= \overline{MN} \times \overline{MT} \times \overline{MB} = lby$. The base of the quadrilateral pyramid BQP is $\overline{BM} . \overline{BR}$, and the height PJ, hence the solid contents of this pyramid $= \frac{1}{3} \overline{BM} . \overline{BR} . \overline{PJ}$. But now:

$$BM = y,\ BR = \frac{BP}{BG} . BH = \frac{BM}{BA} . BH = \frac{y}{h} . b = \frac{by}{h},$$

and likewise:

$$PJ = \frac{BM}{BA} . GW = \frac{y}{h} c = \frac{cy}{h},$$

hence the contents of both pyramids are:

$$= 2 . \tfrac{1}{3} . y . \frac{by}{h} . \frac{cy}{h} = \tfrac{2}{3} \frac{bcy^3}{h^2}.$$

The transverse section of the triangular prism

$$RNO \text{ is} = \tfrac{1}{2} \overline{RQ} . \overline{PJ} = \tfrac{1}{2} y . \frac{cy}{h} = \frac{cy^2}{2h}, \text{ and the side}$$

$$RH = QN = b - \frac{by}{h} = b\left(1 - \frac{y}{h}\right),$$

hence the solid contents of both prisms are:

$$= 2 . \frac{cy^2}{2h} . b \left(1 - \frac{y}{h}\right) = \frac{bcy^2}{h}\left(1 - \frac{y}{h}\right).$$

By addition of the three volumes found, the volume of the water displaced is known:

$$V = bly + \tfrac{2}{3} \frac{bcy^3}{h^2} + \frac{bcy^2}{h} - \frac{bcy^3}{h^2} = \left(l + \frac{cy}{h} - \tfrac{1}{3} . \frac{cy^2}{h^2}\right) by.$$

Now the gross weight of the boat $= G$, we then have to put:

$$\left(l + \frac{cy}{h} - \tfrac{1}{3} . \frac{cy^2}{h^2}\right) by\gamma = G, \text{ or,}$$

$$y^3 - 3hy^2 - \frac{3lh^2}{c} . y + \frac{3h^2G}{bc\gamma} = 0.$$

The depth of immersion y is determined from the loading by the solution of the last cubic equation.

Examples.—1. If the length of the middle portion $l = 50$ feet, the length of each end $c = 15$ feet, the breadth $b = 12$ feet, and the depth $h = 4$ feet, with a depth of immersion $y = 2$ feet, the whole weight amounts to

$G = [50 + 15 \,.\, \frac{2}{4} - \frac{1}{3} \,.\, 15 \,.\, (\frac{2}{4})^2] \,.\, 12 \,.\, 2 \,.\, 62{,}5 = (50 + 7{,}5 - 1{,}25) \,.\, 24 \,.\, 62{,}5 = 87235$ lbs.

2. If the clear weight of the former boat amount to 50000 lbs., we shall have for the depth of immersion: $y^3 - 12y^2 - 160y + 202{,}02 = 0$. By trial, it is easily found that this equation may be answered pretty accurately by $y = 1{,}17$, whence the depth of immersion sought may be taken as great.

Remark. To know the weight of the load of a ship, a scale is attached to both sides, which is called a water-gauge. The divisions are made from experiment, while it is observed what loads correspond to definite immersions.

§ 288. *Stability.*—The floating of bodies takes place either in an upright or an oblique position; and further, with or without stability. A body, a ship, for example, floats uprightly, if one plane through the axis of symmetry is a plane of symmetry of the body; and a body floats obliquely if it is not divided by any of the planes, which may be carried through the axis of floatation into two congruent halves. A body floats with stability, if it strives to maintain its state of equilibrium (compare § 130); if, therefore, mechanical effect is to be expended to bring it out of this position, or if it returns of itself into a position of equilibrium after having been drawn out of one. On the other hand, a body floats without stability if it passes into a new position of equilibrium after having been brought out of one by a shock or blow.

If a body ABC, Fig. 376, floating at first uprightly, is brought into an inclined position, the centre of gravity S of the water displaced passes from the plane of symmetry EF, and assumes a position S_1 on the larger half immersed. The buoyancy applied at S: $P = V\gamma$, and the weight applied at the centre of gravity C of the body, viz., $G = -P$ form a couple by which (§ 90) a revolution is produced. About whatever point this revolution may take place, the point C, yielding to the weight G, will always go down, and S_1, or another point M of the vertical S_1P, obedient to the force P, will rise, therefore the plane of symmetry, or of the axis EF of the ship, will be drawn downwards at C, and upwards at M, and hence it will remain upright if M, as in the figure, lie above C, or incline itself still more as in Fig. 377, if M lie below C. From this, then, the stability of a floating body, or ship, is dependent on the point M, in which the vertical through the centre of gravity S_1 of the displaced water intersects the plane of symmetry. This point is called the *metacentre.*

Fig. 376.

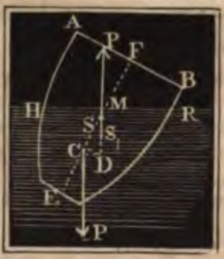
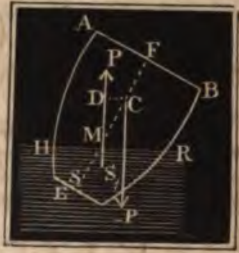


Fig. 377.

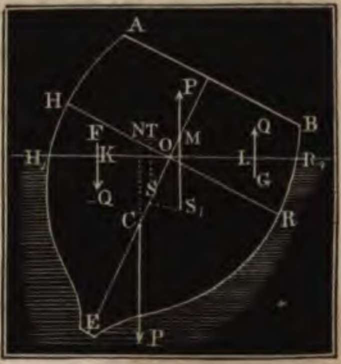


It follows, therefore, from this that a ship or other body floats with stability if its metacentre lies above the centre of gravity of the ship, and without stability if it lies below, lastly if the two points coincide, it is in a state of indifferent equilibrium.

The horizontal distance *CD* of the metacentre *M* from the centre of gravity *C* of the ship, is the arm of the force of the couple constituted of *P*, and $G = -P$, and hence the moment of the last is the measure of its stability $= P \,.\, \overline{CD}$. If we represent the distance *CM* by *c*, and the angle of revolution SMS_1 of the ship, or of the plane of its axis, by $\phi°$, we obtain for the measure of stability $S = Pc \sin \phi$; and this is, therefore, the greater, the greater the weight, the greater the distance of the metacentre from the centre of gravity of the ship, and the greater the angle of inclination of this last.

§ 289. In the last formula, $S = P\, c \sin.\, \phi$, the stability of the ship depends principally on the distance of the metacentre from the centre of gravity of the ship; it is hence of importance to obtain a formula for the determination of this distance.

Fig. 378.



In the transit of the ship *ABE*, Fig. 378, from the upright into the inclined position, the centre of gravity *S* advances to S_1, the space HOH_1 is drawn out of the water, and that of ROR_1 sinks below, and the buoyancy on the one side is thereby diminished by a force *Q* acting at the centre of gravity *F* of the space HOH_1, and on the other side increased by an equal force *Q* applied at the centre of gravity *G* of the space ROR_1. Therefore, the buoyancy *P* applied at S_1 replaces the buoyancy *P*, originally applied at *S*, and the couple $(Q, -Q)$, or what comes to the same thing, an opposite force applied at S_1 keeps in equilibrium a force $-P$ applied at *S* together with a couple $(Q, -Q)$, or more simply, the couple $(P, -P)$ is in equilibrium with the couple $(Q, -Q)$. If now the transverse section $HER = H_1ER_1$ of the part of the ship immersed $= F$, and the section $HOH_1 = ROR_1$ of the space by which the ship is drawn up on the one side, and down on the other $= F_1$; if, further, the horizontal distance *KL* of the centre of gravity of these spaces $= a$, and that of *MT* of the centres of gravity *S* and S_1, or the horizontal projection of the space SS_1 which *S* describes during the rolling $= s$, we have then from the conditions of equilibrium of the two couples:

$$Fs = F_1 a, \text{ hence } s = \frac{F_1}{F} a \text{ and } SM = \frac{MT}{\sin.\,\phi} = \frac{s}{\sin.\,\phi} = \frac{F_1\, a}{F \sin.\,\phi}.$$

The line $CM = c$, appearing as factor in the measure of the stability is $= CS + SM$; hence, if further, we represent the distance *CS* of the centre of gravity C of the ship from the centre of gravity of the displaced water by e, we obtain the measure of the stability

$$S = Pc \sin. \phi = P\left(\frac{F_1 a}{F} + e \sin. \phi\right).$$

If the angle of revolution be small, the transverse sections HOH_1 and ROR_1 may be regarded as equally small triangles; if we represent the breadth $HR = H_1R_1$ of the ship at the place of immersion by b, we may then put

$$F_1 = \tfrac{1}{2} \cdot \tfrac{1}{2} b \cdot \tfrac{1}{2} b \phi = \tfrac{1}{8} b^2 \phi, \text{ and } KL = a = 2 \cdot \tfrac{2}{3} \frac{b}{2} = \tfrac{2}{3} b,$$

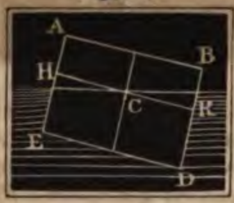
as also $\sin. \phi = \phi$, from which the stability is:

$$S = P\left(\tfrac{1}{12} \frac{b^3 \phi}{F} + e \phi\right) = \left(\frac{b^3}{12\,F} + e\right) P \phi.$$

If the centre of gravity C of the ship coincides with the centre of gravity S of the displaced water, we then have $e = 0$, hence: $S = \frac{b^3}{12\,F} \cdot P\phi$, and if the centre of gravity of the ship lies below that of the displaced water, we then have e negative; hence $S = \left(\frac{b^3}{12\,F} - e\right) P\phi$. It follows also that the stability of a ship is nothing, if e be negative and at the same time $e = \frac{b^3}{12\,F}$.

It is seen from the results obtained, that the stability comes out greater, the broader the ship is, and the lower its centre of gravity lies.

Fig. 379.



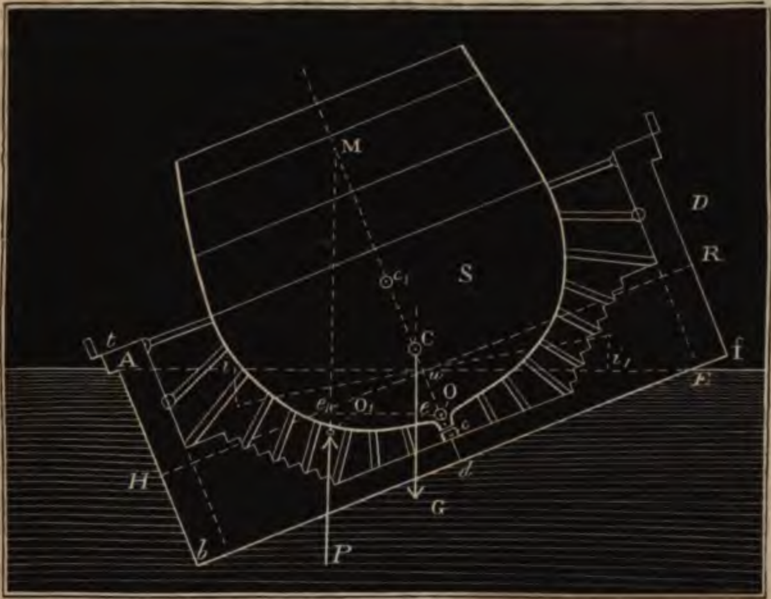
Example. A rectangular figure AD, Fig. 379, of the breadth $AB = b$, height $AE = h$, and depth of immersion $EH = y$, $F = by$, and $e = -\frac{h-y}{2}$; hence, the amount of stability is: $S = P\phi\left(\frac{b^3}{12\,by} - \frac{h}{2} + \frac{y}{2}\right)$, or if the specific gravity of the mass of the body be put $= \varepsilon$,

$$S = P\phi\left(\frac{b^2}{12\,h\varepsilon} - \frac{h}{2}(1 - \varepsilon)\right).$$

Hence, the stability ceases if $b^2 = 6\,h^2 \varepsilon (1 - \varepsilon)$, i. e., if $\frac{b}{h} = \sqrt{6\,\varepsilon\,(1 - \varepsilon)}$. For $\varepsilon = \frac{1}{2}$ $\frac{b}{h} = \sqrt{\frac{6}{2} \cdot \frac{1}{2}} = \sqrt{\frac{3}{2}} =$ 1,225; if, therefore, the breadth is not 1,225 of the height, the body will float without any stability.

[The principles explained in this section apply not only to the construction and use of vessels of every description, and to the ballasting and lading of ships themselves, but likewise to the loading of floating docks with vessels, including their cargoes or armaments. The form of the floating mass, and the position of its centre of gravity, together with its absolute weight, must be taken into account, as well as the density of the liquid in which it floats. Thus, a floating dock D, Fig. 379_1, in the form of a rectangular prism, capable of being closed at the ends after having received the ship S, and of being freed from water, will be subject to exact calculation, if its weight and centre of gravity be known, together with the weight of the vessel and its centre of gravity. *Example.*—Admitting that *D* is 90 feet wide, 36 feet high from *b* to *t*, and 250 feet in length—that its weight, including ballast, is 3860 tons, and that its centre of gravity *c* is four feet above the bottom *d*; also that the ship *S*, weighing 5200 tons, has been received and securely shoared in place, having its centre of gravity c_1 25 feet vertically above the bottom of the keel, supposed to be 1 foot above *c*, then the common centre of gravity *C* is $\frac{26 \times 5200}{9060} = 14{,}92$ ft. above the point *c*, and 18,92 feet above the bottom of the dock. The total displacement in sea water (64 lbs. per cubic foot) will be 337500 cubic feet; and, consequently, in a state of repose, it will sink to the depth of $bH = \frac{337500}{250{,}90} = 15$ feet. *HR* will be the "load line," and the common centre of gravity of ship and dock *C* will be 3,92 feet above it. Should any force acting in a horizontal direction careen the dock so as to make the angle $AwH = \phi = 22° 25'$, depressing the side *AH*, 18 feet, what force must be applied 25 above the water line to keep the dock in this position, *i. e.* what is now the ***restoring power***?

Fig. 379_1.



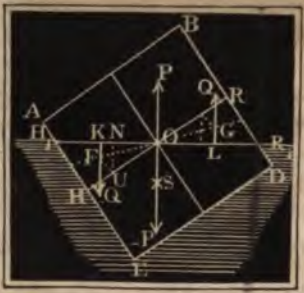
The centre of gravity of the original *quadrangular* prism of displacement *HRfb* is at *O*, that of the new *triangular* prism *EAb* is at O_1. The weight of ship and dock *C* acting at the point *C* is 9060 tons, which is also the force *P* of the prism of water acting upward at

O_1—tending to restore the position of the dock. The original centre of displacement O is $18{,}92 - 7{,}5 = 11{,}42$ feet below the centre of gravity of ship and dock.

The area of the immersed section which is *transferred* by the careening from one side to the other is 401,5 sq. feet, and the distance transferred $ii_1 = 58{,}5$ feet, hence $\frac{401{,}5 \,.\, 58{,}5}{90 \,.\, 15} = 17{,}4 = e_1 O =$ the horizontal distance, the centre of the whole displacement has been removed by the inclination supposed; and $sin.\ 22^\circ\ 25' : 17{,}4 = rad. : 45{,}63$ ft. = height of the metacentre above the original centre of buoyancy O. Again, putting $r_1 = 11{,}42$, we have $r_1 : sin.\ \phi = 11{,}42 : 4{,}35$ feet; hence, ee_1, the "equilibrating lever," or distance apart of CG and PO_1 is $17{,}4 - 4{,}35 = 13{,}05$ feet, and the statical moment is, therefore, $9060 \,.\, 13{,}05 = 116058{,}6$ ft.-tons; which, for a distance of 25 feet from the centre of oscillation, C gives a stability or restoring power of 4642 tons.]

§ 290. *Oblique Floatation.*—The formula $S = P\left(\frac{F_1 a}{F} \pm e\ sin.\ \phi\right)$ for the stability of a floating body may be also applied to find the different positions of floating bodies, for if we put $S = 0$ we obtain the equation for a second position of equilibrium, whose solution leads to the determination of the corresponding angle of inclination. The equation, therefore, $\frac{F_1 a}{F} \pm e\ sin.\ \phi = 0$, must be solved with respect to ϕ.

Fig. 380.



The transverse section of a parallelopiped AD, Fig. 380, is $F = HRDE = H_1R_1DE = by$, if b be the breadth $AB = HR$, and y the perpendicular depth $EH = DR$; further, the transverse section $F_1 = HOH_1 = ROR_1$ as a rectangular triangle with the leg $OH = OR = \frac{1}{2} b$, and the leg:

$$HH_1 = RR_1 = \tfrac{1}{2}\, b\ tang.\ \phi,\ F_1 = \tfrac{1}{8}\, b^2\ tang.\ \phi.$$

If, further, the centre of gravity F is distant from the base $FU = \frac{1}{3} HH_1 = \frac{1}{6}\, b\ tang.\ \phi$, and if from O about $OU = \frac{2}{3} OH = \frac{1}{3} b$, it follows that the horizontal distance of the centre of gravity F from the middle O, $= OK = ON + NK = OU\ cos.\ \phi + FU\ sin.\ \phi = \frac{1}{3}\, b\ cos.\ \phi + \frac{1}{6}\, b\ tang.\ \phi\ sin.\ \phi$, and the arm:

$$a = KL = 2\ OK = \tfrac{2}{3}\, b\ cos.\ \phi + \tfrac{1}{3}\, b\, \frac{sin.\ \phi^2}{cos.\ \phi}.$$

According to this the equation for the oblique position of equilibrium is:

$$\frac{\frac{1}{8}\, b^2\ tang.\ \phi\ (\frac{2}{3}\, b\ cos.\ \phi^2 + \frac{1}{3}\, b\ sin.\ \phi^2)}{by\ cos.\ \phi} - e\ sin.\ \phi = 0,$$

or, $\frac{sin.\ \phi}{cos.\ \phi} = tang.\ \phi$ being substituted,

$$sin.\ \phi\ [(\tfrac{1}{12} + \tfrac{1}{24}\ tang.\ \phi^2)\ b^2 - ey] = 0;$$

which equation will be satisfied by:

$$sin.\ \phi = 0 \text{ and by } tang.\ \phi = \sqrt{2}\ \sqrt{\frac{12\,e\,y}{b^2} - 1}.$$

The first equation, when $\phi = 0$, corresponds to upright, and the second to oblique floatation. The possibility of the latter requires that $\frac{ey}{b^2} > \frac{1}{12}$. If now h be the height of the parallelopiped, and ε its specific gravity, we then have:

$$y = \varepsilon\, h \text{ and } e = \frac{h - y}{2} = (1 - \varepsilon)\frac{h}{2}, \text{ hence it follows that}$$

$$tang.\ \phi = \sqrt{2}\sqrt{\frac{6\,\varepsilon\,(1-\varepsilon)\,h^2}{b^2} - 1},$$

and the equation of condition of oblique floatation is:

$$\frac{h}{b} > \sqrt{\frac{1}{6\,\varepsilon\,(1-\varepsilon)}}.$$

Examples.—1. If the floating parallelopiped is as high as it is broad, and has a specific gravity $\varepsilon = \frac{1}{2}$, then the $tang.\ \phi$ is $= \sqrt{2}\sqrt{3 \cdot \frac{1}{2} - 1} = \sqrt{3 - 2} = 1$; hence, $\phi = 45°$.

2. If the height $h = 0{,}9$ of the breadth b, and the specific gravity $\frac{1}{2}$, we have then $tang.\ \phi = \sqrt{3 \cdot 0{,}81 - 2} = \sqrt{0{,}43} = 0{,}6557$; hence, $\phi = 33° \ 15'$.

§ 291. *Specific Gravity.*—The law of buoyancy of water may be applied to the determination of the density, or the specific gravity of bodies. From § 284, the upward pressure of water is equal to the weight of liquid displaced; hence if V is the volume of a body and γ_1 the density of the liquid, we then have the buoyancy $P = V\gamma_1$. If now γ_2 be the density of the mass of the bodies, we then have the weight of the body $G = V\gamma_2$; hence the ratio of the densities $\frac{\gamma_2}{\gamma_1} = \frac{G}{P}$, i. e. *the density of the body immersed is to the density of the fluid as the absolute weight of the body to the buoyancy or loss of weight by immersion.*

Therefore, $\gamma_2 = \frac{G}{P}\gamma_1$, and $\gamma_1 = \frac{P}{G}\gamma_2$; or if γ be the density of water, ε_1 the specific gravity of the liquid, and ε_2 that of the body, then will $\gamma_1 = \varepsilon_1\gamma$, and $\gamma_2 = \varepsilon_2\gamma$, $\varepsilon_2 = \frac{G}{P}\varepsilon_1$, and $\varepsilon_1 = \frac{P}{G}\varepsilon_2$. If, therefore, the weight of a body ~~or~~ and its loss of weight by immersion is known, then the density or the specific gravity of the mass of a body may be found from the density or specific gravity of the liquid, and inversely, the density or specific gravity of the first, from the density or specific gravity of the last. /?

If the fluid in which the solid body is weighed is water, we then have $\varepsilon_1 = 1$, and $\gamma_1 = \gamma = 1000$ kilogrammes, or 62,5 lbs., according as we take the cubic metre or cubic foot for unit of volume, hence for this case the density of the body is:

$$\gamma_2 = \frac{G}{P}\gamma = \frac{\text{absolute weight}}{\text{loss of weight}} \text{ times the density of water,}$$

and the specific gravity:

$$s_2 = \frac{G}{P} = \frac{\text{absolute weight}}{\text{loss of weight}}.$$

To estimate the buoyancy or loss of weight, as well as to determine the weight C, we make use of an ordinary balance, only that below one of the scale pans of this balance there is appended a hook, to which the body may be suspended by a fine thread or fine wire, whilst it dips into the water contained in a vessel underneath. A balance arranged for the weighing of bodies in water is commonly called a *hydrostatic balance*.

If the body whose specific gravity we wish to determine is lighter than water, we may connect it mechanically with another heavy body, so as to make it sink. If this heavy body loses the weight P_2, and the system the weight P_1, the loss of weight of the lighter body: $P = P_1 - P_2$, now if G represents the [redacted] weight of the lighter body, we have then its specific gravity:

$$s_2 = \frac{G}{P} = \frac{G}{P_1 - P_2}.$$

If the specific gravity of a mechanical combination, or a composition of two bodies, and the specific gravities of their constituents s_1 and s_2 are known, from the weight of the whole, the weights G_1 and G_2 may be estimated. In every case $G_1 + G_2 = G$, and also the volume $\frac{G_1}{s_1\gamma}$ + volume $\frac{G_2}{s_2\gamma}$ = volume $\frac{G}{s\gamma}$, therefore: $\frac{G_1}{s_1} + \frac{G_2}{s_2} = \frac{G}{s}$. By combining these equations we have:

$$G_1 = G\left(\frac{1}{s} - \frac{1}{s_2}\right) + \left(\frac{1}{s_1} - \frac{1}{s_2}\right), \text{ and}$$

$$G_2 = G\left(\frac{1}{s} - \frac{1}{s_1}\right) + \left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

Examples.—1. If a piece of limestone, weighing 310 grains, becomes 121,5 grains lighter when under water, its specific gravity is $s = \frac{310}{121,5} = 2,55$.—2. To find the specific gravity of a piece of oak, round which a piece of lead has been wrapped, and which has lost by being weighed in water 10,5 grains; if now the wood itself weighed 426,5 grains, and the system under water was 484,5 grains lighter than in the air, the specific gravity of the mass of wood would be:

$$s = \frac{426,5}{484,5 - 10,5} = \frac{426,5}{474} = 0,9.$$

3. An iron vessel, completely filled with quicksilver and perfectly closed, has a net weight of 500 lbs., and has lost 40 lbs. in the water; if now the specific gravity of cast-iron = 7,2, and that of quicksilver is 13,6, the weight of the empty vessel is:

$$G_1 = 500\left(\frac{40}{500} - \frac{1}{13,6}\right) + \left(\frac{1}{7,2} - \frac{1}{13,6}\right)$$

$$= 500\,(0,08 - 0,07353) + (0,1388 - 0,0735)$$

$$= \frac{500\,.\,0,00647}{0,0653} = \frac{3235}{65,3} = 49,5 \text{ lbs.,}$$

and the weight of the enclosed quicksilver:

$$G_2 = 500\,(0{,}08 - 0{,}1388) : (0{,}07353 - 0{,}1388) = \frac{500\,.\,0.0588}{0{,}0653}$$

$$= \frac{2940}{6{,}53} = 450{,}2 \text{ lbs.}$$

Remark 1. For the determination of the specific gravities of liquids, meal, corn, &c., the mere weighing in open air is sufficient, because we may give to the bodies any volume at will, by filling vessels with them. If an empty bottle weighs $= G$, and the same filled with water G_1, and the weight G_2 if it contain any other substance, we shall then have the specific gravities of masses of these : $\varepsilon = \frac{G_2 - G}{G_1 - G}$. For example, to find the specific gravity of rye (not rye grains), a bottle is filled with the grains, and after much shaking, then weighed. After deduction of the weight of the empty bottle, the weight of the rye was $= 120{,}75$ grms., and the weight of an equal quantity of water $= 155{,}65$; the weight of the rye is accordingly $= \frac{120{,}75}{155{,}65} = 0{,}776$; and, therefore, 1 cubic foot of this grain weighs

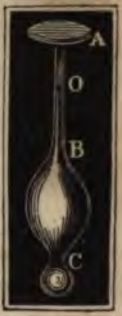
$$= 0{,}776\,.\,62{,}5 = 48{,}5 \text{ lbs.}$$

Remark 2. The problem solved by Archimedes of finding the ratio of the constituents from the specific gravity of a mixture, and from the specific gravity of its constituents, admits only of a limited application to chemical combinations, metallic alloys, &c., because a contraction or expansion of the mass generally takes place, so that the volume of the mixture is no longer equal to the sum of the volumes of the constituents.

Remark 3. The further extension of this subject, namely, its application to the measurement of volume, &c., belongs to physics and chemistry.

§ 292. *Areometer.*—Areometers are principally used to determine the density of liquids. These instruments are hollow bodies, formed about a symmetrical axis, whose centres of gravity lie very low, and by floating perpendicularly in liquids, give their density. They are made of glass, brass, &c., and are called, according to the various purposes for which they are intended, hydrostatic balances, salzometers, hydrometers, alcoholometers, &c. There are two kinds of hydrometers, viz., the weight and the scale hydrometer. The first are often used for the determination of the weights, as was the specific gravity of solid bodies.

Fig. 381.



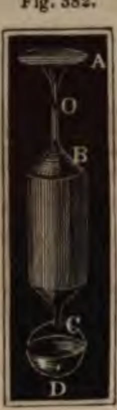
1. If V be the volume of the portion of a hydrometer ABC, Fig. 381, floating freely, and immersed up to a certain mark O in the water, G the weight of the whole balance, P the weight placed upon the plate while floating in the water, whose density may be $= \gamma$, and P_1 the weight required to be put on to make it float in any other liquid of the density γ_1, we shall then have

$$V\gamma = P + G \text{ and } V\gamma_1 = P_1 + G; \text{ hence,}$$

$$\frac{\gamma_1}{\gamma} = \frac{P_1 + G}{P + G}.$$

2. If P be the weight which must be put upon the plate to make the hydrometer ABC, Fig. 382, sink up to a mark O, and P_1 the weight which must be put upon A, together with the body to be weighed, to obtain the same immersion, we shall then have simply the weight of this body $G_1 = P - P_1$. But if P_1 must be augmented by P_2, when the body to be weighed is put into the cup D under the

Fig. 382.



surface, to preserve the depth of immersion unchanged, P_2 will then be the buoyancy, and hence the specific gravity of the body:

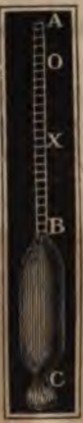
$$\varepsilon = \frac{G_1}{P_2} = \frac{P - P_1}{P_2}.$$

Those hydrometers which have a cup suspended below for the determination of the specific gravities of solid bodies, minerals for instance, are called Nicholson's hydrometers.

3. Let the weight of a hydrometer ABC, Fig. 383, $= G$, and the volume immersed, if this balance floats in water, $= V$, then $G = V\gamma$. If the balance rise by $OX = x$, when immersed in a heavier liquid, for the transverse section F of the stem, the volume immersed is $= V - FX$, and hence $G = (V - Fx)\,\gamma_1$; the two formulæ, divided by one another, give the density of the liquid:

$$\gamma_1 = \frac{V}{V - Fx} \cdot \gamma = \gamma \div (1 - \frac{F}{V}x) = \gamma \div (1 - \mu x).$$

Fig. 383.



If the liquid in which the hydrometer is immersed, be lighter than the water, it will sink in it to a depth x, for which reason, $G = (V + Fx)\,\gamma$, and hence we must put $\gamma_1 = \gamma \div (1 + \mu x)$.

To find the co-efficient $\mu = \frac{F}{V}$, the balance is loaded with a weight P of quicksilver, which is poured in and takes the lowest position, so that while floating in water, a considerable length l of the stem to which the scale is applied, sinks lower down. If now we put $P = Fl\gamma$, we shall then obtain:

$$\mu = \frac{F}{V} = \frac{P}{Vl\gamma} = \frac{P}{Gl}.$$

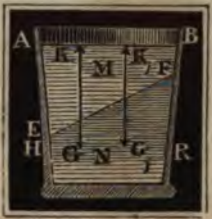
Examples.—1. If a Nicholson's hydrometer weighs 65 grains, 13,5 grains must be taken off the plate, that it may sink to the same depth in alcohol as it does in water; the specific gravity of alcohol is $= \frac{65 - 13.5}{65} = 1 - 0{,}208 = 0{,}792$.—2. The normal weight of a Nicholson's balance is 1500 grains, *i. e.* 1500 grains require to be put on to make the instrument sink to 0; from this 1030 grains must be taken by the weighing of a piece of brass placed upon the upper plate, and 121,5 to be added if this body is placed on the lower plate. The absolute weight of this piece of brass is therefore = 1030 grains, and its specific gravity $= \frac{1030}{121{,}5} = 8{,}47$.—3. A scale areometer, of 1162 grains weight, after having been lightened by 465 grains, rises 6 inches, and has therefore the co-efficient $\mu = \frac{465}{1162.6} = \frac{465}{6772} = 0{,}00686$. After complete filling and restoration of the weight of 1162 grains, it ascends, when floating in a saline solution, $2\,\frac{5}{12}$ inches; hence, the specific gravity of this is:

$$= 1 \div \left(1 - 0{,}00686 \times \frac{29}{12}\right) = 1 \div 0{,}983 = 1{,}02.$$

Remark. The further extension of this subject belongs to physics, chemistry, and technology.

§ 293. *Liquids of Different Densities.*—If several liquids, of different densities, are in the same vessel, without their exerting any chemical action upon each other, from the ready displacement of their particles, they arrange themselves above each other, according to their specific gravities, viz. the densest below, then the less dense, and then the lightest. The limiting surfaces are also in a state of equilibrium, as likewise the free surface horizontal; for as long as the surface of limitation *EF* between the masses *M* and *N*, Fig. 384, is inclined, columns of fluid, of different densities, like *GK*, G_1K_1, rest on the horizontal stratum *HR*, and hence the pressure on this stratum will not be everywhere the same; and lastly, no equilibrium will subsist.

Fig. 384.



In communicating tubes *AB* and *CD*, Fig. 385, the liquids arrange themselves one above the other, according to their densities, only their surfaces *A* and *D* do not lie in one and the same level. If *F* be the area *HR* of the transverse section of a piston, Fig. 386, in the one branch *AB* of two communicating tubes, and the height of pressure or the height *EH* of the surface of the water in the second tube *CD* above *HR*, $= h$, we then have the pressure against the surface of the piston $P = F h\gamma$. On the other hand, if we replace the pressure of the piston by a column of liquid *AH*, Fig. 386, of the height $AH = h_1$ and the density γ_1, we then have $P = Fh_1 \gamma_1$; and equating both expressions, we obtain the equation $h_1 \gamma_1 = h \gamma$ or the proportion $\frac{h_1}{h} = \frac{\gamma}{\gamma_1}$.

Fig. 385.

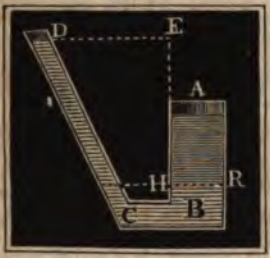
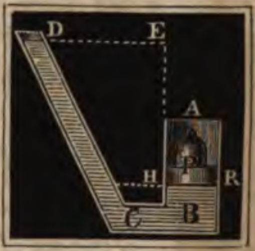


Fig. 386.



Therefore, the heights of pressure in communicating tubes, for the subsistence of equilibrium between two different liquids, or the heights of the columns of liquid measured from the common plane of contact, are inversely as the densities or specific gravities of these liquids.

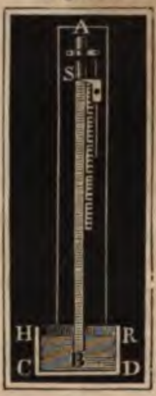
As mercury is of about 13,6 times the density of water, a column of mercury, in communicating tubes, will hold in equilibrium a column of water of 13,6 times the height.

CHAPTER III.

ON THE EQUILIBRIUM AND PRESSURE OF AIR.

§ 294. *Tension of Gases.*—The atmospheric air which surrounds us, as well as all kinds of air or gases, possesses, in virtue of the repulsive force of its parts or molecules, a tendency to occupy a greater and greater space; hence, we can obtain a limited mass of air only by confining it in perfectly closed vessels. The force with which gases endeavor to dilate themselves is called their ***elasticity***, ***tension***, or ***expansive force***. It exhibits itself by pressure against the sides of the vessels which enclose it, and so far differs from the elasticity of solids and liquids, that it manifests its action in every condition of density, while the elasticity of the last-mentioned bodies in a certain state of expansion, is nothing. The pressure or tension of air and other gases is measured by the ***barometer***, the ***manometer***, and the ***valve***. The barometer is chiefly used for determining the pressure of the atmosphere. The common, or as it is called, the ***cistern barometer***, Fig. 387, consists of a glass tube, closed at one end *A* and open at the other *B*, which, when filled with mercury, is inverted, and its open end immersed in a cistern likewise containing mercury. By the inversion of this instrument, there remains in the tube a column of mercury *BS*, which (§ 393) is sustained in equilibrium by the pressure of the air on the surface of mercury *HR*. The space *AS* above the mercurial column is deprived of air, or a vacuum; hence, there is no pressure on this column from above, for which reason, the height of the mercurial column above the surface of mercury *HR* in the cistern, serves for a measure of the air's pressure.

Fig. 387.



To measure this height with precision and convenience, an accurately divided scale is appended, which runs lengthwise along the tube. A more particular description of the different barometers, and an explanation of their uses, &c., belong to the department of physics.

§ 295. It has been found by the barometer, that for a certain mean state of the atmosphere, and at places very little above the level of the sea, the air's pressure is held in equilibrium by a column of mercury, 76 centimetres, or about 28 Paris inches = 29 Prussian inches = 30 English inches nearly, (29,994 exactly.) As the specific gravity of mercury is nearly 13,6 (13,598), it follows that the pressure of the air is equivalent to the weight of a column of water,

$0{,}76 \,.\, 13{,}6 = 10{,}336$ metres $= 31{,}73$ Paris feet $= 32{,}84$ Prussian feet $= \dfrac{13{,}598 \,.\, 29{,}99\cancel{8}4}{12} = 33{,}988$ English feet.

The tension of the air is very often measured by the pressure it exerts upon a unit of surface. Since a cubic centimetre of mercury weighs 0,0136 kilogrammes, the pressure of the atmosphere, or the weight of a column of mercury 76 centimetres high on a base of 1 centimetre square, $= 0{,}0136 \,.\, 76 = 1{,}0336$ kilogrammes, and since a cubic inch of mercury weighs $\dfrac{66 \,.\, 13{,}6}{1728} = 0{,}5194$ Prussian lbs., or $\dfrac{62{,}5 \,.\, 13{,}6}{1728} = 0{,}491$ lbs. English, the mean pressure of the atmosphere is then $= 29 \,.\, 0{,}5194 = 15{,}05$ Prussian lbs. on the square inch, $= 2167$ lbs. on the square foot, and in English measure $= 30 \,.\, 0{,}491 = 14{,}73$ on the square inch, $= 2131{,}12$ lbs. avd. on the square foot. 14,76 lbs. per square inch is the standard usually adopted.

In mechanics, the mean pressure of the atmosphere is commonly taken as unity, and other expansive forces referred to this and assigned in atmospheric pressures, or *atmospheres*. Hence, to a pressure of n atmospheres corresponds a mercurial column of $30 \,.\, n$ inches, or a weight of 14,73 lbs. on each square inch; and inversely, to a mercurial column of h inches corresponds a tension of $\dfrac{h}{28} = 0{,}03571\, h$ or $= \dfrac{h}{30} = 0{,}0333\, h$ atmospheres, and to a pressure of p lbs. on the square inch, a tension of

$$\frac{h}{15{,}05} = 0{,}0644\, p \text{ or } \frac{h}{14{,}72} = 0{,}0678 \text{ atmospheres.}$$

The equation $\dfrac{h}{28} = \dfrac{p}{15{,}05}$ or $\dfrac{h}{30} = \dfrac{p}{14{,}73}$ give the formulæ of reduction $h = 1{,}8604\, p$ inches and $p = 0{,}5375\, h$ lbs., or $h = 2{,}036\, p$ inches, and $p = 0{,}491\, h$ lbs. English. For a tension h inches $= p$ lbs., the pressure against a plane surface of F square inches: $P = Fp = 0{,}491\, Fh$ lbs. English, or $= 0{,}5375\, Fh$ lbs. Prussian.

Examples.—1. If the water in a water-pressure engine stands 250 feet above the surface of the piston, the pressure against the surface will then be $= \dfrac{250}{33{,}988} = 7.35$ atmospheres.—2. If the blast of a cylindrical bellows has a tension of 1,2 atmospheres, its pressure on every square inch $= 1{,}2 \,.\, 14{,}73 = 17{,}676$ lbs., and on the surface of the piston of 50 inches diameter $= \dfrac{\pi \,.\, 50^2}{4} \,.\, 17{,}676 = 34707$ lbs. As the atmosphere exerts a counter-pressure $\dfrac{\pi \,.\, 50^2}{4} \,.\, 14{,}73 = 28922{,}3$ lbs., the pressure on the piston is $= 34695 - 28922{,}3 = 5784{,}7$ lbs.

§ 296. *Manometer.*—To find the tension of gases or vapors enclosed in vessels, instruments similar to the barometer are made use of, which are called *manometers.* These instruments are filled with mercury or water, and are either open or closed; but in the latter case, the upper part is either a vacuum or full of air. The vacuum manometer, Fig. 388, differs little from the ordinary barometer. To measure by this instrument the tension of air in a reservoir, a tube *GK* is fitted in, one end of which *G* passes into the reservoir, and the other *K* projects above the surface of mercury *CE* in the cistern of the instrument. The space *EFHR* above the mercury is hereby put into communication with the air-holder, and the air in it assumes the tension of the air in the holder, and forces into the tube a column of mercury *OS*, which sustains in equilibrium the pressure of the air which is to be measured.

The siphon manometer, *ABC*, Fig. 389, open above, gives the excess of tension, above the pressure of the atmosphere, in the vessel *MN*, because the pressure of the atmosphere on *S*, joined to that of the mercurial column *RS*, is in equilibrium with the tension. If b be the height of the barometer, and h that of the manometer, or the difference of heights *RS* of the surfaces of mercury in both branches of the manometer, we shall then have the tension of the air communicating with the shorter branch measured by the height of a column of mercury: $b_1 = b + h$, or the pressure measured on a square inch $p = 0{,}491\ (b + h)$ lbs.; or if b be the mean height of the barometer, $p = 14{,}73 + 0{,}491\ h$ lbs.

Fig. 388.

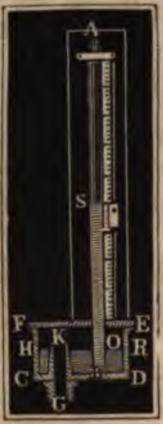


Fig. 389.

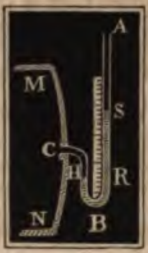
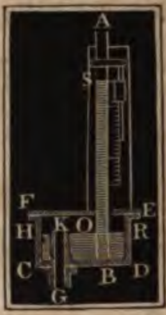
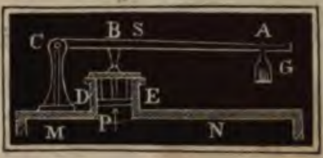


Fig. 390.



Cistern manometers, Fig. 390, *ABCE*, are more common than siphon manometers. As the air here acts through a greater quantity of mercury or water, as it may be, upon the column of fluid, its oscillations do not so quickly affect the column of fluid, and its measurement, when thus at rest, is rendered both easier and more accurate. For the sake of convenience of measuring by, or reading off from the scale, a float is not unfrequently attached to it, which rests on the mercury, and is connected with an index hand, accompanying the scale by means of a thread passing over a small roller.

Fig. 391.



The expansive force of a gas or vapor enclosed in MN may be likewise determined, but with less accuracy, by the help of a valve DE, Fig. 391, if the sliding weight is so placed that it is in equilibrium with the pressure of the air or vapor. If $CS = s$ be the distance of the centre of gravity of the lever from the fulcrum C, $CA = a$ the arm of the weight, and Q the weight of the lever with its valve, we then have the statical moment with which the valve is pressed down by the weight $= Ga + Qs$;[*] if now the pressure of the gas or vapor from below $= P$, the pressure of the atmosphere from above $= P_1$, and lastly, the arm CB of the valve $= d$, we then have the statical moment with which the valve strives to lift itself up $= (P - P_1)\,d$, and by equating the moments of both:

$$Pd - P_1 d = Ga + Qs, \text{ and } P = P_1 + \frac{Ga + Qs}{d}.$$

If r represent the radius $\frac{1}{2}\,DE$ of the valve, p the internal and p_1 the external tension, measured by the pressure on a square inch, we then have: $P = \pi r^2 p$ and $P_1 = \pi r^2 p_1$; hence, $p = p_1 + \dfrac{Ga + Qs}{\pi r^2 d}$.

Examples.—1. If the height of mercury of a manometer, open above, is 3,5 inches, but that of the barometer 27 inches, the corresponding expansive force is then $h = b + h_1 = 27 + 3{,}5 = 30{,}5$ inches, or $p = 0{,}491 \,.\, h = 0{,}491 \,.\, 30{,}5 = 14.97$ lbs.—2. If the height of a water-manometer is 21 inches, the expansive force corresponding to this, with the height of the barometer at 27 inches, is:

$$h = 27 + \frac{21}{13{,}6} = 28{,}54 \text{ inches} = 14{,}01 \text{ lbs. English.}$$

3. If the statical moment of an unloaded safety-valve is 10 inch lbs., the statical moment of a 10 lbs. sliding weight $15 \,.\, 10 = 150$ inch lbs., the arm of the valve measured, from the valve to the fulcrum, 4 inches, and the radius of the valve 1,5 inches, then the difference of the pressures on both surfaces of the valve is:

$$p - p_1 = \frac{150 + 10}{\pi\,(1{,}5)^2 \,.\, 4} = \frac{160}{9\,\pi} = 5{,}66 \text{ lbs.}$$

Were the pressure of the atmosphere $p_1 = 14$ lbs., the tension of the air below the valve would from this amount to $p = 19{,}66$ lbs.

§ 297. *Law of Mariotte.*—The tension of gases increases with their density; the more a certain quantity of air is compressed or condensed, the greater is its tension; and the greater its tension, the more it is allowed to expand or become rarefied, the less does its expansive force exhibit itself. The ratio in which the tension and the density, or the volume of the gases, stand to each other, is expressed by the law discovered by Mariotte, and named after him. This law assumes that *the density of one and the same quantity of air or gas is proportional to its tension;* or, as the spaces which are occupied by one and the same mass are inversely proportional to the densities,

---

[*] If the weight of the lever and valve be counterpoised by a weight attached to a cord, passing upwards and over a pulley above S, the statical moment is reduced to Ga.—AM. ED.

that the volume of one and the same mass of gas is inversely as its expansive force. Accordingly, if a certain quantity of air becomes compressed to one-half its original volume, its density is therefore doubled, its tension is also as great again as at first; and on the other hand, if a certain quantity of air be expanded to ~~one-third~~ three times of its original bulk, therefore, its density reduced to one-third of its original amount, its elasticity will be equal to one-third only of its original tension. If atmospheric air, for example, under the piston *EF* of a cylinder *AC*, Fig. 392, be supposed to press with 15 lbs. on every square inch, it will press on the piston with a force of 30 lbs., if this piston be pushed to E_1F_1, and the enclosed air compressed to one half its original volume, and this force will amount to $3 . 15 = 45$ lbs., if the piston come to E_2F_2, and describes two-thirds of the whole height. If the area of the piston be 1 square foot, the pressure of the atmosphere against it will amount to $= 144 . 15 = 2160$ lbs.; hence, to press down the piston one-half the height of the cylinder, it will require 2160 lbs., and to push it down two-thirds of this height $2 . 2160 = 4320$ lbs. to be exerted.

Fig. 392.

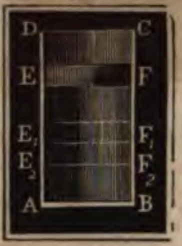
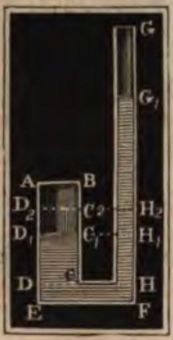


Fig. 393.



The law of Mariotte may be likewise proved by pouring mercury into the tube *GH* communicating with the air of a cylinder *AC*, Fig. 393. If a column of air *AC* be originally enclosed by the quantity of mercury *DEFH*, which has the same tension as the external air, and afterwards be compressed to one-half or one-fourth its volume by the addition of fresh mercury, we shall then find that the distances of the surfaces of G_1H_1, G_2H_2, &c., of mercury are equivalent to the single and treble height of the barometer *b*, &c., that, therefore, if we add to this the single height, corresponding to the external pressure of the air, the tension will be twice or four times as great as that due to its original volume.

The tensions are h and h_1 or p and p_1, γ and γ_1 the corresponding densities, and V and V_1 the volumes appertaining to one and the same quantity of air, we then have, according to the law laid down:

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{h}{h_1} = \frac{p}{p_1}; \text{ hence}$$

$$\gamma_1 = \frac{h_1}{h}\gamma = \frac{p_1}{p}\gamma \text{ and } V_1 = \frac{h}{h_1}V = \frac{p}{p_1}V.$$

From this the density and also the volume may be reduced from one tension to another.

Examples.—1. If in a blowing machine, the manometer stand at 3 inches, whilst the barometer is at 28 inches, the density of the wind is $= \frac{28 + 3}{28} = \frac{31}{28} = 1{,}107$ times

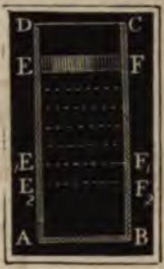
To find an expression for the pressure upon a superficial unit of the interior surface of the piston of a steam engine, also a value for its velocity at any point of the stroke, having given the whole load upon the piston the length of stroke the distance passed over by the piston before the steam is cut off and the area of the piston itself.

Let (a) represent the distance passed over by the piston during the admission of the steam; the pressure upon the piston during this period may be regarded as constant for the amount of steam in the boiler is so great as compared with the amount let into the cylinder at any one stroke that no difference of pressure [illegible]ed by the steam's expanding this

that of the external air.*—2. If a cubic foot of atmospheric air, with the barometer at 28 inches, weighs $\frac{62,5}{770}$ lbs.; with the barometer at 34 inches it will weigh:

$$\frac{62.5}{770} \cdot \frac{34}{28} = \frac{21250}{21560} = 0,985 \text{ lbs.}$$

Fig. 394.



§ 298. The mechanical effect which must be expended to condense a certain quantity of air to a certain degree, and the effects which the air by its expansion will produce, cannot be directly assigned, because the expansive force varies at every moment of condensation or extension; we must therefore endeavor to find a special formula for the calculations of this value. Let us imagine a certain quantity of air AF, enclosed in a cylinder AC, Fig. 394, by a piston EF, and let us inquire what effect must be expended to push forward the piston through a certain space $EE_1 = FF_1$. If the original tension $= p$, and the original height of the capacity of the cylinder $= s_0$, and the tension after describing the space $EE_1 = p_1$, and the residuary volume of air $= s_1$, the proportion $p_1 : p = s_0 : s_1$ then holds true, and gives $p_1 = \frac{s_0}{s_1} p$.

While describing a very small space $E_1E_2 = x$, the tension p_1 may be regarded as invariable, and hence the mechanical effect to be expended is $= Ap_1x = \frac{Aps_0x}{s_1}$, when A represents the surface of the piston.

It follows from the properties of logarithms, that a very small magnitude $y = hyp.\ log.\ (1 + y) = 2,3026\ Log.\ (1 + y)$, if $hyp.\ log.$ represents the hyperbolic, and $Log.$ the common logarithms; we may consequently put

$$Aps_0 \frac{x}{s_1} = Aps_0\ hyp.\ log.\ \left(1 + \frac{x}{s_1}\right)$$
$$= 2,3026\ Aps_0\ log.\ \left(1 + \frac{x}{s_1}\right)$$

But now:

$$hyp.\ log.\left(1 + \frac{x}{s_1}\right) = hyp.\ log.\left(\frac{s_1 + x}{s_1}\right) = hyp.\ log.\ (s_1 + x) - hyp.\ log.\ s_1;$$

hence the elementary mechanical effect is:

$$= Aps_0\ [hyp.\ log.\ (s_1 + x) - hyp.\ log.\ s_1].$$

Let us imagine the whole space EE_1 to be made up of very small parts, such as x, and therefore put $EE_1 = nx$, we shall find the mechanical effects corresponding to all these parts, if in the last formula we substitute for

$s_1, s_1 + x, s_1 + 2\,x, s_1 + 3\,x, \ldots$ to $s_1 + (n-1)\,x$, and for
$s_1 + x, s_1 + 2\,x, s_1 + 3\,x,$ &c., to $s_1 + nx$, or s_0,

---

[*] For comparisons with the manometer, the division of a barometer ought not to be into either inches or metres, but into 1000th parts of 1 *atmosphere*.—AM. ED.

and by summation the whole expenditure of mechanical effect in describing the space s_0—s_1:

$$L = \mathfrak{A}ps_0 \left\{ \begin{array}{l} hyp.\ log.\ (s_1+x) - hyp.\ log.\ s_1 \\ hyp.\ log.\ (s_1+2x) - hyp.\ log.\ (s_1+x) \\ hyp.\ log.\ (s_1+3x) - hyp.\ log.\ (s_1+2x) \\ \quad \cdot \qquad\qquad\qquad \cdot \\ \quad \cdot \qquad\qquad\qquad \cdot \\ \quad \cdot \qquad\qquad\qquad \cdot \\ hyp.\ log.\ (s_1+nx) - hyp.\ log.\ [s_1+(n-1)x] \end{array} \right.$$

$$= \mathfrak{A}ps_0\ [hyp.\ log.\ (s_1+nx) - hyp.\ log.\ s_1]$$

$$= \mathfrak{A}ps_0\ (hyp.\ log.\ s_0 - hyp.\ log.\ s_1) = \mathfrak{A}ps_0\ hyp.\ log.\ \left(\frac{s_0}{s_1}\right),$$

since one member in the one line always cancels one member in the following one.

Since $\frac{s_0}{s_1} = \frac{h_1}{h} = \frac{p_1}{p}$, this mechanical effect may be put:

$$L = \mathfrak{A}ps_0\ hyp.\ log. \left(\frac{h_1}{h}\right) = \mathfrak{A}ps_0\ hyp.\ log.\ \left(\frac{p_1}{p}\right).$$

If we put the space described by the piston $s_0 - s_1 = s$, we shall hence find that the mean force of the piston p condensing the air is in the proportion

$$\frac{h_1}{h} = \frac{p_1}{p},\ P = \frac{L}{s} = \mathfrak{A}p\ \frac{s_0}{s}\ hyp.\ log.\ \left(\frac{p_1}{p}\right).$$

Let $\mathfrak{A} = 1$ (square foot) and $s_0 = 1$ (foot), we obtain the mechanical effect produced

$$L = p\ hyp.\ log.\ \left(\frac{p_1}{p}\right) = 2{,}3026\ p\ log.\ \left(\frac{p_1}{p}\right).$$

This formula gives the mechanical effect which must be expended to convert a unit or cubic foot of air of a lower pressure or tension p into a higher tension p_1, and to reduce it thereby to the volume $\left(\frac{p}{p_1}\right)$ cubic feet. On the other hand:

$$L = p_1\ hyp.\ log.\ \left(\frac{p_1}{p}\right) = 2{,}3026\ p_1\ log.\ \left(\frac{p_1}{p}\right)$$

expresses the effect which a unit of volume of gas gives out or produces when it passes from a higher pressure p_1 to a lower p.

To reduce by condensation a mass of air of the volume V, and tension p to the volume V_1, and the tension $p_1 = \frac{V}{V_1}\ p$, the mechanical effect requisite to be expended is $Vp\ hyp.\ log.\ \left(\frac{V}{V_1}\right)$, and when, inversely, the volume V_1 at a tension p_1 is converted by expansion into the volume V, and into the tension $p = \frac{V_1}{V}\ p_1$, the effect:

$$V\ p\ hyp.\ log.\ \left(\frac{V}{V_1}\right) = V_1 p_1\ hyp.\ log.\ \left(\frac{V}{V_1}\right)$$

will be given out.

Examples.—1. If a blast converts 10 cubic feet of air per second, of 28 inches tension, into air of 30 inches tension, the effect to be expended upon this for every second will be $= 17280 \,.\, 0{,}491 \,.\, 28$ *hyp. log.* $\left(\frac{30}{28}\right) = 237565$ (*hyp. log.* 15 — *hyp. log.* 14) $= 237565$ $(2{,}708050 - 2{,}639057) = 237565 \,.\, 0{,}068993 = 16390$ inch lbs. $= 1365.8$ ft. lbs.— 2. If a mass of vapor in a steam engine below the surface of a piston $A = \pi 8^2 = 201$ square inches, stands 15 inches high, and with a tension of three atmospheres, pushes up the piston 25 inches, the mechanical effect evolved, and which is expended on the piston, is:

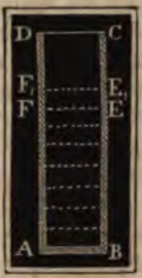
$$L = 201 \,.\, 3 \,.\, 14{,}73 \,.\, 15 \; hyp.\, log \left(\frac{15+25}{15}\right) = 133232 \; hyp.\, log.\, \tfrac{8}{3}$$

$= 133232 \,.\, 0{,}98083 = 130567$ inch lbs. $= 10881$ feet lbs., and the mean force of the piston, without regard to its friction and the counter pressure, is:

$$= \frac{130567}{25} = 5222 \text{ lbs.}$$

§ 299. *Strata of Air.*—Air enclosed in a vessel is at different depths of different density and tension, for the upper strata press together the lower on which they rest, so that there are only one and the same density and tension in one and the same horizontal stratum, and both increase with the depth. But in order to discover the law of this increase of density downwards, or the decrease upwards, we must adopt a method very similar to that of the former paragraph.

Fig. 395.



Let us imagine a vertical column of air AE, Fig. 395, of the transverse section $AB = 1$, and of the height $AF = s$. Let the density of the lower stratum $= \gamma$, and the tension $= p$, and the density of the upper stratum $EF = \gamma_1$, and the tension $= p_1$, we shall then have $\frac{\gamma_1}{\gamma} = \frac{p_1}{p}$. If x is the height EE_1 of the stratum E_1F, we have its weight, and hence also the diminution of its tension corresponding to:

$$y = 1 \,.\, x \,.\, \gamma_1 = \frac{x\gamma p_1}{p}, \text{ and inversely,}$$

$x = \frac{p}{\gamma} \,.\, \frac{y}{p_1}$, or as in the former paragraph:

$$x = \frac{p}{\gamma}\, hyp.\, log.\, \left(1 + \frac{y}{p_1}\right) = \frac{p}{\gamma}\, [hyp.\, log.\, (p_1 + y) - hyp.\, log.\, p_1].$$

Let us put for p_1, successively

$$p_1 + y,\ p_1 + 2\,y,\ p_1 + 3\,y, \text{ \&c., to } p_1 + (n-1)\,y,$$

and add the corresponding heights of the strata or values of x, and we shall then obtain the height of the entire column of air, as in the former §.

$$s = \frac{p}{\gamma}\, (hyp.\, log.\, p - hyp.\, log.\, p_1) = \frac{p}{\gamma}\, hyp.\, log.\, \left(\frac{p}{p_1}\right), \text{ also}$$

$$s = \frac{p}{\gamma}\, hyp.\, log.\, \left(\frac{b}{b_1}\right) = 2{,}302 \frac{p}{\gamma}\, log.\, \left(\frac{b}{b_1}\right),$$

if b and b_1 are the heights of the barometer corresponding to the tensions p and p_1.

If, inversely, the height s is given, the expansive force and density of the air corresponding to it may be calculated. It is:

$\frac{p}{p_1} = \frac{\gamma}{1} = e^{\frac{s\gamma}{p}}$, therefore $\gamma_1 = \gamma e^{\frac{-s\gamma}{p}}$, where $e = 2{,}71828$ is the base of the hyperbolic system of logarithms.

Remark. This formula is applicable to the measurement of heights. Leaving the temperature out of consideration, we may put $s = 58604 \,.\, Log. \left(\frac{b}{b_1}\right)$; for the English measure $= 60000\, Log. \left(\frac{b}{b_1}\right)$.

Examples.—1. If the height of the barometer at the foot of a mountain be 28 inches, and at the top 25 inches, the height of this mountain will be: $s = 58604 \,.\, Log. \left(\frac{30}{25}\right) = 58604 \,.\, 0791813 = 4640$ Prussian feet.—2. The density of the air on a mountain 10,000 feet high is: $Log. \frac{\gamma}{\gamma_1} = \frac{10000}{58604} = 0{,}1706$; hence, $\frac{\gamma}{\gamma_1} = 1{,}481$, and $\frac{\gamma_1}{\gamma} = \frac{1}{1{,}481} = 0{,}675$; it is therefore only 67½ per cent. of the density of that at the foot.

§ 300. *Gay-Lussac's Law.*—Heat or temperature has an essential influence on the density and expansive force of gases. The more air enclosed in a vessel becomes heated, the greater does its expansive force exhibit itself, and the higher that the temperature of the air enclosed by a piston in a vessel is raised, the more it expands, and pushes against the piston. From the experiments of Gay-Lussac, which in later times have been repeated by Rudberg, Magnus and Regnault, it results that for equal densities the expansive force, and for equal expansive forces the volume of one and the same quantity of air increases as the temperature. We may place this law by the side of that of Mariotte, and name it, for distinction's sake, Gay-Lussac's law.

According to the latest experiments, the expansive force of a definite volume of air increases by being heated from the freezing to the boiling point, by 0,367 of its original value, or for this increase of temperature the volume of a definite quantity of air increases, the tension remaining the same, by 36,7 per cent. If the temperature is given in centigrade degrees, of which there are 100 between the freezing and boiling point, it follows that the expansion for each degree is $= 0{,}00367$, and for t degrees temperature $= 0{,}00367 \,.\, t$; if we make use of Fahrenheit's thermometer, which contains between the freezing and boiling point 180°, for each degree the expansion is $= .002039$, and for t degrees temperature $= .002039 \,.\, t$. This co-efficient is true only for atmospheric air; slightly greater values correspond to other gases, and even for atmospheric air, this co-efficient increases slightly with the temperature.

If a mass of air of the original volume V_0, and of the temperature (centigrade) 0°, be heated t degrees without assuming a different tension, the new volume is then $V = (1 + 0{,}00367\, t)\, V_0$, and if it acquire the temperature t_1, it will then assume the volume: $V_1 = (1 + 0{,}00367\, t_1)\, V_0$, and by dividing the ratio of the volumes:

$$\frac{V}{V_1} = \frac{1 + 0{,}00367\, t}{1 + 0{,}00367\, t_1},$$

on the other hand, the corresponding ratio of density:

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{1 + 0{,}00367\, t_1}{1 + 0{,}00367\, t}.$$

If, moreover, a change take place in the tensions, if p_0 is the tension at zero, p that at the temperature t, and p_1 that at t_1, we then have:

$$V = (1 + 0{,}00367\, t) \frac{p_0}{p} V_0, \text{ further } V_1 = (1 + 0{,}00367\, t_1) \frac{p_0}{p_1} V_0,$$

hence:

$$\frac{V}{V_1} = \frac{1 + 0{,}00367\, t}{1 + 0{,}00367\, t_1} \cdot \frac{p_1}{p}, \text{ and } \frac{\gamma}{\gamma_1} = \frac{1 + 0{,}00367 \,.\, t_1}{1 + 0{,}00367 \,.\, t} \cdot \frac{p}{p_1}, \text{ or,}$$

$$\frac{\gamma}{\gamma_1} = \frac{1 + 0{,}00367 \,.\, t_1}{1 + 0{,}00367 \,.\, t} \cdot \frac{b}{b_1}.$$

Example. If a mass of air, of 800 cubic feet, and of 10 lbs. tension, and 10° (centigrade) temperature, is raised by the blast, and by the warming apparatus of a blast-furnace to a tension of 19 lbs. and to a temperature of 200°[*], it will at length assume the greater volume:

$$V_1 = \frac{1 + 0{,}00367 \,.\, 200}{1 + 0{,}00367 \,.\, 10} \cdot \frac{15}{19} \,.\, 800 = \frac{1{,}734}{1{,}0367} \cdot \frac{12000}{19} = 1056 \text{ cubic feet (Prussian).}$$

§ 301. *Density of the Air.*—By aid of the formula at the end of the former paragraph, γ may now be calculated by the density corresponding to a given temperature and tension of the air. By accurate weighings and measurements we have the weight of a cubic metre of atmospheric air at a temperature of 0°, and 0,76 metre height of barometer = 1,2995 kilogrammes. Since a cubic foot (Prussian) = 0,030916 cubic metre and 1 kilogramme = 2,13809 lbs. The density of the air for the relations given is: = 0,030916 . 2,13809 . 1,2995 = 0,08590 lbs. If now the temperature is = t° cent., the density for the French measure: $\gamma = \dfrac{1{,}2995}{1 + 0{,}00367\, t}$ kilogrammes; and for the Prussian measure $\gamma = \dfrac{0{,}08590}{1 + 0{,}00367 \,.\, t}$ lbs., and for the English: $\gamma = \dfrac{0{,}081241}{1 + 0{,}00204\, t}$ lbs. If now the expansive force varies from the mean, if, for example, the height of the barometer is not 0,76 metres, but b, we shall obtain:

$$\gamma = \frac{1{,}2995}{1 + 0{,}00367 \,.\, t} \cdot \frac{b}{0{,}76} = \frac{1{,}71 \,.\, b}{1 + 0{,}00367\, t} \text{ kilog.}$$

or if b, as is commonly the case, be given in Paris inches:

$$\gamma = \frac{0{,}003058 \,.\, b}{1 + 0{,}00367\, t} \text{ lbs.}$$

Very often the expansive force is expressed by the pressure p, on a square centimetre or square inch, for this reason the factor $\dfrac{p}{1{,}0336}$, or $\dfrac{p}{14{,}73}$, or $\dfrac{p}{15{,}05}$ must be introduced, and it then follows that:

---

[*] 10° C. = 50 F., and 200° C. = 392° F.,—the co-efficient will then be .002039.—Am. Ed.

$$\gamma = \frac{1,2995}{1+0,00367\,.\,t} \cdot \frac{p}{1,0336} = \frac{1,2572p}{1+0,00367\,.\,t} \text{ kilog. or}$$

$$\gamma = \frac{0,08565}{1+0,00367\,.\,t} \cdot \frac{p}{15,05} = \frac{0,005691p}{1+0,00367\,.\,t} \text{ lbs. Prussian.}$$

For the same temperature and tension, the density of steam is $\frac{5}{8}$ of the density of atmospheric air, for which reason we have for steam:

$$\gamma = \frac{0,8122}{1+0,00367\,t} \cdot \frac{p}{1,0336} = \frac{0,78577}{1+0,00367\,t} \text{ kilog. or}$$

$$\gamma = \frac{0,05353}{1+0,00367\,.\,t} \cdot \frac{p}{15,05} = \frac{0,003557p}{1+0,00367\,.\,t} \text{ lbs. Prussian.}$$

$$= \frac{0,050775}{1+0,00204\,t} \cdot \frac{p}{14,73} = \frac{0,003447p}{1+0,00204\,t} \text{ lbs. English.}$$

Examples.—1. What weight has the wind contained in a cylindrical regulator of 40 feet length and 6 feet width, at a temperature of 10° and 18 lbs. pressure? The density of this wind is:

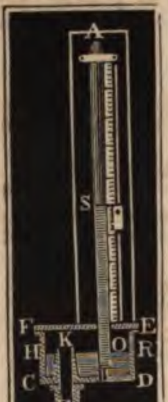
$$= \frac{0,005691\,.\,18}{1,0367} = \frac{0,10244}{1,0367} = 0,0988 \text{ lbs. (Prussian);}$$

the capacity of the regulating vessel is $= \pi\,.\,3^2\,.\,40 = 1131$ cubic feet; hence, the quantity of wind $= 0,0988\,.\,1131 = 112$ lbs.—2. A steam engine uses per minute 500 cubic feet of vapor, of 107° C. temperature and 36 inches pressure, how many pounds of water are required for the generation of this steam? The density of this steam is:

$$= \frac{0,05353}{1+0,00367.107^\circ} \cdot \frac{36}{28} = \frac{0,05353\,.\,36}{1,393\,.\,28} = 0,0494 \text{ lbs.;}$$

hence, the weight of 500 cubic feet, or the weight of the corresponding quantity of water, $= 500\,.\,0,0494 = 24,7$ lbs.

Fig. 396.



§ 302. By aid of the results obtained in the last paragraph, the theory of the air manometer may be explained. This instrument consists of a barometer tube of uniform bore AB, Fig. 396, filled above with air and below with mercury, and of a vessel CE likewise containing mercury, which is put in communication with the gas or vapor whose tension we wish to find. From the height of the columns of mercury and of air, the expansive force may be estimated as follows. The instrument is commonly so arranged, that the mercury in the tube stands at the same level as the mercury in the cistern, when the temperature of the enclosed air $t = 50°$ (10° C.), and the tension in the space EH equal to the mean atmospheric pressure $b = 0,76$ metres $= 30$ inches.

But if for a height of the barometer b, from EH a column of mercury h_1 has ascended into the tube, and the length of the column of the residuary air is h_2, we have then its tension

$$= \frac{h_1 + h_2}{h_2}\, b, \text{ and hence } b_1 = h_1 + \frac{h_1 + h_2}{h_2}\, b.$$

If a change of temperature takes place, the temperature from observation of h_1 and h_2 is not as at first $= t$, but t_1, we then have the tension of the column of air:

[illegible] lbs. English.

[illegible] of 20° and 15 lbs. pressure? [illegible]

[illegible] = 0.0000 lbs. (Possibly) [illegible]

[illegible] The density of the steam is [illegible]

[illegible] is the weight of the corresponding [illegible]

[illegible] of the weights obtained in the [illegible] the theory of the air manometer [illegible] This instrument consists of a [illegible] tube AB, Fig. 288, filled above [illegible] with mercury, and of a vessel CD [illegible] containing mercury, which is put in communication with the gas or vapour whose tension [illegible] the height of the column of mercury [illegible] the pressure may be estimated [illegible] This instrument is commonly so arranged [illegible] mercury in the tube stands at the same [illegible] the mercury in the cistern, when the temperature [illegible] is measured at t = 50° (10° C.), and the [illegible] equal to the mean atmospheric [illegible] h = 0.76 metres = 30 inches.

$$\mathcal{A}S = \frac{h_1 + h_2}{h_2} \cdot \frac{1+0{,}00367 \cdot t_1}{1+0{,}00367 \cdot t} \cdot b,$$

and hence the height of the barometer in question:

$$b_1 = h_1 + \frac{h_1 + h_2}{h_2} \cdot \frac{1+0{,}00367 \cdot t_1}{1+0{,}00367 \cdot t} \cdot b.$$

For $b = 28$ inches (Paris), and $t = 10^\circ$ C., it follows that

$$b_1 = h_1 + 27\,(1 + 0{,}00367\, t_1)\,\frac{h}{h_2},\text{ whereby } h = h_1 + h_2,$$

represents the whole length of the tube measured to the surface of mercury *HR*.

From the height of the barometer b_1, it follows that the pressure on the square inch (Prussian) is

$$p = \frac{15{,}6}{28}\,h_1 + 15{,}6 \cdot \frac{27}{28}\,(1 + 0{,}00367\, t_1 \cdot \frac{h}{h_2} = 0{,}538\, h_1$$
$$+ 14{,}51\,(1+0{,}00367\, t)\,\frac{h}{h_2} \text{ lbs.}$$

Example.—If an air manometer, of 25 inches length, at 21° temperature C., indicates a column of air of 12 inches in length, then the corresponding height of the barometer is:

$$b_1 = 25 - 12 + 27\,(1 + 0{,}00367 \cdot 21)\,\frac{25}{12} = 13 + 9 \cdot 1{,}07707 \cdot \frac{25}{4}$$

$= 13 + 60{,}58 = 73{,}58$ inches, and the pressure on a square inch
$= 0{,}538 \cdot 73{,}58 = 39{,}59$ lbs.

SECTION VI.

DYNAMICS OF FLUID BODIES.

CHAPTER I.

THE GENERAL LAWS OF THE EFFLUX OF WATER FROM VESSELS.

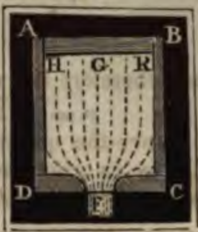
§ 303. *Efflux.*—The doctrine of the efflux of fluids from vessels constitutes the first principal division of hydro-dynamics. We distinguish first between the efflux of *air* and the efflux of *water*, and then again between the efflux under *uniform* and that under *variable pressure.* We next treat of the efflux of water under constant pressure. The pressure of water may be assumed as constant when the same quantity of water is admitted on one side as flows out from the other, or when the quantity of water flowing out in a certain time is small compared with the size of the vessel. The chief problem, whose solution will be here treated, is that of determining the *discharge* through a given *orifice*, under a given pressure, in a definite time.

If the discharge in each second $= Q$, we then have the expenditure, after the lapse of t seconds, under variable pressure: $Q_1 = Qt$. ? But to obtain the efflux per second, it is necessary to know the dimensions of the orifice, and the velocity of the particles of fluid issuing from it. For the sake of simplicity of investigation, we shall for the present assume that the particles of water flow out in straight and parallel lines, and on this account form a *prismatic vein* or stream of fluid. If, now, F be the transverse section of the vein and v the velocity of the water, or of each particle of water, the discharge per second will form a prism of the base F and height v, and, therefore, Q will be $= Fv$ units of volume and $G = Fv\gamma$ units of weight, γ being the density of the water or the effluent liquid.

Examples.—1. If water flows through a sluice, of 1,7 square feet aperture, with a 14 feet velocity, the discharge will be $Q = 14 \cdot 1{,}7 = 23{,}8$ cubic feet, and hence the discharge per hour will be $= 23{,}8 \cdot 3600'' = 85680$ cubic feet.—2. If 264 cubic feet of water were to be discharged through an orifice of 5 square inches in 3 minutes and 10 seconds, the mean velocity of efflux would amount to

$$v = \frac{Q_1}{Ft} = \frac{264}{\frac{5}{144} \cdot 190''} = \frac{264 \cdot 144}{5 \cdot 190} = 40 \text{ feet.}$$

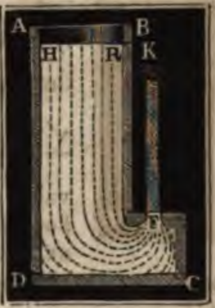
Fig. 397.



§ 304. *Velocity of Efflux.*—Let us suppose a vessel AC, Fig. 397, filled with water, having a horizontal exit orifice, F, rounded from the inside, which forms but a small part of the transverse section or bottom surface CD. Let the *head of water* FG, supposed invariable during the efflux, $= h$, the velocity of efflux $= v$, and the discharge in each second $= Q$, and therefore its weight $= Q\gamma$. The mechanical effect which this mass of water produces by sinking from the height h, is $= Q\,h\,\gamma$, and the mechanical effect which the effluent mass $Q\gamma$ accumulates in its transit from a state of rest into that of the velocity v, is $\frac{v^2}{2g}\,Q\gamma$ (§ 71). If no loss of mechanical effect take place in its passage through the orifice, both mechanical effects will be equal, and therefore $h\,Q\gamma = \frac{v^2}{2g}\,Q\gamma$; *i. e.*, $h = \frac{v^2}{2g}$, and, inversely, $v = \sqrt{2\,g\,h}$, or in feet, $h = 0{,}0155\,v^2$, and $v = 8{,}02\,\sqrt{h}$.

The velocity, therefore, of water issuing through an orifice is equivalent to the final velocity of a body falling freely from the height of the water.

Fig. 398.



The correctness of this law may be proved by the following experiments. If we apply an orifice directed upwards to the vessel AC, Fig. 398, the jet FK will ascend vertically, and nearly attain the level HR of the water in the vessel, and we may assume that it would exactly attain this height, were all resistances, such as those of the air, friction at the sides of the vessel, disturbances from the descending water, &c., entirely removed. But since a body ascending to a perpendicular height h, has the initial velocity $v = \sqrt{2\,g\,h}$ (§ 17), it accordingly follows that the velocity of efflux is $v = \sqrt{2\,g\,h}$.

For a different head of water h_1, the velocity is $v_1 = \sqrt{2\,g\,h_1}$, hence we have $v : v_1 = \sqrt{h} : \sqrt{h_1}$; *therefore, the velocities of efflux are to each other as the square roots of the heads of water.*

Examples.—1. The discharge which takes place in each second through an orifice 10 inches square, under a pressure of 5 feet, is:

$$Q = Fv = 10\,.\,12\,\sqrt{2\,g\,h} = 120\,.\,8{,}02\,\sqrt{5} = 962{,}4\,.\,2{,}236 = 2151{,}9 \text{ cubic inches.}$$

2. That 252 cubic inches may be discharged through an orifice of 6 square inches in each second, the head of water required is:

$$h = \frac{1}{2g}\left(\frac{Q}{F}\right)^2 = 0{,}0155\left(\frac{252}{6}\right)^2 = 0{,}0155\,(42)^2 = 27{,}34 \text{ inches.}$$

§ 305. *Velocity of Influx and Efflux.*—If water flows in with a certain velocity c, the mechanical effect $\frac{c^2}{2g}$, corresponding to the ve- $\frac{c^2}{2g}$ ty locity due to the height $h_1 = \frac{c^2}{2g}$, must be added to the mechanical effect $h \,.\, Q\gamma$; hence we have to put:

$$(h + h_1)\, Q\gamma = \frac{v^2}{2g}\, Q\gamma, \text{ or } h + h_1 = \frac{v^2}{2g},$$

and therefore the velocity of efflux:

$$v = \sqrt{2\,g\,(h + h_1)} = \sqrt{2\,g\,h + c^2}.$$

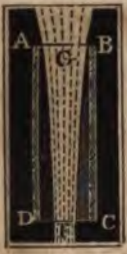
Since the quantity of water flowing into a vessel kept constantly full is as great as that Q which flows out, we may put $Gc = Fv$, where G represents the area of the transverse section HR (Fig. 397) of the water pouring in. Accordingly if we put $c = \frac{F}{G}\, v$, we shall then obtain:

$$h = \frac{v^2}{2g} - \left(\frac{F}{G}\right)^2 \frac{v^2}{2g} = \left[1 - \left(\frac{F}{G}\right)^2\right] \frac{v^2}{2g};$$

$$\text{and hence: } v = \frac{\sqrt{2\,g\,h}}{\sqrt{1 - \left(\frac{F}{G}\right)^2}}.$$

According to this formula, the velocity increases, the greater the ratio of the sections $\frac{F}{G}$ becomes; the velocity is least, viz., $= \sqrt{2gh}$, if the transverse section F of the orifice of efflux is small compared with the transverse section G of the orifice of influx, and it approximates more and more to an infinitely great velocity, the smaller the difference is between these orifices. If $F = G$, therefore $\frac{F}{G} = 1$, then $v = \frac{\sqrt{2\,g\,h}}{0} = \infty$, and therefore also $c = \infty$. This infinite value must be understood to express that the water must flow to and from a bottomless vessel AC, Fig. 399, with an infinite velocity, in order that the stream of fluid CF may entirely fill up the orifice of discharge F. If we put $v = \frac{Gc}{F}$, we shall then obtain:

Fig. 399.



$$h = \left[\left(\frac{G}{F}\right)^2 - 1\right] \frac{c^2}{2g}, \text{ hence } F = \frac{G}{\sqrt{1 + \frac{2\,g\,h}{c^2}}},$$

which expression indicates that the transverse section F of the stream flowing out, for an infinite velocity of influx, is constantly less than the transverse section G of the stream flowing in, and hence, that the discharging orifice is not quite filled when it it is greater than

$$\frac{G}{\sqrt{1 + \frac{2\,g\,h}{c^2}}}.$$

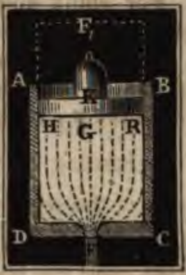
Remark. The accuracy of this formula, given by Daniel Bernoulli, $v = \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{F}{G}\right)^2}}$, has of late been brought into doubt by many philosophers. I have endeavored to prove how unfounded are the objections made, in an article "Efflux," in the "Allgemeinen Maschinenencyclopädie."

Example. If water runs from a circular orifice, 5 inches in width, in the bottom of a prismatic vessel of 60 square inches transverse section, under a pressure of 6 feet, the velocity is then:

$$v = \frac{8,02\sqrt{h}}{\sqrt{1 - \left(\frac{25\pi}{4.60}\right)^2}} = \frac{8,02 . 2,449}{\sqrt{1-(0,327)^2}} = \frac{19,641}{\sqrt{0,8931}} = \frac{19,641}{0,945} = 20,78 \text{ feet.}$$

§ 306. *Velocity of Efflux, Pressure, and Density.*—The above formulæ are only true if the pressure of the air on the fluid surface is as great as its pressure against the orifice; but if these pressures are different from one another, we have then to extend these formulæ. If the upper surface *HR*, Fig. 400, is pressed by a piston *K* with a force P_1, which case, for example, presents itself in that of the fire-engine, we may then imagine it to be replaced by the pressure of a column of water. If h_1 be the height of this column, and γ the density of the liquid, we may therefore put $P_1 = G h_1 y$. If we substitute for h the head of water augmented by $h_1 = \frac{P_1}{G\gamma}$, $h + h_1 = h + \frac{P_1}{G\gamma}$, we then obtain for the velocity of efflux:

Fig. 400.



$v = \sqrt{2\ g\ \left(h + \frac{P_1}{G\gamma}\right)}$, when, moreover, we suppose $\frac{F}{G}$ to be very small. If, further, we represent the pressure on each unit of surface of G by p_1, we have more simply $\frac{P_1}{G} = p_1$, and hence

$v = \sqrt{2\,g\left(h + \frac{p_1}{\gamma}\right)}$. Again, if we represent the pressure of water at the level of the orifice by p, we may then put

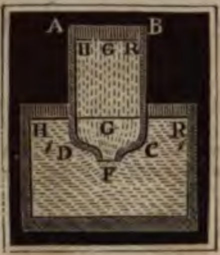
$p = \left(h + \frac{p_1}{\gamma}\right)\gamma$; therefore, $h + \frac{p_1}{\gamma} = \frac{p}{\gamma}$, whence $v = \sqrt{2\,g\frac{p}{\gamma}}$.

The velocity of efflux, therefore, increases as the square root of the pressure on each unit of surface, and inversely as the square root of the density of the fluid. Under equal pressures, therefore, a fluid of a density represented by 4, runs out half as fast as one of density 1. Since the air is 770* times lighter than water, it would,

---

* According to the experiments of Prout, 100 cubic inches of air at 60° temperature and 30 inches barometer, weigh 31,0117 grains troy, and consequently at 32° a cubic foot will weigh $31,0117 \times \frac{521}{493} \times 17,28 = 566,1$ grains; and since a cubic foot of water weighs 62,5 lbs. avoirdupois $= 62,5 \times 7000 = 437500$ grains, the relative weights of

Fig. 401.



if it were an inelastic body, flow out $\sqrt{770} = 27\frac{3}{4}$ faster than water. If the water does not flow freely, but under water, in consequence of a counter-pressure, a diminution of the velocity of efflux then takes place. If the mouth of the vessel AC, Fig. 401, is the depth $FG = h$ below the surface of the upper water HR, and $FG_1 = h_1$ below the surface H_1R_1 of the lower water, we then have the pressure downwards $p = h\gamma$, and the counter pressure upwards $p_1 = h_1\gamma$, hence the force of efflux is: $p - p_1 = (h - h_1)\gamma$, and the velocity of efflux

$$v = \sqrt{2g\left(\frac{p - p_1}{\gamma}\right)} = \sqrt{2g(h - h_1)}.$$

For efflux under water the difference of level $h - h_1$ between the surfaces must be regarded as the head of water.

If the water on the side of the outer orifice be pressed by the force p, and on the side of the inner orifice or of the surface of water by the force p_1, we have then generally:

$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}.$$

Examples.—1. If the piston of a 12 inch cylinder, or that of a fire-engine, were pressed down with a force of 3000 lbs., and there were no obstacles in the tubes or pipes, the water would then issue through the mouth-piece of the tube and be directed vertically upwards with a velocity:

$$v = \sqrt{2g\frac{p_1}{\gamma}} = \sqrt{2g\frac{P_1}{G\gamma}} = 8{,}02\sqrt{\frac{3000}{\frac{\pi}{4}\,.\,62{,}5}} = 8{,}02\sqrt{\frac{600\,.\,4}{\pi\,12{,}5}}$$

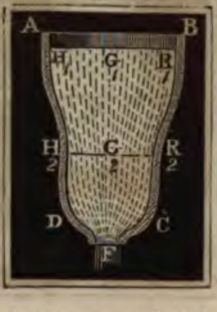
$= 62{,}697$ feet, and ascend to the height $h = 0{,}0155\,.\,v^2 = 60{,}93$ feet.—2. If water rushes into a rarefied space; for example, into the condenser of a steam-engine, whilst it is pressed from above or on its exposed surface by the atmosphere, the last formula $v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}$ for the velocity of efflux is then to be applied. If the head of water $h = 3$ feet, and the external barometer stand at 27 inches, and the internal at 4 Paris inches, we shall now have $\frac{p_1 - p}{\gamma} = 27 - 4 = 23$ Paris inches $= \frac{23}{12}\ 1{,}035 = 1{,}9837$ Prussian feet $= 2{,}042$ English feet, or a column of water $= 13{,}5\,.\,2{,}042 = 27{,}57$ feet, and the velocity of the water rushing into the vacuum $v = 8{,}02\,.\sqrt{3 + 27{,}57} = 44.34$ feet.—3. If the water in the feed-pipe of a steam-engine boiler stands 12 feet above the surface of the water in the boiler, and the pressure of steam be 20 lbs., and the pressure of the atmosphere only 15 lbs. on the square inch, the velocity with which the water enters into the boiler will be:

$$v = 8{,}02\sqrt{12 + \frac{(15 - 20)\,.\,144}{62{,}5}} = 8{,}02\sqrt{12 - \frac{5\,.\,144}{62{,}5}} = 8{,}02\sqrt{0{,}48} = 5{,}55 \text{ feet.}$$

---

air and water at that temperature is $\frac{566{,}1}{437500} = \frac{1}{772}$. At 60° the relation will be $535{,}88 : 437500 = 1 : 816$. And as $\sqrt{816} = 28{,}5$, the velocity of efflux will, under this condition, be $\frac{0.75}{27.75}$ part more rapid than in that supposed in the text.—Am. Ed.

Fig. 402.



§ 307. *Hydraulic Pressure.*—When water enclosed in a vessel is in motion, it then presses more feebly against the sides than when at rest. We must, therefore, distinguish the hydrodynamic or hydraulic pressure from the hydrostatic pressure of water. If p_1 be the pressure on each unit of surface $H_1R_1 = G_1$, Fig. 402, p the pressure without the orifice F, and h the head of water FG_1, we then have for the velocity of efflux

$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)} \div \sqrt{1 - \left(\frac{F}{G_1}\right)^2}, \text{ or}$$

$$h + \frac{p_1 - p}{\gamma} = \left[1 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g};$$ if, further, in another transverse section $H_2R_2 = G_2$, which lies at a height $FG_2 = h_1$ above the orifice, the pressure $= p_2$, we then have likewise:

$$h_1 + \frac{p_2 - p}{\gamma} = \left[1 - \left(\frac{F}{G_2}\right)^2\right]\frac{v^2}{2g}.$$

If we subtract one expression from the other, it then follows that:

$$h - h_1 + \frac{p_1 - p_2}{\gamma} = \left[\left(\frac{F}{G_2}\right)^2 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g},$$

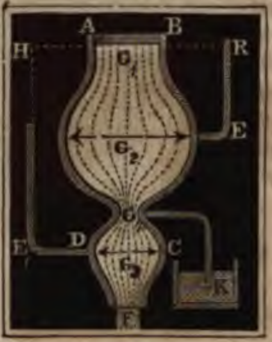
or, if the head of water G_1G_2 of the stratum $H_2R_2 = G_2$ be represented by h_2, the measure of the hydraulic pressure of water at H_2R_2 is:

$$\frac{p_2}{\gamma} = h_2 + \frac{p_1}{\gamma} - \left[\left(\frac{F}{G_2}\right)^2 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g}.$$

But now $\frac{Fv}{G_1}$ is the velocity v_1 of the water at the surface G_1, and $\frac{Fv}{G_2}$ the velocity v_2 of the water at the section G_2; hence more simply we may put $\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + h_2 - \left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right)$.

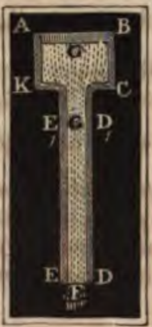
Therefore, from this it follows that the hydraulic head of water $\frac{p_2}{\gamma}$ *at any place in the vessel is equivalent to the hydrostatic head of water* $\frac{p_1}{\gamma} + h_2$ *diminished by the difference of the height due to the velocity at this point, and at the place of entrance.* If the upper surface of the water G_1 is great, we may neglect the velocity of influx, and hence may put $\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + h_2 - \frac{v_2^2}{2g}$, and the *hydraulic head of water is less by the height due to the velocity than the hydrostatic head of water.* The faster, therefore, water flows in conduit pipes, the less it presses against the sides of the pipes. From this cause, pipes very often burst, or begin to leak, when its motion in them is checked, or when the pipes are stopped up, &c.

Fig. 403.



By means of the apparatus of efflux *ABCD*, Fig. 403, we may have ocular demonstration of the difference between hydraulic and hydrostatic pressure. If we carry upwards a small tube *ER* from the transverse section G_2, it will become filled with water, which will ascend in it above the level of the fluid surface if G_2 is $> G_1$, therefore $v_2 < v_1$; for as the pressure p_1 on the fluid surface is counteracted by the pressure of the air against the mouth of the tube, we may put for the height which measures the pressure at G_2 viz. $x = \frac{p_2}{\gamma} = h_2 - \left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right)$, and therefore x is $> h_2$ if $\frac{v_2^2}{2g}$ is $< \frac{v_1^2}{2g}$. If, on the other hand, the transverse section G_3 be $< G_1$, and the water therefore flow through G_3 quicker than through G_1, we shall then have the height of the column of water in the small tube E_1 whose inner orifice is at G_3, $y = h_3 - \left(\frac{v_3^2}{2g} - \frac{v_1^2}{2g}\right)$ less than h_3, and hence it will not reach to the level *HR* of G_1. Again, if G_4 be very small, and therefore the corresponding velocity v_4 very great, then $\frac{v_4^2}{2g} - \frac{v_1^2}{2g}$ may be $> h_4$, and hence the corresponding hydraulic head of water z may be negative, *i. e.* the air may press more from without than the water from within. A column of water will therefore ascend in the tube E_2K, which is inserted below, and whose outer orifice is under water, which in conjunction with the pressure of the water, will balance that of the external atmosphere. If this small tube be short, the water, which may be colored for this purpose, will ascend from the vessel *K* underneath, through the tube, enter the reservoir of efflux, and will arrive at *F* and be discharged.

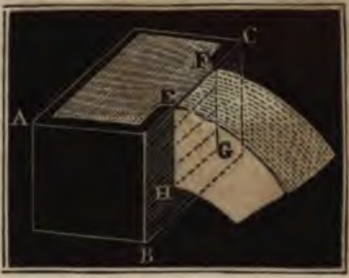
Fig. 404.



Remark. If the discharging vessel *ACE*, Fig. 404, consists of a wide reservoir *AC* and of a narrow vertical tube *CE*, the hydraulic pressure at all places in this tube is then negative. If we disregard the pressure of the atmosphere p_1, the pressure of the water in the vicinity of the mouth *F* may be put $= 0$, because the whole head of water here $GF = h$ will be expended in generating the velocity $v = \sqrt{2gh}$; on the other hand, at a place D_1E_1 at the height $G_1G = h_1$ below the surface of water, the hydraulic pressure $= h_1 - h = -(h - h_1)$ negative; if, therefore, a hole be bored in this tube, no water will run out, but air will rather be drawn in, which will arrive at *F* and flow out. This negative pressure will be greatest directly below the water, in *G* because h_2 is there least.

§ 308. By means of the formula $Q = Fv = F\sqrt{2gh}$, the discharge issuing in one second can only then be calculated directly when the orifice is horizontal, because here only the velocity throughout the whole transverse section F is the same; but if the transverse section of the orifice has an inclination to the horizon, for example, if it is at the side of the vessel, the particles of water at different depths will then flow out with different velocities, and the discharge Q can no longer be considered as a prism, and hence, therefore, the formula $Q = Fv = F\sqrt{2gh}$ cannot be applied directly. The most simple case of this kind is presented in the efflux through a cut in the side of a vessel, or in what is called a weir, Fig. 405. This cut forms a rectangular aperture of efflux $EFGH$, whose breadth $EF = GH$ is represented by b, and height $EH = FG$ by h. If we divide this surface bh by horizontal lines into a great number n of equally broad laminæ, we may suppose the velocity in each of these to be the same. Since the heads of water of these laminæ from above downwards are $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c., we then have the corresponding velocities $\sqrt{2g\frac{h}{n}}, \sqrt{2g.\frac{2h}{n}}, \sqrt{2g.\frac{3h}{n}}$; and since, further, the area of a lamina $= b.\frac{h}{n} = \frac{bh}{n}$, we then have the discharges: $\frac{bh}{n}\sqrt{2g\frac{h}{n}}, \frac{bh}{n}\sqrt{2g.\frac{2h}{n}}, \frac{bh}{n}\sqrt{2g.\frac{3h}{n}}$, &c.; consequently the discharge through the entire section:

Fig. 405.



$$Q = \frac{bh}{n}\left(\sqrt{2g\frac{h}{n}} + \sqrt{2g.\frac{2h}{n}} + \sqrt{2g.\frac{3h}{n}} + ..\right)$$

$$= \frac{bh\sqrt{2gh}}{n\sqrt{n}}(\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}).$$

But now:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + .. + \sqrt{n}, \text{ or}$$

$$1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + .. + n^{\frac{1}{2}} = \frac{n^{1+\frac{1}{2}}}{1+\frac{1}{2}} = \frac{2}{3}n^{\frac{3}{2}} = \frac{2}{3}n\sqrt{n};$$

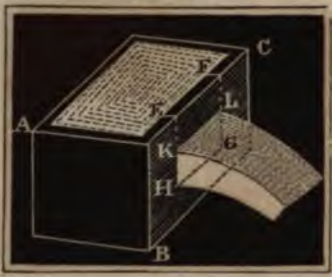
hence, the discharge required is:

$$Q = \frac{bh\sqrt{2gh}}{n\sqrt{n}} \cdot \frac{2}{3}n\sqrt{n} = \frac{2}{3}bh\sqrt{2gh} = \frac{2}{3}b\sqrt{2gh^3}.$$

If by the term *mean velocity* (v) *be understood that velocity which must subsist at all places, that as much water, in consequence, does issue as with the variable velocities of efflux within the whole profile;* we may then put: $Q = bh.v$, and, consequently, $v = \frac{2}{3}\sqrt{2gh}$, i. e.

the mean velocity of water issuing through a rectangular cut in the side of a vessel is $\frac{2}{3}$ of the velocity at the sill or lower edge of the cut.

Fig. 406.



If the rectangular aperture of efflux KG, Fig. 406, with horizontal sill does not reach the surface of the water, we may find the discharge by regarding the aperture as the difference of the two cuts $EFGH$ and $EFLK$. Hence, if h_1 is the depth HE of the lower, and $KE = h_2$ that of the upper edge, we then have the discharge from these apertures $\frac{2}{3}\, b \sqrt{2\, g\, h_1^3}$ and $\frac{2}{3}\, b \sqrt{2 g h_2^3}$, and hence the quantity of water for the rectangular orifice $GHKL$:

$$Q = \tfrac{2}{3}\, b \sqrt{2\, g\, h_1^3} - \tfrac{2}{3}\, b \sqrt{2\, g\, h_2^3} = \tfrac{2}{3}\, b \sqrt{2 g}\, (h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}),$$ and the mean velocity of efflux:

$$v = \frac{Q}{b(h_1 - h_2)} = \tfrac{2}{3} \sqrt{2\, g} \,.\, \frac{h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}}{h_1 - h_2}.$$

If h is the mean head of water $\frac{h_1 + h_2}{2}$, or the depth of the centre of the orifice below the surface of water, and a the height of the orifice $HK = h_1 - h_2$, we may then put:

$$v = \tfrac{2}{3} \sqrt{2 g} \,.\, \frac{\left(h + \frac{a}{2}\right)^{\frac{3}{2}} - \left(h - \frac{a}{2}\right)^{\frac{3}{2}}}{a},$$ or approximately:

$$= \left[1 - \tfrac{1}{96} \left(\frac{a}{h}\right)^2\right] \sqrt{2 g h}.$$

Example. If a rectangular orifice is 3 feet wide and $1\frac{1}{4}$ feet high, and the lower edge lies $2\frac{3}{4}$ feet below the surface of water, the discharge is then:

$Q = \frac{2}{3} . 8{,}02 . 3\, (2{,}75^{\frac{3}{2}} - 1{,}5^{\frac{3}{2}}) = 16{,}04\, (4{,}560 - 1{,}837) = 16{,}04 . 2{,}723 = 43{,}67$ cubic feet. From the formulæ of approximation the mean velocity of efflux is:

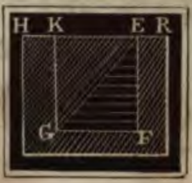
$$v = \left[1 - \tfrac{1}{96} \left(\frac{1{,}25}{2{,}125}\right)^2\right] . \; 8{,}02 \sqrt{2{,}125} = (1 - 0{,}0036) . 11{,}685 = 11{,}685 - 0{,}042 =$$

11,643 feet, and hence the discharge $Q = 3 . \frac{5}{4} . 11{,}643 = 43{,}65$ cubic feet.

Remark.—If the cut in the side is inclined to the horizon at an angle δ, we shall then have to substitute the height of the aperture $\frac{h_1 - h_2}{\sin. \delta}$ for its vertical projection, whence we must put $Q = \frac{2}{3} \frac{b \sqrt{2g}}{\sin. \delta} (\sqrt{h_1^3} - \sqrt{h_2^3})$. If the transverse section of the reservoir parallel to the aperture be not considerably greater than the section of the aperture, we shall then have to take into account the velocity $v_1 = \frac{F}{G}\, v$ with which the water flows to it, and for this reason put:

$$Q = \tfrac{2}{3}\, b \sqrt{2g} \left[\left(h_1 + \frac{v_1^2}{2g}\right)^{\frac{3}{2}} - \left(h_2 + \frac{v_1^2}{2g}\right)^{\frac{3}{2}}\right].$$

Fig. 407.



§ 309. *Triangular Lateral Orifice.* — Besides rectangular lateral orifices, we have in practice triangular and circular. Let us first consider the efflux through a triangular orifice *EFG*, Fig. 407, with horizontal base, whose vertex *E* lies in the surface of the water. Let the base $FG = b$, and the height $EF = h$, let us divide the last into n equal parts, and carry through the points of division lines parallel to the base, we then resolve the entire surface into small elements of the areas:

$$\frac{b}{n} \cdot \frac{h}{n}, \frac{2b}{n} \cdot \frac{h}{n}, \frac{3b}{n} \cdot \frac{h}{n}, \text{ \&c.,}$$

and the heads of water:

$$\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}, \text{ \&c.}$$

The discharges for these are:

$$\frac{bh}{n^2} \sqrt{2g\frac{h}{n}}, \frac{2bh}{n^2} \sqrt{2g \cdot \frac{2h}{n}}, \frac{3bh}{n^2} \sqrt{2g \cdot \frac{3h}{n}}, \text{ \&c.,}$$

and we obtain the discharge for the whole orifice:

$$Q = \frac{bh}{n^2} \sqrt{2g\frac{h}{n}} (1 + 2\sqrt{2} + 3\sqrt{3 +} \ldots + n\sqrt{n})$$

$$= \frac{bh\sqrt{2gh}}{n^2\sqrt{n}} (1 + 2^{\frac{3}{2}} + 3^{\frac{3}{2}} + \ldots + n^{\frac{3}{2}}),$$

or since the series in the parenthesis

$$= \frac{n^{\frac{3}{2}} + 1}{\frac{3}{2} + 1} = \tfrac{2}{5} n^{\frac{5}{2}}, \; Q = \tfrac{2}{5}\, bh \sqrt{2gh} = \tfrac{2}{5}\, b \sqrt{2gh^3}.$$

If the base of the orifice *EGK* lies in the surface and the vertex lower by h, we then have the discharge $\frac{2}{3}\, bh \sqrt{2gh}$ flowing through the rectangle *EFGK*,

$$Q_1 = \tfrac{2}{3}\, bh \sqrt{2gh} - \tfrac{2}{5}\, bh \sqrt{2gh} = \tfrac{4}{15}\, bh \sqrt{2gh}.$$

Through the trapezium *ABCD*, Fig. 408, whose upper base $AB = b_1$, lies in the surface of the water, and whose lower base is $CD = b_2$, and height $DE = h$, we may find the discharge by regarding the orifice as composed of a rectangle and two triangles, viz.,

$$Q = \tfrac{2}{3}\, b_2 h \sqrt{2gh} + \tfrac{4}{15}\, (b_1 - b_2)\, h \sqrt{2gh} = \tfrac{2}{15}\, (2b_1 + 3b_2)\, h \sqrt{2gh}.$$

Fig. 408.

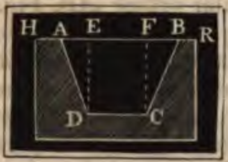
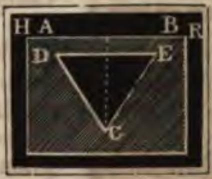


Fig. 409.



Further, the discharge for a triangle *CDE*, Fig. 409, of the base $DE = b_1$, and of the height h_1, and whose vertex *C* is distant h from the surface of water: $Q =$ discharge through ABC less the discharge through AE

$$= \tfrac{4}{15}\, bh \sqrt{2\, gh} - \tfrac{2}{15}\, (2\, b + 3\, b_1)\, h_1 \sqrt{2\, gh_1}$$
$$= \tfrac{2}{15} \sqrt{2\, g}\, [2\, b\, (h^{\frac{3}{2}} - h_1^{\frac{3}{2}}) - 3\, b_1 h_1^{\frac{3}{2}}].$$

As the breadth $AB = b$ may be determined by the proportion $b : b_1 = h : (h - h_1)$, it follows that

$$Q = \frac{2\sqrt{2g}\, .\, b_1}{15} \left(\frac{2\, h\, (h^{\frac{3}{2}} - h_1^{\frac{3}{2}})}{h - h_1} - 3\, h_1^{\frac{3}{2}}\right)$$
$$= \frac{2\sqrt{2\, gb_1}}{15} \left(\frac{2\, h^{\frac{5}{2}} - 5\, h\, h_1^{\frac{3}{2}} + 3\, h_1^{\frac{5}{2}}}{h - h_1}\right).$$

Lastly, for a triangle ACD, Fig. 410, whose vertex lies above the base, the quantity discharged is

$$Q = \tfrac{2}{3}\sqrt{2g}\, .\, b_1\, (h^{\frac{3}{2}} - h_1^{\frac{3}{2}}) - \frac{2\sqrt{2g}\, .\, b_1}{15} \left(\frac{2\, h^{\frac{5}{2}} - 5\, h\, h_1^{\frac{3}{2}} + 3\, h_1^{\frac{5}{2}}}{h - h_1}\right)$$
$$= \frac{2\sqrt{2g}\, .\, b_1}{15} \left(\frac{3\, h^{\frac{5}{2}} - 5\, h_1\, h^{\frac{3}{2}} + 2\, h_1^{\frac{5}{2}}}{h - h_1}\right).$$

Fig. 410.

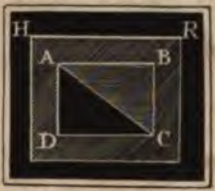
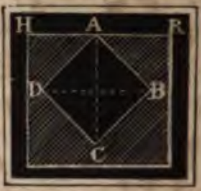


Fig. 411.



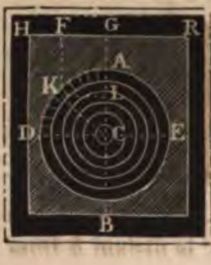
Example. What quantity of water flows through the square $ABCD$, Fig. 411, whose vertical diagonal $AC = 1$ foot, if the angular point A reaches the surface of the water? The upper half of this square gives the expenditure:

$Q = \frac{2}{5}\, b \sqrt{2\, gh^3} = \frac{2}{5}\, 1\, .\, 8{,}02 \sqrt{\frac{1}{8}} = 0{,}4\, .\, 8{,}02\, .\, 0{,}353 = 3{,}21\, .\, 0{,}353 = 1{,}1331$ cubic feet, but the lower water expenditure:

$$Q_1 = \frac{2\, b\sqrt{2g}}{15} \left(\frac{2\, h^{\frac{5}{2}} - 5\, h\, h_1^{\frac{3}{2}} + 3\, h_1^{\frac{5}{2}}}{h - h_1}\right) = \frac{2\, .\, 8{,}02}{15} \left(\frac{2 - 5\, (\frac{1}{2})^{\frac{3}{2}} + 3\, (\frac{1}{2})^{\frac{5}{2}}}{1 - \frac{1}{2}}\right)$$

$= \frac{32{,}08}{15}\, (2 - 1{,}7678 + 0{,}5303) = \frac{32{,}08\, .\, 0{,}7625}{15} = 1{,}6307$ cubic feet; the discharge through the entire orifice is $Q = 1{,}1331 + 1{,}6307 = 2{,}7638$ cubic feet.

Fig. 412.



§ 310. *Circular Lateral Orifices.*—The discharge through a circular orifice AB, Fig. 412, may be determined by an approximate formula in the following manner. Let us decompose the orifice by concentric circles into equally small annuli, and each annulus into very small elements, which may be regarded as parallelograms. If now r is the radius of such an annulus, b its breadth and n the number of its elements, we have the magnitude of an element

$K_1 = \frac{2\pi r b}{n}$. If h is the depth CG of the centre C below the surface of water HR, and ϕ the angle ACK, by which an element K is distant from the highest point A of the annulus, we have then the head of water of this element:

$$KF = CG - CL = h - r\cos.\phi,$$

and hence the discharge of this element

$= \frac{2\pi r b}{n} \sqrt{2g(h - r\cos.\phi)}$. Now $\sqrt{h - r\cos.\phi}$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\cos.\phi - \tfrac{1}{8}\left(\frac{r}{h}\right)^2 \cos.\phi^2 + \ldots\right]$$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\cos.\phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 (1 + \cos.2\phi) + \ldots\right];$$

hence the discharge of an element:

$$= \frac{2\pi r b}{n}\sqrt{2gh}\left[1 - \tfrac{1}{2}\cdot\frac{r}{h}\cos.\phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2(1 + \cos.2\phi) - \ldots\right].$$

The discharge of an entire annulus is now known, if we put in the parenthesis for 1, $n \,.\, 1 = n$, for $\cos.\phi$ the sum of all the cosines of ϕ from $\phi = 0$ to $\phi = 2\pi$, and for the cosine of 2ϕ, the sum of all the cosines of 2ϕ from $2\phi = 0$ to $2\phi = 4\pi$. But as the sum of all the cosines of a complete circle is $= 0$, these cosines vanish, and the discharge for the annulus:

$$= \frac{2\pi r b}{n}\sqrt{2gh}\left[n - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 . n - \ldots\right]$$

$$= 2\pi r b\sqrt{2gh}\left[1 - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 - \ldots\right].$$

If now for b we substitute $\frac{r}{m}$, and for r, $\frac{r}{m}, \frac{2r}{m}, \frac{3r}{m}$, to $\frac{mr}{m}$, we then obtain the discharge of all the annuli which make up the circular surface, and lastly, the quantity of efflux of the whole circle

$$Q = 2\pi r\sqrt{2gh}\left[\frac{r}{m^2}(1 + 2 + 3 + \ldots + m)\right.$$

$$\left. - \tfrac{1}{16}\frac{r^3}{m^4 h^2}(1^3 + 2^3 + 3^3 + \ldots + m^3)\right]$$

$$= 2\pi r\sqrt{2gh}\,.\left(\frac{r}{m^2}\cdot\frac{m^2}{2} - \tfrac{1}{16}\frac{r^3}{m^4 h^2}\cdot\frac{m^4}{4}\right)$$

$$= \pi r^2\sqrt{2gh}\left[1 - \tfrac{1}{32}\left(\frac{r}{h}\right)^2 - \ldots\right],$$

or more accurately:

$$Q = \pi r^2\sqrt{2gh}\left[1 - \tfrac{1}{32}\left(\frac{r}{h}\right)^2 - \tfrac{5}{1024}\left(\frac{r}{h}\right)^4 - \ldots\right].$$

If the circle reaches the surface of the water, then

$$Q = \tfrac{987}{1024}\pi r^2\sqrt{2gh} = 0{,}964\, F\sqrt{2gh},$$

if F represents the area of the circle.

It is besides easy to conceive that in all cases where the head of water at the centre is equal to or greater than the diameter, we may put the whole series = 1, and take $Q = F\sqrt{2gh}$. This rule may also be applied to other orifices, and, therefore, in all cases where the centre of gravity of an orifice lies at least as deep below the fluid surface as the figure is high, the depth h of this point may be regarded as the head of water, and Q put $= F\sqrt{2gh}$.

If we consider that the mean of all the cosines of the first quadrant $= \frac{\pi}{4}$, and that all the cosines of the second $= -\frac{\pi}{4}$, and likewise that the mean of the first and of the second vanishes, we may then, by the method adopted above, find the discharge of the upper semicircle:

$$Q_1 = \frac{\pi r^2}{2}\sqrt{2gh}\left[1 - \frac{\pi}{12}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)^2\right],$$

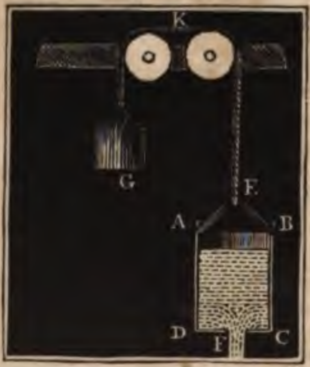
and that of the lower:

$$Q_2 = \frac{\pi r^2}{2}\sqrt{2gh}\left[1 + \frac{\pi}{12}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)^2 \cdot\cdot\right].$$

Example. What quantity of water flows hourly through a circular orifice 1 inch in diameter, above which the fluid surface stands $\frac{7}{12}$ inch high?
$\frac{r}{h} = \frac{6}{7}$, hence $\left(\frac{r}{h}\right)^2 = \frac{36}{49} = 0{,}735$; further, $1 - \frac{1}{32}\left(\frac{h}{r}\right)^2 = 1 - 0{,}023 = 0{,}977$, and consequently the discharge per second:
$Q = \frac{\pi \cdot 1^2}{4} 12 \cdot 8{,}02 \sqrt{\frac{7}{144}} \cdot 0{,}977 = \frac{\pi}{4} \cdot 8{,}02 \cdot 0{,}977 \sqrt{7} = 16{,}26$ cubic inches, which, per minute, = 973, and per hour = 33,78 cubic feet.

§ 311. *Discharging Vessels in Motion.*—The velocity of efflux varies if a vessel previously at rest or in uniform motion changes its condition of motion, because in this case every particle acts by its own weight, as well as by its inertia against the surrounding medium.

Fig. 413.



If we move the vessel AC, Fig. 413, upwards with a vertical accelerating force, whilst the water flows through the bottom by the hole F, an increase takes place, and if it be moved downwards vertically by an accelerating force, a diminution of the velocity of efflux ensues. If p is the accelerating force, each element of water M presses not only by its own weight Mg, but also by its inertia Mp; consequently the force of each element in the one case, must be put $(g + p)M$, and in the other $(g - p)M$, therefore instead of g, $g \pm p$. From this it follows then that $\frac{v^2}{2} = (g \pm p)h$, and hence for the velocity:

$$v = \sqrt{2(g \pm p)h}.$$

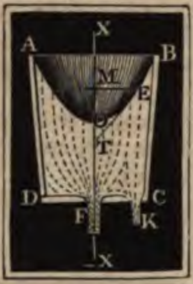
If the vessel ascends with the accelerating force g, then is

$v = \sqrt{2 \,.\, 2\, gh} = 2 \sqrt{gh}$, therefore the velocity of efflux 1,414 times that of a vessel at rest. If the vessel falls by its own weight, therefore, with the accelerated motion g, there is $v = \sqrt{0} = 0$, no water therefore flows out. If the vessel moves uniformly up or down, there remains $v = \sqrt{2\, gh}$, but if it ascends with a retarded motion, then will $v = \sqrt{2\,(g - p)\, h}$, and if it descends with the same retardation, then $v = \sqrt{2\,(g + p)\, h}$.

If the vessel moves horizontally, or at an acute angle to the horizon (§ 274), the fluid surface will be inclined to the horizon, and a change in the velocity of flow will take place.

By the rotation of a vessel AC, Fig. 414, about its vertical axis $\overline{XX}$, the concave surface forms a parabolic funnel AOB, hence there will be over the middle F of the bottom a lesser head of water than at the edges, and hence the water will flow through the orifice F, in the axis, more slowly than through any other orifice K at the bottom. If h represent the head of water in the middle, then the velocity of efflux at the middle will be $= \sqrt{2\, gh}$, if y be the distance $FK = ME$ of any other orifice K from the axis, and ω the angular velocity, we shall then have the corresponding elevation of the water above the middle:

Fig. 414.



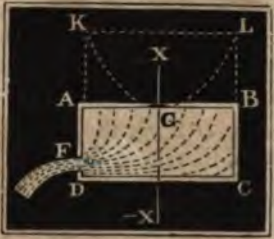
$$OM = \tfrac{1}{2}\, TM = \tfrac{1}{2}\, ME \text{ cotang. } T = \tfrac{1}{2}\, y \,.\, \frac{\omega^2 y}{g} = \frac{\omega^2 y^2}{2g} = \frac{w^2}{2g},$$

if w be the velocity of rotation of the orifice K. Hence then the velocity of efflux for this is

$$v = \sqrt{2g\left(h + \frac{w^2}{2g}\right)} = \sqrt{2\, gh + w^2}.$$

This formula is true for every arbitrarily shaped vessel, and also for one closed above, as AC, Fig. 415, so that the funnel cannot be formed. Its application to wheels of reaction and to turbines will be found in the sequel.

Fig. 415.



Examples.—1. If a vessel full of water AC, Fig. 413, weighs 350 lbs., and by means of a rope passing over a roller K is drawn by a weight G of 450 lbs., it will ascend with an accelerating force $p = \frac{450 - 350}{450 + 350} \cdot g = \frac{100}{800}\, g = \frac{1}{8}\, g$, and hence the velocity of efflux will be $v = \sqrt{2\,(g + p)\, h} = \sqrt{2 \,.\, \frac{9}{8} \,.\, gh} = \sqrt{\frac{9}{4}\, gh}$. Were the head of water $h = 4$ feet, the velocity of efflux would be $v = 1 \sqrt{9 \,.\, g} = 3 \sqrt{32{,}2} = 16{,}01$ feet.—2. If a vessel AC, Fig. 415, full of water revolves so that it makes 100 revolutions per minute, if the depth of the orifice F below the surface of water in the middle amounts to 2 feet, and the distance from the axis $\overline{XX}$, 3 feet, then the velocity of efflux is

$$v = \sqrt{2\, g\, h + w^2} = \sqrt{64{,}4 \,.\, 2 + \left(\frac{3 \,.\, \pi \,.\, 100}{30}\right)^2} = \sqrt{128{,}8 + 100 \,.\, \pi^2}$$

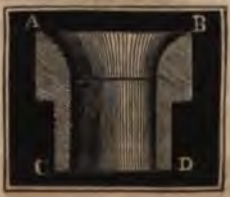
$= \sqrt{128 + 987} = 33{,}4$ feet. If the vessel be at rest the velocity will be $v = \sqrt{128.8}$ $= 11.34$ feet.

CHAPTER II.

ON THE CONTRACTION OF THE FLUID VEIN BY THE EFFLUX OF WATER THROUGH ORIFICES IN A THIN PLATE.

§ 312. *Co-efficient of Velocity.*—The laws of efflux developed in the preceding chapter accord almost entirely with experiment, so long as the head of water is not small compared with the width of the orifice, and as long as the orifice gradually widens inwards without forming corners or edges, and is close at the bottom or sides of the vessel. The experiments made by Michelotti, by Eytelwein, and by the author on this subject, with smoothly polished metallic mouth-pieces, have shown that the effective discharge, or that which actually flows out, amounts to from 96 to 98 per cent. of the theoretical quantity.

Fig. 416.



The mouth-piece *AD*, Fig. 416, represented in half its natural size, gave for a head of water of 10 feet, 97,5 per cent., for 5 ft. 96,9 per cent., and for 1 ft. 95,8 per cent. Since for this efflux the fluid vein has the same transverse section as the orifice, we must then assume that this diminution of discharge is accompanied with a loss of velocity, which is caused by the friction or adhesion of the water to the inner circumference of the orifice, and by the viscidity of the water. In what follows, we shall call the ratio of the effective velocity of efflux to that of the theoretical $= v \sqrt{2\,gh}$, the *co-efficient of velocity*, and represent it by ϕ. From this, therefore, the effective velocity of efflux in the most simple case is $v_1 = \phi\, v = \phi \sqrt{2\,gh}$, and the discharge:

$$Q = Fv_1 = \phi\, Fv = \phi\, F \sqrt{2\,gh}.$$

If we substitute for ϕ the mean value 0,97, we then obtain for the quantity in feet

$$Q = 0{,}97 \,.\, F \sqrt{2\,gh} = 0{,}97 \,.\, 8{,}02\, F \sqrt{h} = 7{,}779\, F \sqrt{h}.$$

A *vis viva* $\frac{Q\gamma}{g} \,.\, v_1^2$, is inherent in a discharge Q issuing with the velocity v_1, by virtue of which it is capable of producing the mechanical effect $Q\gamma \,.\, \frac{v_1^2}{2g}$. But since by its descent from the height $h = \frac{v^2}{2g}$, the weight $Q\gamma$ produces the mechanical effect $Q\gamma \,.\, h = Q\gamma \frac{v^2}{2g}$, it follows that by the efflux of the water, this suffers a loss

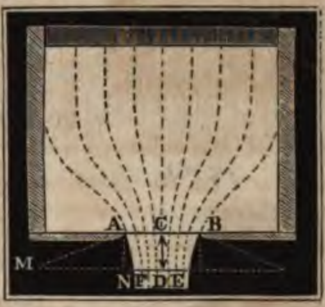
$$Q\gamma\left(\frac{v^2}{2g}-\frac{v_1^2}{2g}\right)=Q\gamma\,.\,\frac{v^2}{2g}(1-\phi^2)=(1-0{,}97^2)\,Q\gamma\,.\,\frac{v^2}{2g}$$
$$=0{,}059\;Q\gamma\,.\,\frac{v^2}{2g},\text{ or } 5{,}9 \text{ per cent.}$$

Therefore, the effluent water produces by its *vis viva* 5,9 per cent. less mechanical effect, than does its weight by falling from the height h.

§ 313. *Co-efficient of Contraction.*—If water flows through an *orifice in a thin plate*, a considerable diminution of the discharge under otherwise similar circumstances takes place, whilst the particles of fluid rushing through the orifice move in convergent directions, and in this way give rise to a *contraction of the fluid vein.* The measurements of the vein made by many, and especially of late by the author, have shown that the vein at a distance which is about equal to one-half of the width of the orifice, has the greatest contraction, and a thickness equal to 0,8 that of the diameter of the orifice. If F_1 is the transverse section of the contracted vein, as also F the transverse section of the orifice, we then have from this $F_1 = (0{,}8)^2\,F = 0{,}64\,F$. The ratio $\frac{F_1}{F}$ of these transverse sections is called the *co-efficient of contraction*, and is represented by α, and accordingly, the mean value for the efflux of water through orifices in a thin plate may be put: $\alpha = 0{,}64$.

As long as we possess no more accurate knowledge on the contraction of the fluid vein, we may assume that the stream flowing through a circular orifice AB, Fig. 417, forms a body of rotation $ABEF$, whose envelope is generated by the revolution of a circular arc AF about the axis CD of the stream. Let the diameter AB of the orifice $= d$, and the distance CD of the contracted section EF, $= \frac{1}{2}\,d$, we then obtain the radius:
$MA = MF = r$ of the generating arc AF by means of the equation:

Fig. 417.



$$\overline{AN}^2 = FN\,(2\,MF - FN),\text{ or}$$
$$\frac{d^2}{4}=\frac{d}{10}\left(2r-\frac{d}{10}\right),\; r = 1{,}3\,d.$$

Orifices made after this figure of the contracted vein give pretty nearly the velocity of discharge $v_1 = 0{,}97\,.\,v$.

The contraction of the fluid vein is caused by the water which lies directly above the orifice flowing out together with that which comes to it from the sides. There takes place, therefore, in the interior of the vessel a convergence of the filaments of water, similar to that represented in the figure, and the contraction of the fluid vein consists in a mere propagation of this convergence. We may convince ourselves of this motion of the water in the vicinity of the orifice by means of a glass apparatus of efflux; if we drop into the fluid minute substances which are either heavier or lighter than water, for example, such as oak saw-dust, bits of sealing wax, &c., and allow them to pass out with it from the orifice.

§ 314. *Contraction of the Fluid Vein.*—If water flows through triangular or quadrilateral orifices, and in a thin plate, the stream then assumes particular figures. The inversion of the jet, or the altered position of its transverse section with respect to that of the orifice, is very striking to the eye, in consequence of which a corner of this section comes to coincide with the middle of one side of the orifice.

Fig. 418.

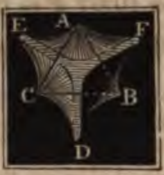


Fig. 419.

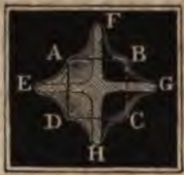
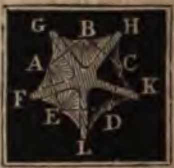
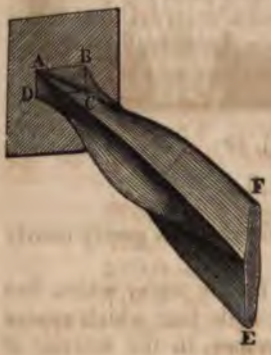


Fig. 420.



Hence, from a triangular orifice *ABC*, Fig. 418, the section of the stream at a certain distance from the orifice forms a treble star-like vein *DEF*, from a quadrilateral orifice *ABCD*, Fig. 419, a star of four veins *EFGH*, from a five-sided orifice *ABCDE*, Fig. 420, a star *FGHKL*, consisting of five veins. These sections vary at different distances from the orifice: at a certain distance they diminish, and at a successive one again increase; hence the vein consists of plates or ribs of variable breadth, and thereby forms, when the efflux is observed under great pressure, bulges and nodes, similar to what is seen in the cactus. If the orifice *ABCD*, Fig. 421, is rectangular; at a lesser distance from the orifice, the section will then form a cross or star; and at a greater one, it will again assume the form of a rectangle *EF*.

Fig. 421.



Observations on various kinds of orifices have been made by Bidone, and accurate measurements of the vein from square apertures also by Poncelet and Lesbros.* The last measurements have led to a small co-efficient of contraction 0,563. The measurements of water issuing through lesser orifices, give us, however, greater co-efficients of contraction; they show, moreover, that these are greater for elongated rectangles than for rectangles which approximate more to the square.

---

* *See* Allgem. Maschinen-encyclopädie, article Ausfluss.

§ 315. *Co-efficient of Efflux.*—If in the flow of water through orifices in thin plates, the effective velocity were equal to the theoretical $v = \sqrt{2gh}$, we should have the effective discharge:

$$Q = \alpha F . v = \alpha F \sqrt{2gh},$$

because αF represents the transverse section of the vein at the place of greatest contraction, where the particles of water move in parallel directions. But this is by no means the case: it is shown rather by experience that Q is smaller than $\alpha F \sqrt{2gh}$, that we must therefore multiply the theoretical discharge $F \sqrt{2gh}$ by a co-efficient which is less than the co-efficient of contraction, in order to obtain the effective discharge. We must hence assume that for efflux from an orifice in a thin plate, a certain loss of velocity takes place, and therefore introduce a co-efficient of velocity ϕ, and hence put the effective velocity of efflux $v_1 = \phi v = \phi \sqrt{2gh}$. From this then we have the effective discharge: $Q_1 = F_1 . v_1 = \alpha F . \phi v = \alpha \phi F v = \alpha \phi F \sqrt{2gh}$. Again, if we call the ratio of the effective discharge to the theoretical or hypothetical quantity, the *co-efficient of efflux*, and represent it in what follows by μ, we then have:

$$Q_1 = \mu Q = \mu F v = \mu F \sqrt{2gh},$$

hence $\mu = \alpha \phi$, i. e. *the co-efficient of efflux is the product of the co-efficients of contraction and of velocity.*

Multiplied observations, but chiefly the measurements of the author, have led to this, that the co-efficient of efflux for orifices in thin plates is not constant; that for small orifices and for small velocities, it is greater than for large orifices and for great velocities: and that it is considerably greater for elongated and small orifices than for orifices which have a regular form, or which approximate to the circle.

For square orifices of from 1 to 9 square inches area, with from 7 to 21 feet head of water, according to the experiments of Bossut and Michelotti, the mean co-efficient of efflux is $\mu = 0{,}610$; for circular ones of from ½ to 6 inches diameter, with from 4 to 21 feet head of water, $\mu = 0{,}615$, or about $\frac{8}{13}$. The single values observed by Bossut and Michelotti vary considerably from one another, but we cannot discover in them any dependence between the dimensions of the orifice and the magnitude of the head of water. From the author's experiments at a pressure of 24 inches, the co-efficient for an orifice of

0,3937 inches or 1 centimetre diameter is $\mu = 0{,}628$
0,7874 " 2 centimetres " $= 0{,}621$
1,1811 " 3 " " $= 0{,}614$
1,5748 " 4 " " $= 0{,}607$.

On the other hand, at a pressure of 10 inches for the round orifice of

1 centimetre diameter $\mu = 0{,}637$
2 centimetres " $= 0{,}629$
3 " " $= 0{,}622$
4 " " $= 0{,}614$

From these it is manifest that the co-efficient of efflux increases when the dimensions of the orifice and the head of water decrease.

If for μ we take the mean value $= 0{,}615$, and for $\alpha = 0{,}64$, we obtain the co-efficient of velocity for the efflux through orifices in a thin plate, $\phi = \frac{\mu}{\alpha} = 0{,}96$, therefore, nearly as great as for efflux through rounded or conoidal orifices.

Remark 1. Buff's experiments (see Poggendorf's Ann. Band 46), show that the co-efficient of efflux for small orifices and for small heads of water or velocities is considerably greater than for large or mean orifices and velocities. An orifice of 2,084 lines diameter, gave for $1\frac{1}{2}$ inch pressure, $\mu = 0{,}692$, for 35 inches $\mu = 0{,}644$; on the other hand, an orifice of 4,848 lines for $4\frac{1}{2}$ inches pressure $\mu = 0{,}682$, and for 29 inches $\mu = 0{,}653$.

Remark 2. According to the author's experiments, the co-efficients for efflux under water are about $1\frac{1}{3}$ per cent. less than for efflux in air.

§ 316. *Rectangular Lateral Orifices.*—The most accurate experiments on efflux through large rectangular lateral apertures are those made at Metz by Poncelet and Lesbros. The widths of these orifices were two decimetres, (nearly 8 inches); the depths, however, varied from one centimetre to two decimetres. In order to produce perfect contraction, a brass plate of four millimetres, $= 0{,}1575$ inches, thickness was used for these orifices. From the results of their experiments, these experimenters have calculated by interpolation the tables at the end of this paragraph for the co-efficients which may be used for the measurement or calculation of the discharge.

If b be the breadth of the orifice, and if h_1 and h_2 are the heads of water above the lowest, and above the uppermost horizontal edge of the orifice, we then have, from § 308, the discharge: $Q = \frac{2}{3} b \sqrt{2g} \, (h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}})$. But if we substitute the height of the aperture a, and the mean head of water $h = \frac{h_1 + h_2}{2}$, we then have approximately $Q = \left(1 - \frac{a^2}{96\, h^2}\right) ab \sqrt{2gh}$, and hence the effective discharge $Q_1 = \mu\, Q = \left(1 - \frac{a^2}{96\, h^2}\right) \mu\, ab \sqrt{2gh}$. If, further, we put $\left(1 - \frac{a^2}{96\, h^2}\right) \mu = \mu_1$, we have then simply $Q_1 = \mu_1 ab \sqrt{2gh}$, and in order to allow of our calculating by this simple or general formula of efflux, not only the values of μ, but also those of μ_1 are given in the following tables.

Since the water in the vicinity of the orifice is in motion, it stands lower directly before the aperture than at a greater distance from the plate in which the aperture is made; on this account two tables have been compiled, the one for heads of water measured at a greater distance from the orifice, and the other for those measured immediately at the side in which the orifice lies. It may be seen, moreover, from both tables, although with certain variations, that the co-efficients of efflux increase the lower the orifice is, and the less the head of water.

If the orifices have different breadths, we are compelled, so long as we have no further experiments, still to use the co-efficients of these tables in like manner for the calculation of the discharge. If, further, the orifices are not rectangular, we must determine their mean breadth and mean depth, and introduce into the calculation the co-efficients corresponding to these dimensions. Lastly, it is always preferable to measure the head of water at a certain distance from the side in which the orifice lies, and to use the first table, than directly at the orifice where the surface of water is curved and less tranquil, than a little above it.

Examples. 1.—What quantity of water flows through a rectangular aperture, 2 decimetres broad and 1 decimetre deep, if the surface of water is 1½ metre above the upper edge? Here $b = 0{,}2$; $a = 0{,}1$, $h = \frac{h_1 + h_2}{2} = \frac{1{,}6 + 1{,}5}{2} = 1{,}55$; hence the theoretical discharge $Q = 0{,}1 \,.\, 0{,}2 \sqrt{2g} \,.\, \sqrt{1{,}55} = 0{,}02 \,.\, 4{,}429 \,.\, 1{,}245 = 0{,}1103$ cubic metre. But now Table I. gives for $a = 0{,}1$ and $h_2 = 1{,}5$, $\mu_1 = 0{,}611$, hence the effective discharge $Q_1 = 0{,}611 \,.\, 0{,}1103 = 0{,}0674$ cubic metre —2. What discharge corresponds to a rectangular orifice in a thin plate of 8 inches breadth, 2 inches depth, with a 15 inches head of water above the upper edge?* The theoretical discharge is $Q = \frac{2}{3} \,.\, \frac{1}{6} \,.\, 7{,}906 \sqrt{\frac{4}{3}} = 0{,}8784 \,.\, 1{,}1547 = 1{,}014$ cubic feet. But now 2 inches is about 0,05 metre, and 15 inches about 0,4 metre; hence, according to the table $a = 0{,}05$ and $h_2 = 0{,}4$, the corresponding co-efficient $\mu_1 = 0{,}628$ is to be taken, and the quantity of water sought is $Q_1 = 0{,}628 \,.\, 1{,}014 = 0{,}637$ cubic feet.—3. If the breadth $= 0{,}25$, the depth $= 0{,}15$, and the head of water $h_2 = 0{,}045$ metre, then is $Q = 0{,}25 \,.\, 0{,}15 \,.\, 4{,}429 \,.\, \sqrt{0{,}12} = 0{,}166 \,.\, 0{,}3464 = 0{,}0575$ cubic metre. To the height 0,15 corresponds for $h_2 = 0{,}04$, the mean value: $\mu_1 = \frac{0{,}582 + 0{,}603}{2} = 0{,}5925$, and $h_2 = 0{,}05$, $\mu_1 = \frac{0{,}585 + 0{,}605}{2} = 0{,}595$; but since h_2 is given $= 0{,}045$, we must then substitute the new mean $\frac{0{,}5925 + 0{,}595}{2} = 0{,}594$ for the co-efficient of efflux, and we therefore obtain the discharge sought: $Q_1 = 0{,}594 \,.\, 0{,}0575 = 0{,}03415$ cubic metre.

---

* In using the following tables, the English measures will be furnished with the proper co-efficients by employing the first, or left-hand column, in which to find the height h, and the column under the number of inches answering to the height of orifice a.—Am. Ed.

TABLE I.

The co-efficients for the efflux through rectangular orifices in a thin vertical plate, from Poncelet and Lesbros. The heads of water are measured at a certain distance back from the orifice, or at a point where the water may be considered as still.

Head of water, or distance of the surface of water from the upper side of the orifice.		HEIGHT OF ORIFICE.					
Eng. in.	Metres.	0,20^m or 8 in.	0,10^m or 4 in.	0,05^m or 2 in.	0,03^m or 1,18 in.	0,02 or ,8 in.	0,01^m or ,4 in.
0,00	0,000	"	"	"	"	"	"
0,19	0,005	"	"	"	"	"	0,705
0,39	0,010	"	"	0,607	0,630	0,660	0,701
0,57	0,015	"	0,593	0,612	0,632	0,660	0,697
0,78	0,020	0,572	0,596	0,615	0,634	0,659	0,694
1,18	0,030	0,578	0,600	0,620	0,638	0,659	0,688
1,57	0,040	0,582	0,603	0,623	0,640	0,658	0,683
1,97	0,050	0,585	0,605	0,625	0,640	0,658	0,679
2,36	0,060	0,587	0,607	0,627	0,640	0,657	0,676
2,75	0,070	0,588	0,609	0,628	0,639	0,656	0,673
3,14	0,080	0,589	0,610	0,629	0,638	0,656	0,670
3,54	0,090	0,591	0,610	0,629	0,637	0,655	0,668
3,93	0,100	0,592	0,611	0,630	0,637	0,654	0,666
4,72	0,120	0,593	0,612	0,630	0,636	0,653	0,663
5,51	0,140	0,595	0,613	0,630	0,635	0,651	0,660
6,29	0,160	0,596	0,614	0,631	0,634	0,650	0,658
7,08	0,180	0,597	0,615	0,630	0,634	0,649	0,657
7,87	0,200	0,598	0,615	0,630	0,633	0,648	0,655
9,84	0,250	0,599	0,616	0,630	0,632	0,646	0,653
11,81	0,300	0,600	0,616	0,629	0,632	0,644	0,650
15,75	0,400	0,602	0,617	0,628	0,631	0,642	0,647
19,68	0,500	0,603	0,617	0,628	0,630	0,640	0,644
23,62	0,600	0,604	0,617	0,627	0,630	0,638	0,642
27,56	0,700	0,604	0,616	0,627	0,629	0,637	0,640
31,49	0,800	0,605	0,616	0,627	0,629	0,636	0,637
35,43	0,900	0,605	0,615	0,626	0,628	0,634	0,635
39,37	1,000	0,605	0,615	0,626	0,628	0,633	0,632
43,30	1,100	0,604	0,614	0,625	0,627	0,631	0,629
47,24	1,200	0,604	0,614	0,624	0,626	0,628	0,626
51,18	1,300	0,603	0,613	0,622	0,624	0,625	0,622
55,11	1,400	0,603	0,612	0,621	0,622	0,622	0,618
59,05	1,500	0,602	0,611	0,620	0,620	0,619	0,615
62,99	1,600	0,602	0,611	0,618	0,618	0,617	0,613
66,93	1,700	0,602	0,610	0,617	0,616	0,615	0,612
70,86	1,800	0,601	0,609	0,615	0,615	0,614	0,612
74,80	1,900	0,601	0,608	0,614	0,613	0,612	0,611
78,74	2,000	0,601	0,607	0,613	0,612	0,612	0,611
118,11	3,000	0,601	0,603	0,606	0,608	0,610	0,609

TABLE II.

Co-efficients of efflux through rectangular orifices in a vertical plate, from Poncelet and Lesbros. The heads of water are measured directly at the orifice.

Head of water, or distance of the surface of water from the upper side of the orifice.		HEIGHT OF ORIFICE.					
Eng. in.	Metres.	0,20^{m} or 8 in.	0,10^{m} or 4 in.	0,05^{m} or 2 in.	0,03^{m} or 1,18 in.	0,02^{m} or ,8 in.	0,01^{m} or ,4 in.
0,00	0,000	0,619	0,667	0,713	0,766	0,783	0,795
0,19	0,005	0,597	0,630	0,668	0,725	0,750	0,778
0,39	0,010	0,595	0,618	0,642	0,687	0,720	0,762
0,57	0,015	0,594	0,615	0,639	0,674	0,707	0,745
0,78	0,020	0,594	0,614	0,638	0,678	0,697	0,729
1,18	0,030	0,593	0,613	0,637	0,659	0,685	0,708
1,57	0,040	0,593	0,612	0,636	0,654	0,678	0,695
1,97	0,050	0,593	0,612	0,636	0,651	0,672	0,686
2,36	0,060	0,594	0,613	0,635	0,647	0,668	0,681
2,75	0,070	0,594	0,613	0,635	0,645	0,665	0,677
3,14	0,080	0,594	0,613	0,635	0,643	0,662	0,675
3,54	0,090	0,595	0,614	0,634	0,641	0,659	0,672
3,93	0,100	0,595	0,614	0,634	0,640	0,657	0,669
4,72	0,120	0,596	0,614	0,633	0,637	0,655	0,665
5,51	0,140	0,597	0,614	0,632	0,636	0,653	0,661
6,29	0,160	0,597	0,615	0,631	0,635	0,651	0,659
7,08	0,180	0,598	0,615	0,631	0,634	0,650	0,657
7,87	0,200	0,599	0,615	0,630	0,633	0,649	0,656
9,84	0,250	0,600	0,616	0,630	0,632	0,646	0,653
11,81	0,300	0,601	0,616	0,629	0,632	0,644	0,651
15,75	0,400	0,602	0,617	0,629	0,631	0,642	0,647
19,68	0,500	0,603	0,617	0,628	0,630	0,640	0,645
23,62	0,600	0,604	0,617	0,627	0,630	0,638	0,643
27,56	0,700	0,604	0,616	0,627	0,629	0,637	0,640
31,49	0,800	0,605	0,616	0,627	0,629	0,636	0,637
35,43	0,900	0,605	0,615	0,626	0.628	0,634	0,635
39,37	1,000	0,605	0,615	0,626	0,628	0,633	0,632
43,30	1,100	0,604	0,614	0,625	0,627	0,631	0,629
47,24	1,200	0,604	0,614	0,624	0,626	0,628	0,626
51,18	1,300	0,603	0,613	0,622	0,624	0,625	0,622
55,11	1,400	0,603	0,612	0,621	0,622	0,622	0,618
59,05	1,500	0,602	0,611	0,620	0,620	0,619	0,615
62,99	1,600	0,602	0,611	0,618	0,618	0,617	0,613
66,93	1,700	0,602	0,610	0,617	0,616	0,615	0,612
70,86	1,800	0,601	0,609	0,615	0,615	0,614	0,612
74,80	1,900	0,601	0,608	0,614	0,613	0,613	0,611
78,74	2,000	0,601	0,607	0,614	0,612	0,612	0,611
118,11	3,000	0,601	0,603	0,606	0,608	0,610	0,609

§ 317. *Wiers.*—If water flows through *wiers*, or through notches in a thin plate, as, for example, *FB*, Fig. 422, the fluid vein then suffers a contraction on three sides, by which a diminution of the discharge is effected, since the quantity discharged from these orifices is $Q_1 = \frac{2}{3} \mu\, b\, h \sqrt{2 g h}$. But here the head of water $EH = h$, or the head of water above the sill, of the wier must not be measured immediately at the sill, but at least two feet before the plate in which the orifice lies, because the fluid surface before the opening suffers a depression, which becomes greater and greater the nearer it is to the orifice, and in the plane of the orifice amounts to a quantity *GR* of from 0,1 to 0,25 the head of water *FR*, so that the thickness *FG* of the stream in this plane is only 0,9 to 0,75 of the head of water. Experiments instituted by many philosophers on the flow of water through notches in thin plates, have afforded a multiplicity of results, but not always of the desired accordance. The following short table contains the results of the experiments of Poncelet and Lesbros on wiers of two decimetres, or about 8 inches breadth.

Fig. 422.

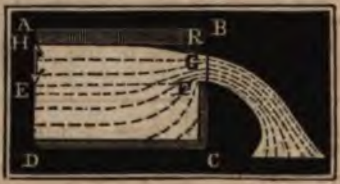


TABLE OF THE CO-EFFICIENTS OF EFFLUX FOR WIERS OF 2 DECIMETRES, = 7.87 INCHES BREADTH, ACCORDING TO PONCELET AND LESBROS.

Head of water h.	metrs. 0,01 or ,4 in	metrs. 0,02 ,8 in.	metrs. 0,03 1,2 in.	metrs. 0,04 1,6 in.	metrs. 0,06 2,4 in.	metrs. 0,08 3,2 in.	metrs. 0,10 4 in.	metrs. 0,15 6 in.	metrs. 0,20 8 in.	metrs. 0,22 9 in.
Co-efficient of efflux $\mu_1 = \frac{2}{3} \mu$.	0,424	0,417	0,412	0,407	0,401	0,397	0,395	0,393	0,390	0,385

From the average of determinations, we may here put $\mu_1 = 0,4$. Experiments on wiers of greater breadth gave Eytelwein the mean $\mu_1 = \frac{2}{3} \mu = 0,42$, and Bidone $\mu_1 = \frac{2}{3} \,.\, 0,62 = 0,41$, &c. The most extensive experiments are those of d'Aubuisson and Castel. From these, d'Aubuisson asserts that for wiers whose breadth is no more than the third part of the breadth of the canal or side in which the wier lies, the mean of μ is $= 0,60$, therefore, we may put $\frac{2}{3} \mu = 0,40$: but, on the other hand, for wiers which extend over the whole side, or have the same breadth as the water-course: $\mu = 0,665$, therefore, $\mu_1 = 0,444$; lastly, for other relations between the breadth of the wier and that of the canal, the co-efficients of efflux are very different, and lie between 0,58 and 0,66. Experiments made by the author, reduce the variability of these co-efficients to certain laws (§ 322).

[During his investigations in the summer of 1845, to determine the relative value of the several sources for supplying water to the city of Boston, the editor had an opportunity of making extensive series of experiments on the passage of water through wiers of 1, 2, and 3,01 feet in breadth, and from 0,066 foot to 2,087 feet in depth above the bottom of the notch. The water was measured in a cubical box, 6 feet on a side, to which was attached, on the exterior, a glass gauge tube with a scale extending to the top of the receptacle. In like manner, a gauge tube was inserted in the dam which contained the notch, and several feet distant from it, with the 0 of its scale accurately adjusted to the level of the bottom of the notch. A scale sliding vertically was placed immediately over the centre of the wier by which the depth over the edge of the notch-board could be ascertained. The reservoir from which the water was drawn was at least 6 times as wide as the opening of the notch. The following are some of the co-efficients ($\mu_1 = \frac{2}{3}\mu$) for the several breadths of wier:

1.—Wier 3,01 feet in breadth, cut in 2 inch planks—

Full depth h, over bottom of notch.	Co-efficients of discharge $= \frac{2}{3}\mu$.	Depression of surface at the notch.
0,075 feet	0,3667	0,021 feet
0,189 “	0,3794	0,040 “
0,280 “	0,3973	0,070 “
0,316 “	0,4211	0,079 “
0,360 “	0,4307	0,086 “
0,480 “	0,4349	0,097 “
0,545 “	0,4376	0,120 “
0,689 “	0,4301	0,149 “
0,755 “	0,4294	0,155 “
0,801 “	0,4208	0,158 “
1,023 “	0,4129	0,167 “
2. Wier 2 feet wide, in a 1 inch board—		
0,199 feet	0,4195	unc. “
1,020 “	0,4344	0,196 “
1,062 “	0,4408	0,206 “
1,232 “	0,4477	0,228 “
1,280 “	0,4460	0,230 “
3. Wier 1 foot wide, in 1 inch board—		
0,329 feet	0,4144	unc. “
0,333 “	0,4166	unc. “
0,339 “	0,4191	unc. “
0,352 “	0,4265	0,068 “
0,360 “	0,4265	0,070 “
2,060 “	0,4149	0,118 “
2,087 “	0,4130	0,125 “

The “full depth over the notch” here signifies, of course, that of the general level of the reservoir, above the edge of the wier. In each case it will be observed that the above co-efficients increase with the increase of depth up to a certain point, and then diminish gradually as far as the observations were extended. As these experiments were made with a view to determine an important practical and economical question, they were conducted with great care, and are believed to be worthy of reliance as the basis of computation for works on an extended scale.]

Examples.—1. A wier, 0,25 metre broad and 0,15 head of water, gives per second the discharge $Q = 0{,}393 \,.\, bh\sqrt{2gh} = 0{,}393 \,.\, 4{,}429 \,.\, 0{,}25 \,.\, (0{,}15)^{\frac{3}{2}} = 0{,}435 \,.\, 0{,}0581 =$ 0,02527 cubic metres.—2. What breadth must be given to a wier which, with a head of water of 8 inches, will allow 6 cubic feet per second to pass through? It is

$$b = \frac{Q_1}{\mu_1 \sqrt{2gh^3}} = \frac{6}{0{,}4 \,.\, 8{,}02 \sqrt{(\frac{2}{3})^3}} = \frac{6}{3{,}208 \,.\, 0{,}5443} = 3{,}436 \text{ feet.}$$

If, according to Eytelwein, we take $\mu_1 = 0{,}42$, it follows that:

$$b = \frac{6}{3{,}368 \,.\, 0{,}5443} = 3{,}27 \text{ feet.}$$

§ 318. *Maximum and Minimum of Contraction.*—In the flow of water through orifices in a plane side, the axis of the stream is perpendicular to the surface of the side, and therefore the amount of the contraction is a mean, but if the axis of the orifice or of the fluid stream forms an acute angle with the portion of the side containing the orifice, the contraction will be less; and if the angle between this axis, and the inner surfaces of the edges of the aperture, be obtuse, the contraction will be still greater. The one case is represented in Fig. 423, and the other in Fig. 424. Without doubt this difference of contraction is caused by the particles of water, which flow towards the orifice from the sides, deviating less from their direction in the one than in the other case, when they pass through the orifice and unite to form a fluid stream.

Fig. 423.

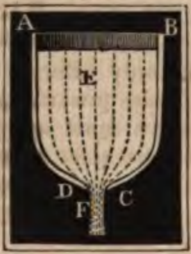
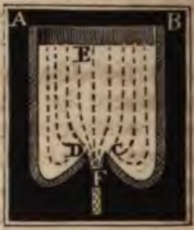


Fig. 424.



The contraction is a minimum, *i. e.* nothing, if by the gradual convergence of the side which embraces the orifice, the lateral flow is entirely prevented, and a maximum if the side has a direction opposite to that of the fluid stream, so that certain particles of water must revolve 180° before arriving at the orifice. Both cases are represented

Fig. 425.

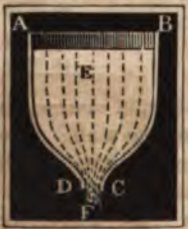
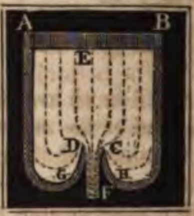


Fig. 426.



in Figs. 425 and 426. In the first case, the co-efficient of efflux is about 1, viz. 0,96 to 0,97; and in the second from the measurements of Borda, Bidone and the author, a mean of 0,53. Changes in the co-efficients of efflux through convergent sides very often present themselves in practice; they occur in dams, which are inclined to the horizon, as in Fig. 427. Poncelet found for a similar opening the co-efficient of efflux $\mu = 0,80$, when the board was inclined 45°, and on the other hand, $\mu = 0,74$ only for an inclination of $63\frac{1}{2}°$, that is, for a slope of $\frac{1}{2}$. For similar wiers, Fig. 428, where, as in the Poncelet sluice-board, contraction takes place at one side only, the author found $\mu = 0,70$, therefore, $\mu_1 = \frac{2}{3}\mu = 0,467$ for an inclination of 45°, and $\mu = 0,67$, therefore, $\mu_1 = 0,447$ for an inclination of $63\frac{1}{2}°$.

Fig. 427.

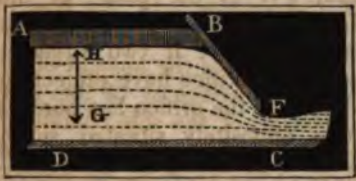
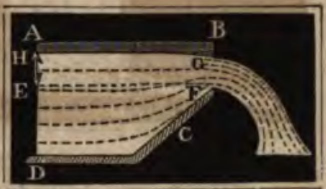


Fig. 428.

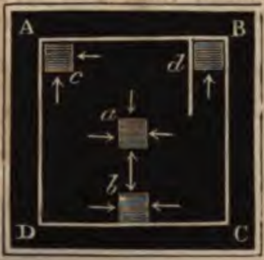


Example. If a sluice-board, inclined at an angle of 50°, which goes across a channel $2\frac{1}{4}$ feet broad, is drawn up $\frac{1}{2}$ foot high, and the surface of water stands 4 feet above the bottom of the channel, the height of the aperture may be put $a = \frac{1}{2}\, sin.\, 50° = 0,3830$ feet, the head of water $h = 4 - \frac{1}{2} \,.\, 0,3830 = 3,8085$ feet, and the co-efficient of efflux $\mu = 0,78$; hence, the discharge $Q_1 = 0,78 \,.\, 2,25 \,.\, 0,3830 \,.\, 8,02 \sqrt{3,8085} =$ 10,49 cubic feet (English).

§ 319. *Partial Contraction.*—We have only hitherto considered those cases where the water flows from all sides towards the aperture, and forms a contracted vein around, and we must now investigate others, where the water flows from one or more sides to the aperture, and therefore produces a stream only partially contracted. To distinguish the circumstances of contraction, we will call the case, where the vein is contracted on all sides, *general;* and the case, where it is only contracted in one part of its circumference, *partial*, or *imperfect contraction*. Partial contraction is induced when an orifice in a plane thin plate is confined by other plates in the direction of the fluid stream at one or more sides.

In Fig. 429, are represented four orifices of equal size a, b, c, d, in the bottom AC of a vessel. The contraction by efflux through the orifice a in the middle of the bottom is general, because the water can flow to it from all sides; the contraction from the efflux through b, c, d, is partial, because the water can only flow to them from one, two, or three sides. Likewise, if a rectangular lateral aperture goes to the bottom of the vessel, the contraction is then partial, because it falls away at the bottom side, if, further, the aperture of the

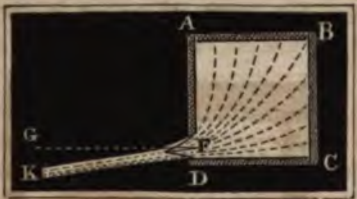
Fig. 429.



dam reaches the bottom in the lateral walls of the channel, there is then only a contraction on one side.

Partial contraction is remarkable in two respects; first, by giving an oblique direction to the stream; and, secondly, by increasing the quantity of discharge.

Fig. 430.



If the lateral aperture *F*, Fig. 430, reaches a second side *CD*, so that no contraction takes place there, the axis *FK* of the fluid stream becomes deflected by an angle *KFG* of about 9° from the normal *FG* to the plane of the orifice. The obliquity of the stream is much greater if two adjacent sides of the orifice have projecting borders. If the orifice has borders in two oppositely situated sides, and contraction at these prevented, such a deviation of course will not take place, but at the other side, the vein at some distance from the orifice, will spread out more than if the border were not there. Although a greater discharge is obtained by a partial contraction, we must, as a rule, endeavor to avoid this, because the fluid stream, in consequence, suffers a deviation in its direction and a greater extension in its breadth.

Experiments on the efflux of water with partial contraction have been made by Bidone and by the author. They allow us to assume that the co-efficients of efflux increase simultaneously with the ratio of the contracted part to the whole perimeter, though it is easy to perceive that this relation is different, if the perimeter is almost or entirely restricted, and the contraction almost or entirely suppressed. Let us put the ratio of this restriction to the entire perimeter $= n$, and let us represent by $\varkappa$, any number deduced from experiment, we may then, although only approximately, put the ratio of the corresponding co-efficient of efflux μ_n of partial contraction to the co-efficient of efflux of perfect contraction:

$$\frac{\mu_n}{\mu_0} = 1 + \varkappa\, n, \text{ and consequently } \mu_n = (1 + \varkappa\, n)\, \mu_0.$$

Bidone's experiments give for circular orifices $\varkappa = 0{,}128$, and for rectangular $\varkappa = 0{,}152$; the author's, however, give for the last, $\varkappa = 0{,}134$. Rectangular orifices with borders, are those which are most frequently met with in practice; we will assume for them the mean value $\varkappa = 0{,}143$, and hence put $\mu_n = (1 + 0{,}143 \,.\, n)\, \mu_0$. For a rectangular lateral orifice of the depth a and breadth b, $n = \frac{b}{2\,(a+b)}$, if the contraction on one side b is suppressed; if, for instance, this side lies in the plane of the bottom; again, $n = \frac{1}{2}$, if a side a and a side b are bordered, and $n = \frac{2\,a + b}{2\,(a+b)}$, if on one side b, and both sides a, the contraction is prevented; if, for example, the orifice takes up the whole breadth of the reservoir, and reaches the plane of the bottom.

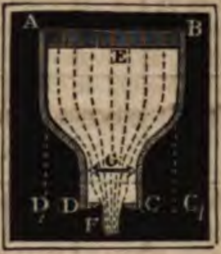
Example. What quantity of water flows through a 3 feet broad and 10 inch deep vertical aperture of a dam at a pressure of 1½ feet above the upper side of the aperture, if the lower one coincides with the bottom of the channel, and hence there is no contraction at the bottom? The theoretical discharge is:

$$Q = \tfrac{10}{12} \,.\, 3 \,.\, 8{,}02 \sqrt{1{,}5 + \tfrac{5}{12}} = \tfrac{5}{2} \,.\, 8{,}02 \sqrt{1{,}9166\,..} = 28{,}11 \text{ cubic feet.}$$

According to Poncelet's table for general contraction $\mu = 0{,}604$, we have, therefore, $n = \frac{3}{2\,(3 + \frac{10}{12})} = \frac{9}{18 + 5} = \frac{9}{23}$; hence, for the above cases of partial contraction $\mu_n\,(= 1 + 0{,}143 \,.\, \frac{9}{23}) \,.\, 0{,}604 = 1{,}056 \,.\, 0{,}604 = 0{,}638$, and the effective discharge is $Q_1 = 0{,}638\, Q = 0{,}638 \,.\, 28{,}11 = 18{,}14$ cubic feet.

§ 320. *Imperfect Contraction.*—The contraction of the fluid vein depends, further, upon whether the water before the orifice is tolerably still, or whether it arrives before it with a certain velocity. The quicker the water flows to the orifice, the less contracted does the vein become, and the greater is the discharge. The relations of contraction and efflux above given and investigated, have reference only to the case where the orifice lies in a large side, and it can only be assumed that the water flows to it with a small velocity; hence we must know the relations of contraction and efflux, when the transverse section of the orifice is not much less than that of the affluent water, and when, consequently, the water arrives at the orifice with a considerable velocity. In order to distinguish these two cases from one another, we shall call the contraction, where the superincumbent water is still, *perfect;* and that where it is in motion, *imperfect contraction.* The contraction, for example, is imperfect in the efflux from a vessel AC, Fig. 431, because the transverse section F of the orifice is not much smaller than that of G of the arriving water, or the area of the side CD, in which this orifice lies. If, on the other hand, the vessel had the form ABC_1D_1, and, therefore, the area of the bottom surface C_1D_1 much greater than the transverse section F of the orifice, the efflux would then go on with perfect contraction. The imperfectly contracted vein is besides distinguishable, not merely by its greater thickness from the perfectly contracted fluid vein, but also by its not having so transparent and crystalline an appearance.

Fig. 431.



If the ratio of the area of the orifice F, and the side containing the orifice G, therefore, $\frac{F}{G} = n$, the co-efficient of efflux for perfect contraction $= \mu_0$, and that for imperfect $= \mu_n$, we may with greater accuracy, according to the experiments and calculations made by the author, put:

1. For circular orifices:

$$\mu_n = \mu_0\,[1 + 0{,}04564\,(14{,}821^{\,n} - 1)], \text{ and}$$

2. For rectangular orifices:

$$\mu^n = \mu_0 \, [1 + 0{,}0760 \, (9^n - 1)].^*$$

To render the calculation easier in cases of application, the corrections $\frac{\mu_n - \mu_0}{\mu_0}$ of the co-efficient of efflux on account of imperfect contraction are compiled in the following short tables.

TABLE I.

CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR CIRCULAR ORIFICES.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,007	0,014	0,023	0,034	0,045	0,059	0,075	0,092	0,112	0,134	0,161	0,189	0,223	0,260	0,303	0,351	0,408	0,471	0,546	0,613

TABLE II.

CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR RECTANGULAR ORIFICES.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,009	0,019	0,030	0,042	0,056	0,071	0,088	0,107	0,128	0,152	0,178	0,208	0,241	0,278	0,319	0,365	0,416	0,473	0,537	0,608

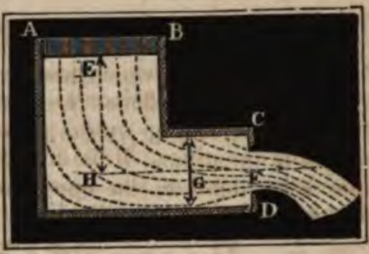
The different values of the ratio of the transverse sections $n = \frac{F}{G}$ stands above in these tables, and immediately below additions to the

---

[*] "Experiments on the Imperfect Contraction of Water," &c., Leipzic, 1843.

co-efficients of efflux, on account of imperfect contraction; for example, for the ratio of the transverse sections $n = 0{,}35$, *i. e.*, for the case where the area of an orifice is 35 hundredths of the area of the whole side of the orifice, we have for circular orifices $\frac{\mu_n - \mu_0}{\mu_0} = 0{,}075$, and for rectangular orifices $= 0{,}088$; therefore, the co-efficient of efflux for perfect contraction in the first case is to be made about 75 thousandths, and in the second about 88 thousandths greater to obtain the corresponding co-efficients of efflux for imperfect contraction. Were the co-efficient of efflux $\mu_0 = 0{,}615$, we should have in the first case $\mu_{0\ 35} = 1{,}075 \,.\, 0{,}615 = 0{,}661$, and in the second, $\mu_{0\ 35} = 1{,}088 \,.\, 0{,}615 = 0{,}669$.

Fig. 432.



Example.—What discharge does a rectangular lateral aperture F, $1\frac{1}{4}$ feet broad and $\frac{1}{2}$ foot deep, give if it be cut in a rectangular wall CD, Fig. 432, 2 feet broad and 1 foot deep, and the head of water $EH = h$ in still water amounts to 2 feet? The theoretical discharge is $Q = 1{,}25 \,.\, 0{,}5 \,.\, 8{,}02 \sqrt{2} = 5{,}012 \,.\, 1{,}414 = 7{,}086$ cubic feet, and the co-efficient of efflux for perfect contraction is, according to Poncelet, $\mu_0 = 0{,}610$; but now the ratio of the transverse sections $n = \frac{F}{G} = \frac{1{,}25 \,.\, 0{,}5}{2 \,.\, 1} = 0{,}312$, and for $n = 0{,}312$, from Table II., $\frac{\mu_n - \mu_0}{\mu_0} = 0{,}071 + \frac{12}{50}(0{,}088 - 0{,}071) = 0{,}071 + 0{,}004 = 0{,}075$; hence it follows, that the co-efficient of efflux for the present case is $\mu_{0{,}312} = 1{,}075 \,.\, \mu_0 = 1{,}075 \,.\, 0{,}610 = 0{,}6557$, and the discharge $Q_1 = 0{,}6557 \,.\, Q = 0{,}6557 \,.\, 6{,}987 = 4{,}581$ cubic feet.

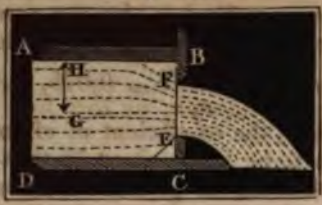
§ 321. *Efflux of Water in Motion.*—We have hitherto assumed that the head of water has been measured in still water; we must now, therefore, consider the case when only the head of water in motion, and flowing with a certain velocity towards the orifice, can be measured. Let us suppose the case of a rectangular lateral orifice, and represent its breadth by b, and the heads of water with respect to both horizontal edges h_1 and h_2, the height due to the velocity c of the affluent water by k, we shall then have the theoretical discharge:

$$Q = \tfrac{2}{3}\, b \sqrt{2g}\, [(h_1 + k)^{\frac{3}{2}} - (h_2 + k)^{\frac{3}{2}}].$$

This formula is not directly applicable to the determination of the discharge, because the height due to the velocity:

$k = \frac{c^2}{2g} = \frac{1}{2g}\left(\frac{Q}{G}\right)^2$ is again dependent on Q, and further transformation leads to a complicated equation of a higher order, hence it is far simpler to put the effective discharge $Q_1 = \mu_1\, ab \sqrt{2gh}$, and understand by μ_1 not the mere co-efficient of efflux, but one especially dependent on the ratios of the transverse sections. Most frequently, this case presents itself when the object is to measure water flowing

Fig. 433.



in canals and courses, because it is seldom possible in this case to dam up the water so high by a transverse section *BC*, Fig. 433, containing the orifice of discharge, that the orifice *EF* becomes only a small part, compared with the transverse section of the stream of water flowing to it; and, hence, the velocity of the last very small compared with the mean velocity.

From experiments made by the author on this subject with Poncelet orifices, where the head of water is measured one metre above the plane of the orifice, the expression: $\frac{\mu_n - \mu_0}{\mu_0} = 0{,}641 \left(\frac{F}{G}\right)^2 = 0{,}641 \ . n^2$, may be taken as tolerably accurate, when $n = \frac{F}{G}$ is the ratio of the transverse section, which, however, should not much exceed $\frac{1}{2}$; further, μ_0 represents the co-efficient for general contraction, taken from Poncelet's table corresponding to the present case. If b be the breadth, a the depth of the orifice, B the breadth and A the depth of the fluid stream, and h the depth of the upper side of the orifice below the surface of water, we have, accordingly, the effective discharge:

$$Q_1 = \left[1 + 0{,}641 \left(\frac{ab}{AB}\right)^2\right] \mu_0 \ . \ ab \sqrt{2g\left(h + \frac{a}{2}\right)}.$$

The following table serves for shortening the calculation in cases of application.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,002	0,006	0,014	0,026	0,040	0,058	0,079	0,103	0,130	0,160

Example. To find the quantity of water conducted through a course 3 feet broad, a board is placed across, with a 2 feet wide and 1 foot deep rectangular orifice, and the water in this way is so dammed up that it at last attains a height of $2\frac{1}{4}$ feet above the bottom, and $1\frac{3}{4}$ above the lower edge of the orifice. The theoretical discharge is $Q = ab \sqrt{2gh} = 1 \ . \ 2 \ . \ 8{,}02 \sqrt{1{,}25} = 16{,}04 \ . \ 1{,}118 = 17{,}93$ cubic feet; the co-efficient of efflux for perfect contraction may be put 0,689, and the ratio of the transverse sections $n = \frac{F}{G} = \frac{ab}{AB} = \frac{1 \ . \ 2}{2{,}25 \ . \ 3} = 0{,}296$; hence, it follows, that the co-efficient of efflux for the present ratio of discharge:
$= (1 + 0{,}641 \ . \ 0{,}296^2)\mu_0 = 1{,}056 \ . \ 0{,}602 = 0{,}6357$, and the effective quantity discharged $= 17{,}93 \ . \ 0{,}6357 = 11{,}31$ cubic feet.

§ 322. Imperfect contraction very often occurs in the efflux through wiers, as in Fig. 422. Wiers may take up a part only of the breadth of the reservoir or canal, or the whole breadth. In the latter case, contraction at the sides of the aperture does not take place, and for this reason more water flows through them than through wiers of the first kind. The author has made experiments also on these circumstances of efflux, and deduced from the results formulas by which the corresponding co-efficients may be estimated with tolerable certainty with the assistance of the ratio of the sections $n = \frac{G}{V} = \frac{hb}{\mathcal{A}B}$. If we retain the denominations of the former paragraph, we then have for the Poncelet wiers:

$$\frac{\mu_n - \mu_0}{\mu_0} = 1{,}718 \left(\frac{F}{G}\right)^4 = 1{,}718 \,.\, n^4,$$

and for wiers occupying the entire breadth of the canal:

$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}041 + 0{,}3693\, n^2,$$

hence, in the first case, the discharge is:

$$Q_1 = \tfrac{2}{3}\left[1 + 1{,}718 \left(\frac{hb}{\mathcal{A}B}\right)^4\right] \mu_0 \,.\, b \sqrt{2gh^3}.$$

And in the second:

$$Q_1 = \tfrac{2}{3}\left[1{,}041 + 0{,}3693 \left(\frac{h}{\mathcal{A}}\right)^2\right] \mu_0 \,.\, b \sqrt{2gh^3},$$

where *h* represents the head of water *EH* above its sill *F*, measured at about 3 feet 6 inches back from the wier.

In the following tables, the corrections $\frac{\mu_n - \mu_0}{\mu_0}$, for the most simple values of *n* are put down.

TABLE I.

CORRECTIONS FOR THE PONCELET WIERS.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,000	0,000	0,001	0,003	0,007	0,014	0,026	0,044	0,070	0,107

TABLE II.

CORRECTIONS FOR WIERS OVER THE ENTIRE SIDE, OR WITHOUT ANY LATERAL CONTRACTION.

n	0,00	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,041	0,042	0,045	0,049	0,056	0,064	0,074	0,086	0,100	0,116	0,133

Example. To determine the quantity of water carried off by a canal 5 feet broad, a waste board is applied, with an outward sloping edge, over which the water is allowed to flow after it has ceased to rise; the head of water above the bottom of the canal is 3½ feet, and above the edge 1½ feet, hence the theoretical discharge is $Q = \frac{2}{3} \,.\, 5 \,.\, 8{,}02$.

$\left(\frac{3}{2}\right)^{\frac{3}{2}} = 48,12$ cubic foot. The co-efficient of efflux is, since $\frac{h}{A} = \frac{1,5}{3,5} = \frac{3}{7}$ and $\mu_0 =$ 0,577,
$\mu_{\frac{3}{7}} = [1,041 + 0,3693 . (\frac{3}{7})^2] . 0,577 = 1,110 . 0,577 = 0,64$, hence the effective discharge $Q_1 = 0,64 . Q = 0,64 . 48,12 = 30,79$ cubic feet.

CHAPTER III.

ON THE EFFLUX OF WATER THROUGH TUBES.

§ 323. *Short Tubes, or Mouth-pieces.*—If water is allowed to flow through *short tubes*, or *mouth-pieces*, other relations take place than when it flows through orifices in a thin plate, or through outwardly sloping orifices in a thick plate. When the tube is prismatic, and its length 2½ to 3 times that of its width, it then gives an uncontracted and opaque stream, which has a small distance of projection, and hence, also, a smaller velocity than that of a jet flowing, under otherwise similar circumstances, through an orifice in a thin plate. If, therefore, the tube *KL* has the same transverse section as the orifice *F*, Fig. 434; and if also the head of water of both is one and the same, we then obtain in *LR* a troubled and uncontracted, and, therefore, a thicker jet, and in *FH* a clear and contracted, and, therefore, thinner one; and, it may be observed, that the distance of the projection *ER* is less than that of *DH*. This ratio of efflux only takes place when the tube is of a given length; if it is shorter, or scarcely as long as it is broad, then the jet *KL*, Fig. 435, will not touch the sides of the tube, the tube will have no influence on the efflux, and the jet will be the same as through orifices in a thin plate.

Fig. 434.

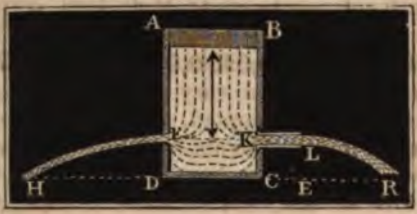
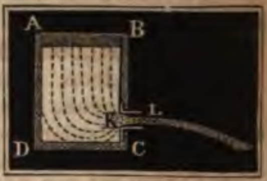
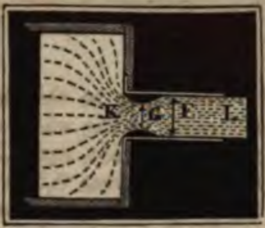


Fig. 435.



Sometimes in tubes of greater length, the fluid stream does not entirely fill the tube, namely: when the water is not allowed to come into contact with the sides of the tube; but if in this case we close the outer orifice by the hand or by a board for a few moments, a stream will then be formed which will entirely fill the tube, and the so-called *full flow* will then take place. Contraction of the fluid vein takes place also in the flow through tubes, but the place of contraction is here in the interior of the tube. We may be convinced of this, if we avail ourselves of glass tubes, such as *KL*, Fig. 436, and color the water, for in this case we shall remark, that there is progressive motion only in the middle of the transverse section *G* close behind the place of entrance *K*, but not at the outside of it, and that it is a sort of eddying motion which takes place. But it is the capillarity, or the adhesion of the water to the sides of the tube, which causes the fluid entirely to fill the end *FL* of the tube. The water flowing from the tube has only a pressure equal to that of the atmosphere, but the contracted section *G* is only a times the size of the section F of the tube, and for this reason the velocity in it $\frac{1}{a}$ times as great as the velocity of efflux v; hence the pressure of the water in the vicinity of G is

Fig. 436.



$$\frac{\left(\frac{1}{a}v\right)^2}{2g} - \frac{v^2}{2g} = \left[\left(\frac{1}{a}\right)^2 - 1\right]\frac{v^2}{2g}$$ (§ 307) less than at its exit, or than the atmospheric pressure. If we bore a narrow hole in the tube at G, no discharge will pass through it, but there will be an absorption of air rather; the full discharge and the action of the tube will at last entirely cease if the hole be made wider, or more holes bored.

§ 323. *Cylindrical Tubes.*—Numerous experiments have been made on the flow of water through cylindrical additional tubes; but the results vary considerably from one another.* The co-efficients of Bossut are those, which from their smallness (0,785) have been found to vary most from others. From the experiments of Michelotti, with tubes from ½ to 3 inches width, and with a head of water of from 3 to 20 feet, the mean of this co-efficient is: $\mu = 0{,}813$. The experiments of Bidone, Eytelwein and d'Aubuisson vary very little from this. The mean, however, which may be adopted, and which corresponds particularly with the author's experiments on the discharge through short mouth-pieces = 0,815. As we have found this for orifices in a thin plate 0,615, it follows that, under otherwise similar circumstances and relations, $\frac{815}{615} = 1{,}325$ times as much water flows through cylindrical additional tubes, as through round orifices in a thin plate. These co-efficients, moreover, increase as the width of tubes becomes less, and but slightly with the increase of the head of water or velocity of efflux. According to the author's experiments, under a pressure of from 9 to 24 inches for tubes three times as long as broad:

---

* A considerable series of experiments on the flow of water through adjutages was, some years since, performed by a committee of the Franklin Institute, which yet await a proper reduction to render the results available.—Am. Ed.

at	1 or ,4 in.	2 or ,8 in.	3 or 1,2	4 centimetres width, or 1,6 inches width.
$\mu =$	0,843	0,832	0,821	0,810

According to this table, therefore, the co-efficients increase considerably as the width of the tubes decreases. Buff found for tubes 2,79 lines wide, and 4,3 lines long, the co-efficients of efflux gradually to increase from 0,825 to 0,855, when the head of water sank from 33 to $1\frac{1}{2}$ inches successively.

The author found a co-efficient of efflux of 0,819 for the flow of water through rectangular additional tubes.

If the additional tubes *KL*, Fig. 437, are on the inside partially confined; if, for instance, one side is contiguous to the bottom, and if a partial contraction is produced thereby, then the co-efficient of efflux, from the author's experiments, does not perceptibly increase, but the water flows away at different parts of the section, with different velocities, and, of course, from the side *BC* faster than from the side opposite to it. If the inner anterior surface of a tube does not coincide with the side surface, but projects, as *a*, *b*, *c*, Fig. 438, then this tube is called an *internal additional tube.* If the anterior surface of this tube is at least $\frac{1}{8}$th as broad as the tube is wide, as for example *a*, then the co-efficient of efflux will remain the same as if this surface were in the plane of the side, but if the anterior surface be less, as *b*, *c*, the co-efficient will then be less. For a very small and almost vanishing anterior surface, according to the experiments of Bidone and the author, this amounts to 0,71 if the vein fills the tube, and 0,53 (compare § 318) if it does not quite fill the inner sides of the tube. In the first case (*b*) the fluid stream is broken and divergent, like a brush; and in the second (*c*) strongly contracted and quite crystalline.

Fig. 437.

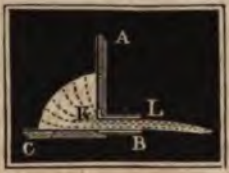
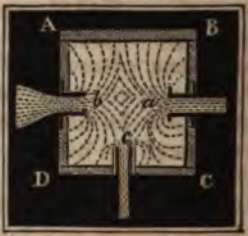


Fig. 438.



§ 324. *Co-efficient of Resistance.*—As water flows without contraction from prismatic additional tubes, it follows that, in its efflux through these mouth-pieces, the co-efficient of contraction = unity, and the co-efficient of velocity ϕ = the co-efficient of efflux μ. A discharge Q with the velocity v, possesses a *vis viva* $\frac{Q\gamma}{g}v^2$, and is capable of producing the mechanical effect $\frac{v^2}{2g}\, Q\,\gamma$ (§ 71). But now the theoretical velocity of efflux $= \frac{v}{\phi}$, hence the mechanical effect $\frac{1}{\phi^2} \cdot \frac{v^2}{2g} \cdot Q\,\gamma$ corresponds to the mass of water flowing out, and the discharge Q accordingly loses by efflux the mechanical effect

$$\left(\frac{1}{\phi^2} \cdot \frac{v^2}{2g} - \frac{v^2}{2g}\right) Q\,\gamma = \left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}\, Q\,\gamma.$$

For efflux through orifices in a thin plate, the mean of $\phi = 0,97$, hence the loss of effect here amounts to

$$\left[\left(\frac{1}{0,97}\right)^2 - 1\right] \frac{v^2}{2g}\, Q\,\gamma = 0,063\, \frac{v^2}{2g}\, Q\,\gamma;$$

for efflux through short cylindrical tubes $\phi = 0,815$, and the corresponding loss of effect

$$= \left[\left(\frac{1}{0,815}\right)^2 - 1\right] \frac{v^2}{2g}\, Q\,\gamma = 0,505\, \frac{v^2}{2g}\, Q\,\gamma,$$

i. e. eight times as great as for efflux through orifices in a thin plate. In rendering available the *vis viva* of flowing water, it is consequently better to let the fluid flow through orifices in a thin plate, than through prismatic tubes. But if the inner edges in which the tube meets the sides of the cistern are rounded, and by this a gradual passage from the cisterns into the tube effected, the co-efficient of efflux will then rise to 0,96, and the loss of mechanical effect will be brought down to 8½ per cent. In shorter adjutages, accurately rounded, having the form of the contracted fluid vein $\mu = \phi = 0,97$, and hence the loss of mechanical effect as for orifices in a thin plate = 6 per cent.

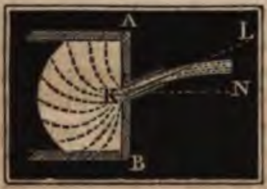
A head of water $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}\, Q\,\gamma$ is due to the loss of mechanical effect $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$; hence we may also suppose that from the obstacles to the efflux, the head of water suffers the loss $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$, and assume after deduction of this loss, that the residuary part of the head of water is expended in generating the velocity. We may call this loss $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$, which increases with the square of the velocity of efflux, the *height due to the resistance*, and the co-efficient $\frac{1}{\phi^2} - 1$, with which the height due to the velocity is to be multiplied to obtain the height due to the resistance, the *co-efficient of resistance*. We shall represent in what follows, the co-efficient, expressing the ratio of the height of resistance to the head of water, by the letter ζ; therefore, the height due to the resistance itself may be expressed by $\zeta \cdot \frac{v^2}{2g}$. By the formula $\zeta = \frac{1}{\phi^2} - 1$ and $\phi = \frac{1}{\sqrt{1+\zeta}}$, the co-efficient of resistance may be calculated from the co-efficient of velocity, and *vice versâ.*

Examples.—1. What discharge will flow through a 2 inch wide tube, under a head of water of 3 feet, which corresponds to a co-efficient of resistance $\zeta = 0{,}4$? $\phi = \frac{1}{\sqrt{1{,}4}} = 0{,}845$; hence, $v = 0{,}845 \,.\, 8{,}02 \sqrt{3} = 12{,}05$ feet; further, $F = \left(\frac{1}{12}\right)^2 \pi = 0{,}02182$ square feet; consequently, the quantity of water sought is $Q = 0{,}263$ cubic feet.—2. If a tube of 2 inches width, under a pressure of 2 feet, deliver in a minute 10 cubic feet of water, its co-efficient of efflux, or of velocity, is then $\phi = \frac{Q}{F\sqrt{2gh}}$ $= \frac{10}{60 \,.\, 0{,}02182 \,.\, 8{,}02 \,.\, \sqrt{2}} = \frac{1}{1{,}050\sqrt{2}} = 0{,}674$, the co-efficient of resistance $= \left(\frac{1}{0{,}674}\right)^2 - 1 = 1{,}201$; and lastly, the loss in head of water produced by the resistances of the tube:

$$= 1{,}201 \frac{v^2}{2g} = 1{,}201 \,.\, 0{,}0155 \left(\frac{Q}{F}\right)^2 = 0{,}0186 \,.\, \frac{1}{0{,}1309^2} = 1{,}085 \text{ feet.}$$

§ 325. *Oblique Additional Tubes.*—Obliquely attached or obliquely cut tubes give a smaller quantity of water than rectangularly attached, or rectangularly cut additional tubes, because the direction of the water in them becomes changed. Experiments conducted upon an extensive scale, have led the author to the following. If δ be the angle which the axis of the tube *KL*, Fig. 439, makes with the normal *KN* to the plane *AB* of the inner orifice of the tube; and if ζ be the co-efficient of resistance for rectangularly cut tubes, we shall then have the co-efficient of resistance for the inclined tube: $\zeta_1 = \zeta + 0{,}303 \, sin. \, \delta + 0{,}226 \, sin. \, \delta^2$. Let us take for ζ the mean value 0,505, and we shall obtain:

Fig. 439.

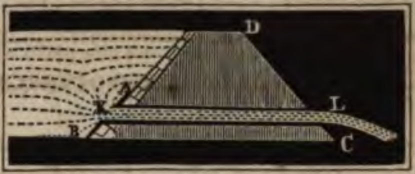


for $\delta^0 =$	0	10	20	30	40	50	60°
The co-efficient of resistance $\zeta_1 =$	0,505	0,565	0,635	0,713	0,794	0,870	0,937
The co-efficient of efflux $\mu_1 =$	0,815	0,799	0,782	0,764	0,747	0,731	0,719

From this, for example, the co-efficient of resistance of an additional tube deviating by 20° from the axis is $\zeta = 0{,}635$, and the co-efficient of efflux $= \frac{1}{\sqrt{1{,}635}} = 0{,}782$, and for 35° deviation, the first $= 0{,}753$, and the last $= 0{,}755$.

In general, these inclined and additional tubes are larger than we have hitherto assumed, and they should be longer too, because the water would not otherwise perfectly fill the tube. The preceding formula represents only that part of the resistance which is due to that portion of tube at the inner orifice, which is three times as long as the tube is wide. The resistance which the remaining portion of tube opposes to the motion of the water, will be given subsequently.

Fig. 440.

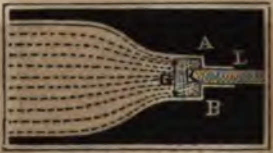


Example. If the plane of the inner orifice *AB* of a horizontally lying pond sluice *KL*, Fig. 440, as likewise the interior surface of the pond dam, is inclined 40° to the horizon, then the axis of the pipe makes, with the normal to this plane, an angle of 50°, and hence the co-efficient of resistance for efflux through the portion of the interior orifice of this tube is $= 0{,}870$; and if now the co-efficient of resistance 0,650 were due to the remaining and longer portion, the co-efficient of resistance of the entire tube would then be $= 0{,}870 + 0{,}650 = 1{,}520$, and hence the co-efficient of efflux $= \frac{1}{\sqrt{1 + 1{,}520}} = \frac{1}{\sqrt{2{,}520}} = 0{,}630$. For a 10 feet head of water and 1 foot width of tube, the following discharge would be given:

$$Q = 0{,}63 \,.\, \frac{\pi}{4} \,.\, 8{,}02 \sqrt{10} = 12{,}55 \text{ cubic feet.}$$

§ 326. *Imperfect Contraction.*—When a cylindrical additional tube *KL*, Fig. 441, is inserted into a plane wall *AB*, whose area G does not much exceed the transverse section F of the tube, the water then comes to the place of insertion with a velocity which must not be disregarded, and it then issues into the tube with imperfect contraction only, on which account the velocity of efflux is again greater than when the water before entrance into the tube is to be assumed as still. Again if $\frac{F}{G} = n$ is the ratio of the section of the tube to that of the area of the side, and further, μ_0 be the co-efficient of efflux for perfect contraction, where $\frac{F}{G}$ may be equated to 0, we shall have, according to the experiments of the author, to put the co-efficient of efflux for imperfect contraction, or for the ratio of the sections n:

Fig. 441.



$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}102\, n + 0{,}067\, n^2 + 0{,}046\, n^3 \text{ or}$$

$$\mu_n = \mu_0 (1 + 0{,}102\, n + 0{,}067\, n^2 + 0{,}046\, n^3).$$

If the transverse section of the tube occupies the sixth part of the whole surface of the side, there is:

$$\mu_{\frac{1}{6}} = \mu_0 (1 + 0{,}102 \,.\, \tfrac{1}{6} + 0{,}067 \,.\, \tfrac{1}{36} + 0{,}046 \,.\, \tfrac{1}{216})$$
$$= \mu_0 (1 + 0{,}017 + 0{,}0019 + 0{,}0002) = 1{,}019\, \mu_0, \text{ or}$$
$$\mu_0 \text{ being put } = 0{,}815,\ \mu_{\frac{1}{6}} = 0{,}815 \,.\, 1{,}019 = 0{,}830.$$

The following useful and convenient table gives somewhat more accurately the values for correction $\frac{\mu_n - \mu_0}{\mu_0}$

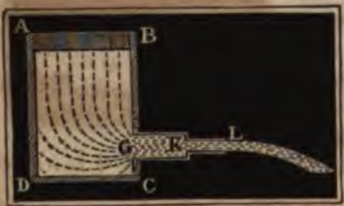
TABLES

OF CORRECTION FOR IMPERFECT CONTRACTION, BY EFFLUX THROUGH SHORT CYLINDRICAL TUBES.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,006	0,013	0,020	0,027	0,035	0,043	0,052	0,060	0,070	0,080
n	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,090	0,102	0,114	0,127	0,138	0,152	0,166	0,181	0,198	0,227

By efflux through short parallelopipedical tubes, these corrections are nearly the same.

Fig. 442.



These co-efficients are especially applicable to the efflux of water through compound tubes; for example, in the case represented in Fig. 442, where the orifice of the short tube *KL* enters the wider short tube *CK*, whose orifice again lies in the cistern *AC*. Imperfect contraction takes place at the entrance of the water from the wider into the narrower tube, and hence the co-efficient of efflux must be determined by the last rule. If we put the co-efficient of resistance corresponding to the co-efficient of efflux $= \zeta_1$, the co-efficient of resistance for the entrance into the wider tube from the cistern $= \zeta$, the head of water $= h$, the velocity of efflux $= v$, and the ratio $\frac{F}{G}$ of the section of the tubes $= n$, therefore, the velocity of the water in the wider tube $= nv$, then the formula gives:

$$h = \frac{v^2}{2g} + \zeta \cdot \frac{(nv)^2}{2g} + \zeta_1 \cdot \frac{v^2}{2g}, \text{ i.e. } h = (1 + n_2 \zeta + \zeta_1) \frac{v^2}{2g}.$$

And hence:

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + n^2 \zeta + \zeta_1}}.$$

Example. What discharge will the apparatus delineated in Fig. 442 deliver, if the head of water $h = 4$ feet, the width of the narrower tube 2 inches, and that of the wider one 3 inches? $n = (\frac{2}{3})^2 = \frac{4}{9}$, hence $\mu_{\frac{4}{9}} = 1{,}069 \,.\, 0{,}815 = 0{,}871$, and the corresponding co-efficient of resistance $\zeta_1 = \left(\frac{1}{0{,}871}\right)^2 - 1 = 0{,}318$; but now we have $\zeta = 0{,}505$ and $n^2 \,.\, \zeta = \frac{16}{81} \,.\, 0{,}505 = 0{,}099$; hence it follows, that $1 + n^2 \zeta + \zeta_1$

$= 1 + 0{,}099 + 0{,}318 = 1{,}417$, and the velocity of efflux $v = \frac{8{,}02 \,.\, \sqrt{4}}{\sqrt{1{,}417}} = \frac{16{,}04}{\sqrt{1{,}417}}$ $= 14{,}34$ feet. Again, since the transverse section of the tube $= 0{,}02182$ square feet, the discharge will be $Q = 14{,}34 \,.\, 0{,}02182 = 0{,}313$ cubic feet.

§ 327. *Conical Tubes.*—Additional conical tubes give a discharge different from that of prismatic or cylindrical tubes; they are either conically convergent, or conically divergent; in the first case the outer orifice is smaller, and in the second case larger than the inner orifice. The co-efficients of efflux for the first tubes are greater, and for the last, less than for cylindrical tubes. One and the same conical tube gives more water when the wider orifice is made the exit orifice, as *K* in Fig. 443, than when it is turned inwards, as *L* in the same figure, except that it does not give a greater quantity in proportion as the wider orifice exceeds the narrower. If many, as Venturi and Eytelwein, give for conically divergent tubes, a greater co-efficient of efflux, than for conically convergent tubes, it must always be borne in mind, that they take the narrower transverse section for the orifice. The following experiments instituted at pressures of from 0,25 to 3,3 metres, with a tube *AD* 9 centimetres long, Fig. 444, bring before us the effect of conicalness in tubes. The width of these tubes at one extremity amounted to $DE = 2{,}468$, at the other $AB = 3{,}228$ centimetres, and the angle of convergence, *i. e.* the angle *AOB*, which the oppositely situated sides *AE* and *BD* of a section in the direction of the longer axis include = 4°, 50′. By efflux through the narrower orifice, the co-efficient was = 0,920, but by efflux through the wider it was = 0,553, and if in the calculation, we take the narrower entrance orifice for the transverse section, it will give = 0,946. In the first case, when the tube was applied as a conically convergent adjutage, the fluid vein was little contracted, thick and smooth; but, in the second case, when the tube served as a conically divergent adjutage, it was strongly divergent, broken, and spouting. Venturi and Eytelwein have experimented further on efflux through conically divergent tubes. Both philosophers have applied these conical tubes to cylindrical and conoidal adjutages, made after the form of the contracted fluid vein. By such a connection as is represented in Fig. 445, where the divergent portion *KL* of the outer orifice is between 12 and $21\frac{1}{2}$ lines wide, and $8\frac{13}{18}$ of an inch long, and the angle

Fig. 443.

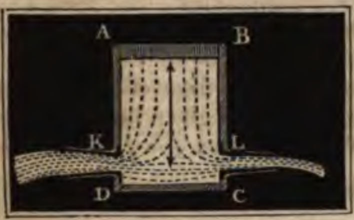


Fig. 444.

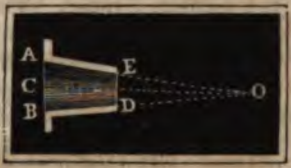
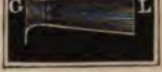


Fig. 445.



of convergence estimated at 5°, 9′. Eytelwein found $\mu = 1,5526$ when he took the narrow end for the orifice, and, on the other hand, $\mu = 0,483$ for the wider end, in which he was right. Through this combined adjutage there certainly flows $\frac{1,5526}{0,615} = 2,5$ times as much as through a simple orifice in a thin plate, and $\frac{1,5526}{0,815} = 1,9$ times as much as through a short cylindrical tube. With small velocities and greater divergence, it is scarcely possible, even by previously closing the tubes, to bring about a full flow.

§ 328. The most ample experiments have been made by d'Aubuisson and Castel on efflux through conically convergent additional tubes. The tubes for this purpose were of great variety, of different lengths, widths, and angles of convergence. The most extensive experiments were those made with tubes of 1,55 centimetres width at the discharging orifice, and of from 2,6 times greater, *i. e.* of 4 centimetres in length, for which reason we will here communicate the results in the following table. The head of water was, throughout, 3 metres. The discharges were measured by a special gauge-cistern; but in order to obtain besides the co-efficients of efflux, those of the velocity and contraction, the amplitude of the jet, due to given heights, were measured, and from these the velocity of efflux (*see* § 38, *Ex.* 2) calculated. The ratio $\frac{v}{\sqrt{2gh}}$ of the effective velocity v to the theoretical $\sqrt{2gh}$ gave the co-efficient of velocity ϕ, as also the ratio $\frac{Q}{F\sqrt{2gh}}$ of the effective discharge Q to the theoretical $F\sqrt{2gh}$ gave the co-efficient of efflux μ, and, lastly, the ratio of both co-efficients, *i. e.* $\frac{\mu}{\phi}$, determined the co-efficient of contraction α.

TABLE

OF THE CO-EFFICIENTS OF EFFLUX AND VELOCITY FOR EFFLUX THROUGH CONICALLY CONVERGENT TUBES.

Angle of convergence	Co-efficient of efflux.	Co-efficient of velocity.	Angle of convergence	Co-efficient of efflux.	Co-efficient of velocity.
0° 0′	0,829	0,829	13° 24′	0,946	0,963
1° 36′	0,866	0,867	14° 28′	0,941	0,966
3° 10′	0,895	0,894	16° 36′	0,938	0,971
4° 10′	0,912	0,910	19° 28′	0,924	0,970
5° 26′	0,924	0,919	21° 0′	0,919	0,972
7° 52′	0,930	0,932	23° 0′	0,914	0,974
8° 58′	0,934	0,942	29° 58′	0,895	0,975
10° 20′	0,938	0,951	40° 20′	0,870	0,980
12° 4′	0,942	0,955	48° 00′	0,847	0,984

From this table it is seen that the co-efficients of efflux attain their maximum 0,946 for a tube of $13\frac{1}{2}°$ lateral convergence; that, on the other hand, the co-efficients of velocity come out always greater and greater, the greater is the angle of convergence. The following example will show how this table may be used in those cases which present themselves in practice.

Example.—What discharge will a short conoidal tube of $1\frac{1}{2}$ inches width at the outer orifice, and of 10° convergence, deliver under a pressure of 16 feet? According to the experiments of the author, a cylindrical tube of this width gives $\mu = 0,810$; d'Aubuisson's tube, however, gave $\mu = 0,829$, therefore about $0,829 - 0,810 = 0,019$ more. Now from the table for a tube of 10° convergence, μ is $= 0,937$; hence, for the given tube we may put $\mu = 0,937 - 0,019 = 0,918$, whence the discharge

$$Q = 0,918 \,.\, \frac{\pi}{4 \,.\, 8^2} \,.\, 8,02 \sqrt{4} = \frac{8,02 \,.\, 0,918\, \pi}{64} = 0,3614 \text{ cubic feet.}$$

§ 329. *Resistance of Friction.*—The longer prismatic or cylindrical tubes are, the more they retard the efflux; hence we must assume that the sides of the tubes offer resistance to the motion of the water by the friction, adhesion, or viscidity of the fluid acting against them. From reason and from numerous observations and measurements, we may assume that this resistance of friction is independent of the pressure, that it increases directly as the length l, and inversely as the width d, and, therefore, proportional to the ratio $\frac{l}{d}$. Moreover, it appears that this resistance is greater for great and less for less velocities, and that it very nearly increases with the square of the velocity v of the water. If we measure this resistance by the height of a column of water, which must be deducted from the entire head h, in order to obtain the height requisite for the generation of the velocity, we may then put for this height, which we shall term *the height due to the resistance*, $h_1 = \zeta_1 \,.\, \frac{l}{d} \,.\, \frac{v^2}{2g}$, where ζ_1 represents a number, from experiment, which we may call the *height due to the resistance of friction.* There is a greater loss, therefore, of pressure or head of water from the friction of the water in the tube, the greater the ratio $\frac{l}{d}$ of the length to the width is, and the greater the height due to the velocity $\frac{v^2}{2g}$. From the discharge Q and the transverse section of the tube $F = \frac{\pi d^2}{4}$ there follows the velocity $v = \frac{4\,Q}{\pi d^2}$, and hence the height due to the friction:

$$h_1 = \zeta_1 \,.\, \frac{l}{d} \,.\, \frac{1}{2g} \left(\frac{4\,Q}{\pi d^2}\right)^2 = \zeta_1 \,.\, \frac{1}{2g} \,.\, \left(\frac{4}{\pi}\right)^2 \,.\, \frac{l\,Q^2}{d^5}.$$

In order to obtain the least possible loss of head of water, or fall, in leading a certain quantity of water Q, the pipe must be made as wide as possible, and not unnecessarily long. A double width requires, for instance, only $(\frac{1}{2})^5 = \frac{1}{32}$ of the fall that the single width does.

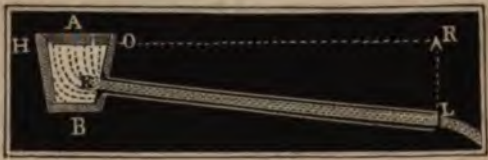
If the transverse section of a tube be rectangular, and of the depth a and the breadth b, we must substitute for

$$\frac{1}{d} = \frac{1}{4} \cdot \frac{\pi d}{\frac{1}{4} \pi d^2} = \frac{1}{4} \cdot \frac{\text{circumference}}{\text{area}} = \frac{1}{4} \cdot \frac{2(a+b)}{ab} = \frac{a+b}{2ab},$$

whence it follows: $h_1 = \zeta \cdot \frac{l(a+b)}{2ab} \cdot \frac{v^2}{2g}$.

By means of this formula for the resistance of friction in pipes, we may also find the velocity and the quantity of efflux which a pipe, of given length and width, will conduct under a given pressure. It is quite the same, whether the tube KL, Fig. 446, is horizontal, inclines, or ascends, if only by the head of water is understood the depth RL of the middle point L of the orifice of the tube below the surface of water HO of the efflux reservoir. If h is the head of water, h_1 the height due to the resistance for the orifice of entrance, and h_2 that for the remaining portion of the tube, we then have:

Fig. 446.



$h - (h_1 + h_2) = \frac{v^2}{2g}$, or $h = h_1 + h_2 + \frac{v^2}{2g}$. If ζ represents the coefficient of resistance for the portion of tube next the cistern, and ζ_1 the co-efficient of the resistance of friction of the rest of the tube, we then have

$$h = \zeta \cdot \frac{v^2}{2g} + \zeta_1 \cdot \frac{l}{d} \cdot \frac{v^2}{2g} + \frac{v^2}{2g},$$

$$\text{or, 1.) } h = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g},$$

$$\text{and 2.) } v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \zeta_1 \cdot \frac{l}{d}}}.$$

From the last formula the discharge $Q = Fv$ is given.

For very long tubes $1 + \zeta$ is small compared with $\zeta_1 \frac{l}{d}$, whence, simply, $h = \zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g}$, and inversely,

$$v = \sqrt{\frac{1}{\zeta_1} \cdot \frac{d}{l} \cdot 2gh}.$$

§ 330. The co-efficient of friction, like the co-efficient of efflux, is not quite constant; it is greater for small, and less for great velocities; *i. e.* the resistance of the friction of water in tubes does not increase exactly with the square of the velocity, but with some other power of it. Prony and Eytelwein have assumed, that the head of water lost by the resistance due to friction ought to increase as the simple velocity and as its square, and have given for it the expression $h_1\,(\alpha\,v + \beta\,v^2)\,\frac{l}{d}$, where α and β are co-efficients deduced from experiment. To determine these co-efficients, 51 experiments, which, at various times, were made by Couplet, Bossut, and Du Buat, on the motion of water through long tubes, were made use of by these hydraulicians.

Prony found from this, that $h_1 = (0{,}0000693\,v + 0{,}0013932\,v^2)\,\frac{l}{d}$, Eytelwein $h_1 = (0{,}0000894\,v + 0{,}0011213\,v^2)\,\frac{l}{d}$; d'Aubuisson assumes $h_1 = (0{,}0000753\,v + 0{,}001370\,v^2)\,\frac{l}{d}$ metres.

A formula, discovered by the author, agrees more accurately with observation. It has the form

$$h_1 = \left(\alpha + \frac{\beta}{\sqrt{v}}\right)\frac{l}{d}\,\frac{v^2}{2g},$$

and is based on the hypothesis, that the resistance of friction increases simultaneously as the square, and as the square root of the cube of the velocity. From this we have the co-efficient of resistance $\zeta_1 = \alpha + \frac{\beta}{\sqrt{v}}$, and the height due to the resistance of friction $h_1 = \zeta_1 \cdot \frac{l}{d}\,\frac{v^2}{2g}$.

For the measurement of the co-efficient of resistance ζ_1, or of the auxiliary constants α and β, not only the determinations of Prony and Eytelwein from the 51 experiments of Couplet, Bossut, and Du Buat were used by the author, but also 11 experiments made by him, and 1 experiment by Gueymard in Grenoble. The older experiments extend only to velocities of from 0,043 to 1,930 metres; in the experiments of the author, however, the extreme limit of velocity reached to 4,648 metres. The widths of the tubes, in the older experiments, were 27 mm. = 1.06 in.; 36 mm. = 1.95 in.; 54 mm. = 2.12 in.; 135 mm. = 5.31 in.; and 490 mm. = 19.29 in.; later experiments were conducted with tubes of 33 mm. = 1.29 in.; 71 mm. = 2.79 in.; and 275 mm. = 5.31 in. By means of the method of least squares, it has been found from the 63 experiments laid down;

$$\zeta_1 = 0{,}01439 + \frac{0{,}0094711}{\sqrt{v}}\text{; therefore,}$$

$$h_1 = \left(0{,}01439 + \frac{0{,}0094711}{\sqrt{v}}\right)\frac{l}{d}\cdot\frac{v^2}{2g}\ \text{metre;}$$

for Prussian measure:

$$h_1 = \left(0{,}01439 + \frac{0{,}016921}{\sqrt{v}}\right)\frac{l}{d}\cdot\frac{v^2}{2g}\ \text{feet,}$$

or for English measure:

$$h_1 = \left(0{,}01439 + \frac{0{,}017963}{\sqrt{v}}\right)\frac{l}{d}\cdot\frac{v^2}{2g}\ \text{feet.}$$

§ 331. For facilitating the calculation, the following table of the co-efficients of resistance has been compiled. We see from this, that the variability of these co-efficients is not inconsiderable, as for 0,1 metre velocity it is = 0,0443, for 1 metre = 0,0239, and for 5 metres = 0,0186.

TABLE OF THE CO-EFFICIENTS OF FRICTION.*

v ft	v	10ths of a metre.									
		0	1 or 4 in.	2 or 8 in.	3 12 in.	4 16 in.	5 20 in.	6 24 in.	7 28 in.	8 32 in.	9 36 in.
ft. in.	Whole metres.										
0	0	∞	0,0443	0,0356	0,0317	0,0294	0,0278	0,0266	0,0257	0,0250	0,0244
3.4	1	0239	0234	0230	0227	0224	0221	0219	0217	0215	0213
6.7	2	0211	0209	0208	0206	0205	0204	0203	0202	0201	0200
9.10	3	0199	0198	0197	0196	0195	0195	0194	0193	0193	0192
13.0	4	0191	0191	0190	0190	0189	0189	0188	0188	0187	0187

We find in this table the co-efficients of resistance due to a certain velocity, when we look for the whole metre in the vertical, and the tenths in the first horizontal column, then proceed from the first number horizontally, and from the last vertically to the place where both motions meet; for example, for $v = 1,3$ metre $\zeta_1 = 0,0227$, for $v = 2,8$, $\zeta_1 = 0,0201$.

For the Prussian measure we may put:

v	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9 ft.
ζ_1	0,0679	0,0522	0,0453	0,0411	0,0383	0,0362	0,0346	0,0333	0,0322

v	1	1¼	1	2	3	4	6	8	12	20 ft.
ζ_1	0,0313	0,0296	0,0282	0,0263	0,0242	0,0229	0,0213	0,0204	0,0192	0,0182

§ 332. *Long Tubes.*—With respect to the motion of water in long tubes or conducting pipes, the three following fundamental problems present themselves for solution.

1. The length l and the width d of the tube and the quantity of water Q to be conducted are given, and the head of water is required. We have first to calculate the velocity $v = \frac{Q}{F} = \frac{4\,Q}{\pi\, d^2} = 1,2732 \,.\, \frac{Q}{d^2}$, then to look for the co-efficient of friction ζ_1 corresponding to this value, in one of the last tables, and, lastly, to substitute the values of

---

* To apply this table to English measures, the velocity, if below 40 inches, will be taken out from the numbers under "10ths of a metre," and if above 36, the number of feet and inches in the left hand column will be used with the inches at the head. Thus, for v = 7 feet 11 inches we have on a line with 6 feet 7 inches at the left, and under 16 inches at top, the co-efficient, 0205.—AM. ED.

d, l, v, ζ and ζ_1, where ζ represents the co-efficient of friction for the portion of the interior orifice, in the last formula

$$h = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g}.$$

2. The length and width of the tube, as well as the head of water or fall, are given to determine the discharge. We must here find the velocity by the formula

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \zeta_1 . \frac{l}{d}}};$$

but since the co-efficient of resistance is not quite constant, but varies somewhat with v, we must know v approximately beforehand, in order to be able to find out ζ_1.

From v it then follows that $Q = \frac{\pi d^2}{4} v = 0{,}7854\, d^2 v.$

3. The discharge, the head of water, and the length of the tube are given, to determine the requisite width of the tube.

$$\text{As } v = \frac{4\,Q}{\pi d^2}, \text{ therefore } v^2 = \left(\frac{4\,Q}{\pi}\right)^2 . \frac{1}{d^4},$$

we then have $2\,gh = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \left(\frac{4\,Q}{\pi}\right)^2 . \frac{1}{d^4}$, or

$2\,g\,h . \left(\frac{\pi}{4\,Q}\right)^2 = (1 + \zeta) \frac{1}{d^4} + \zeta_1 \frac{l}{d^5}$, or

$2\,g\,h . \left(\frac{\pi}{4\,Q}\right)^2 d^5 = (1 + \zeta)\, d + \zeta_1\, l$, hence the width of the tube

$$d = \sqrt[5]{\frac{(1 + \zeta)\, d + \zeta_1\, l}{2\,g\,h} . \left(\frac{4\,Q}{\pi}\right)^2}.$$

But now $\left(\frac{4}{\pi}\right)^2 = 1{,}6212$, $1 + \zeta$ as a mean $= 1{,}505$ and $\frac{1}{2g} = 0{,}0155$, hence we may put:

$$d = 0{,}4787 \sqrt[5]{(1{,}505 . d + \zeta_1\, l) \frac{Q^2}{h}} \text{ feet.}$$

This formula can only be used as a formula of approximation, because the unknown quantity d, and also the co-efficient ζ_1, dependent on $v = \frac{4\,Q}{\pi d^2}$, appear in it.

Examples.—1. What head of water does a conducting pipe, of 150 feet length and 5 inches width, require, if it is to carry off 25 cubic feet of water per minute? Here $v = 1{,}2732 . \frac{25 . 12^2}{60 . 5^2} = 3{,}056$ feet, hence we may put $\zeta_1 = 0{,}0242$, and the head of water or the entire fall of the pipe will be:

$$h = \left(1{,}505 + 0{,}0242 . \frac{150 . 12}{5}\right) . 0{,}0155 . 3{,}056^2$$
$$= (1{,}505 + 8{,}712) . 0{,}0155 . 9{,}339 = 1{,}479 \text{ feet, (English.)}$$

2. What discharge will a conducting pipe, 48 feet long and 2 inches wide, with a 5 feet head of water, deliver? It will be:

$$v = \frac{8{,}02\sqrt{5}}{\sqrt{1{,}505 + \zeta_1 \frac{48\,.\,12}{2}}} = \frac{17{,}88}{\sqrt{1{,}505 + 288\,.\,\zeta_1}}.$$

If we previously take $\zeta_1 = 0{,}020$, we shall obtain $v = \frac{17{,}88}{2{,}7} = 6{,}6$, but $v = 6{,}6$ gives more correctly $\zeta_1 = 0{,}0211$, hence we shall have more correctly:
$v = \frac{17{,}88}{\sqrt{1{,}505 + 288\,.\,0{,}0211}} = \frac{17{,}88}{\sqrt{7{,}582}} = 6{,}50$ feet, and the quantity of water
$Q = 0{,}7854\,.\left(\frac{2}{12}\right)^2.\,6{,}50 = 0{,}137$ cubic feet $= 236{,}7$ cubic inches.

3. What width must be given to a conducting pipe, 100 feet in length, which at a head of water of 5 feet, delivers half a cubic foot of water per second? Here
$d = 0{,}4787\,\sqrt[5]{(1{,}505\,d + 100\,\zeta_1)\,.\,\frac{1}{5}\,(\frac{1}{2})^2} = 0{,}4787\,\sqrt[5]{0{,}075\,d + 5\,\zeta_1}$. If we set out with $\zeta_1 = 0{,}02$, we obtain $d = 0{,}4787\,\sqrt[5]{0{,}075\,d + 0{,}100}$, or approximately
$= 0{,}4787\,\sqrt[5]{0{,}100} = 0{,}30$, therefore more correctly,
$d = 0{,}4787\,\sqrt[5]{0{,}0225 + 0{,}100} = 0{,}4787\,\sqrt[5]{0{,}1225} = 0{,}31456$ feet $= 3{,}77$ inches. To this width corresponds the transverse section $F = 0{,}7854\,.\,0{,}31456^2 = 0{,}0777$ square feet, the velocity $v = \frac{Q}{F} = \frac{0{,}5}{0{,}0777} = 6{,}43$ feet, and to this again the co-efficient of resistance $\zeta_1 = 0{,}0211$. If we substitute the last more accurate value, we then obtain $d = 0{,}4787\,\sqrt[5]{0{,}1280} = 0{,}3173$ feet.

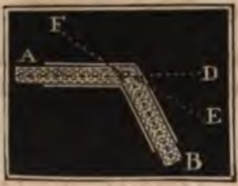
Remark. Experiments with $2\frac{1}{2}$ and $4\frac{1}{2}$ inch wide common wooden pipes have given the author the co-efficient of resistance 1,75 times as great as for metallic pipes, to which refer the values given in the table of the former §. Whilst, therefore, for example, for a velocity of 3 feet, $\zeta_1 = 0{,}0242$ for metallic pipes, we must put it for wooden pipes $= 0{,}0242\,.\,1{,}75 = 0{,}04235$; whilst we have found in *Example* 1, the head of water for a metallic pipe 150 feet long, 1,527 feet, it will amount, under the same circumstances, for a wooden one to,

$$= (1{,}505 + 0{,}04235\,.\,360)\,.\,0{,}0155\,.\,9{,}339 = 16{,}75\,.\,0{,}1494 = 244 \text{ feet.}$$

§ 332. *Bent Tubes.*—Particular resistances are opposed to the motion of water in pipes when they are *bent* or *knee-shaped.* Experiments have been made by the author on both kinds of resistances, on which account it is necessary here to communicate the results.

If a pipe ACB, Fig. 447, forms a knee, or if, as it is termed, it be *angled*, a loss of pressure ensues, which increases uniformly with the height $\frac{v^2}{2g}$ due to the velocity, and, further, increases with half the angle of deflexion $ACF = BCE = \delta$. The height of water lost through the knee, or the height due to the resistance corresponding to its transit through the knee, may be given by the expression $h = \zeta_2 \frac{v^2}{2g}$, where ζ_2 expresses the co-efficient of the knee resistance, dependent on the magnitude of the angle of deviation of the tube. Experiments made on different knees have led to the expression

Fig. 447.

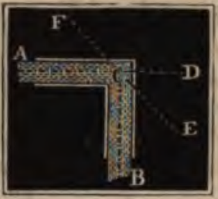


$$\zeta_2 = 0{,}9457\ \sin.\ \delta^2 + 2{,}047\ \sin.\ \delta^4,$$

and from this the following table has been calculated.

δ^0	10	20	30	40	45	50	55	60	65	70
ζ_2	0,046	0,139	0,364	0,740	0,984	1,260	1,556	1,861	2,158	2,431

Fig. 448.



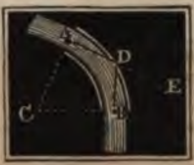
It is seen from this, that a considerable loss of *vis viva* is produced by knee tubes; for example, a rectangular knee ACB, Fig. 448, gives, since the angle of deviation, amounts to 45^0, the loss of head $= 0,956 \frac{v^2}{2g}$, therefore, pretty nearly equal to the height due to the velocity; a knee of 125^0, for which $\delta = 62\frac{1}{2}^0$, diminishes the head of water by so much as double the height due to the velocity $2 \,.\, \frac{v^2}{2g}$. By putting in the height due to the resistance of the knee $\zeta^2 \frac{v^2}{2g}$, we obtain the complete formula for the motion of water in tubes:

$$h = (1 + \zeta_1 \frac{l}{d} + \zeta_2) \frac{v^2}{2g}.$$

Example. If the conducting pipe in the first example of the preceding paragraph, 150 feet long and 5 inches wide, which is to deliver 25 cubic feet of water per minute, contains two rectangular knees (Fig. 451), we then have the required head of water:

$$h = (1,505 + 8,712 + 2 \,.\, 0,956) \frac{v^2}{2g} = 12,129 \,.\, 0,0155 \,.\, 9,339 = 12,129 \,.\, 0,14475$$
$$= 1,7557 \text{ feet.}$$

Fig. 449.



§ 334. *Curved Tubes.*—Curved tubes AB, Fig. 449, offer, under otherwise similar circumstances, far less resistance than unrounded knee tubes. The height due to the resistance which measures this obstacle increases likewise as the square of the velocity, but at the same time also as the simple angle of deflexion or curvature $ACB = BDE = \beta$, and may be expressed, therefore, by the formula:

$$h = \zeta \,.\, \frac{\beta^0}{180^0} \,.\, \frac{v^2}{2g} = \zeta \,.\, \frac{\beta}{\pi} \,.\, \frac{v^2}{2g}.$$

The corresponding co-efficient of resistance is by no means constant, it depends much more on the ratio of the width of the tube to the radius of curvature of its axis, and is the less, the less is this ratio. An extensive series of experiments made by the author, and the well known experiments of Du Buat, have, by their combinations, led to the expression $\zeta = 0,131 + 1,847 \left(\frac{r}{R}\right)^{\frac{7}{2}}$, for tubes with circular transverse sections, and for tubes with quadrangular or rectangular transverse sections $\zeta = 0,124 + 3,104 \left(\frac{r}{R}\right)^{\frac{7}{2}}$, where r represents half the width of the tube, and R the radius of curvature of the axis.

The two following tables have been calculated accordingly.

TABLE I.

CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH CIRCULAR TRANSVERSE SECTIONS.

$\frac{r}{R}$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
ζ	0,131	0,138	0,158	0,206	0,294	0,440	0,661	0,977	1,408	1,978

TABLE II.

CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH RECTANGULAR TRANSVERSE SECTIONS.

$\frac{r}{R}$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
ζ	0,124	0,135	0,180	0,250	0,398	0,643	1,015	1,546	2,271	3,228

It is from hence seen, that for a round tube whose radius of curvature is twice as great as the radius of the tube, the co-efficient of resistance is = 0,294, and for a tube whose radius of curvature is at least ten times as great as the radius of the transverse section, this co-efficient = 0,131.

Example.—1. If the conducting pipe in the second example of § 332 has five small curves of 90° curvature, and of the ratio $\frac{r}{R} = \frac{1}{2}$ (Fig. 452), what quantity of water will it deliver? The height due to the resistance of the one curve

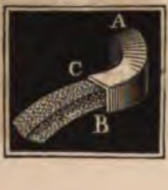
$= 0{,}294 \,.\, \frac{90^\circ}{180^\circ} \frac{v^2}{2g} = 0{,}147 \,.\, \frac{v^2}{2g}$; hence, for all five curvatures, it

$= 5 \,.\, 0{,}147 \frac{v^2}{2g} = 0{,}735 \frac{v^2}{2g}$, and, accordingly, the velocity sought:

$v = \frac{17{,}88}{\sqrt{7{,}584 + 0{,}735}} = \frac{17{,}88}{\sqrt{8{,}319}} = 6{,}153$ feet, and the quantity of water:

$Q = 0{,}7854 \,.\, \frac{1}{36} \,.\, 6{,}130 = 0{,}1337$ cubic feet = 231 cubic inches.

Fig. 450.



2. If the curved buckets of a turbine form channels 12 inches long, 2 inches broad, and 2 inches deep, as *ABC*, Fig. 450, and if the water flows through them with a velocity of 50 feet, and the mean radius of curvature R of this axis of the channels amounts to 8 inches, then is $\frac{r}{R} = \frac{1}{8}$, the co-efficient of the resistance of curvature = 0,127; further, $\frac{\beta}{\pi} = \frac{12}{\pi r} = \frac{12}{8\,\pi} = 0{,}4774$; and lastly, the height due to the resistance corresponding to the curvature of the scoop

$$= 0{,}127 \,.\, 0{,}4774 \frac{v^2}{2g} = 0{,}0606 \frac{v^2}{2g} = 0{,}0606 \,.\, 0{,}0155 \,.\, 50^2 = 2{,}348 \text{ feet.}$$

Therefore, by the resistance of curvature, 2,348 feet in fall are lost.

Remark. The earlier formula given by Du Buat, Gerstner, and Navier for the resistance of curvature, are quite useless. An extended account of the experiments of the author on this subject will be published in the third number of his "Investigations in Mechanics and Hydraulics."

Fig. 451.

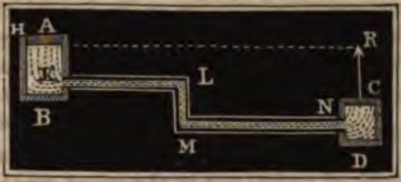


Fig. 452.

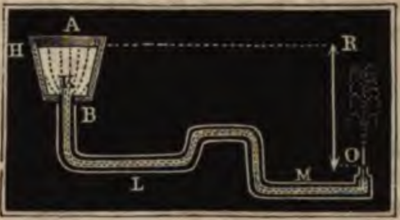
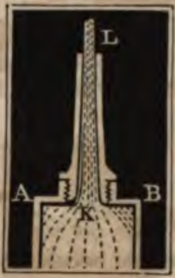


Fig. 453.



§ 335. *Jets d'Eau.*—A conducting tube either discharges into the air or under water. The discharge under water is applied when the tube at its outer orifice is so wide that the entrance of air may be feared. Here of course the head of water *RC*, Fig. 451, must be taken from the surface *H* of the upper water to that of *C* of the lower water. If the tube, for example, *KLM*, Fig. 452, discharges into the open air, it will give a stream of water *OR*, which, when allowed to ascend, is called a *jet d'eau*. We shall here consider what is most required for these jets. That a jet may ascend to the utmost possible height, it is necessary that the water should flow from the adjutage with great velocity; hence such adjutages must be applied which offer the fewest obstacles to the water in its passage, to which, therefore, the greatest co-efficients of velocity are due. Orifices in a thin plate, short tubes fashioned like the contracted fluid vein, and long and conically convergent ones, are those which give the greatest velocities of efflux. Orifices in a thin plate are little suitable, because a jet formed by them presents nodes and bulgings, and, therefore, is sooner scattered by the external air than the prismatic jet. The same takes place in a certain degree with short mouth-pieces, shaped like the contracted vein. Hence, for fountains and fire-engines, mostly long and slightly conically convergent, similar to those which d'Aubuisson used for his experiments, are very properly made use of. Sometimes entirely cylindrical jets are used. Where these mouth-pieces, as, for example, *KL*, Fig. 453, are screwed on to the conducting tube *AB*, they should gradually widen, that no contraction may occur in passing into them. If these mouth-pieces or discharging tubes are very long, like those of fire-engines, the friction of the water in them will then cause a considerable loss of pressure, because the water has here a great velocity. For great velocities we may properly put the co-efficient of resistance $\zeta = 0,016$, and, therefore, the loss of head of water $= 0,016 \frac{l}{d} \cdot \frac{v^2}{2g}$. If now the length of a hose is twenty times as great as the mean width, we shall then obtain the height of the resistance due to friction

$$= 0,016 \,.\, 20 \,.\, \frac{v^2}{2g} = 0,32 \,.\, \frac{v^2}{2g};$$

thus, from this cause, above 32 per cent. of the height of ascent is lost. These tubes are generally much longer, hence this loss is greater.

The velocity with which water passes out of a mouth-piece or hose, and on which the jet or the height of ascent principally depends, may be estimated by means of the above principles. If we put this velocity of efflux $= v$, the width of the mouth-piece at the exit orifice $= d$, and the mean width of the conducting tube $= d_1$, we shall then obtain the velocity of the water in it $v_1 = \frac{d^2}{d_1^2} v$. If ζ represent the co-efficient of resistance at the inner orifice of the tube, ζ_1 that of the resistance of friction in the pipe, and ζ_2 the co-efficient for the knees or curvature of the pipe, the height due to the resistance for the motion of water in the conduit pipes will be:

$$h = (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{v_1^2}{2g} = (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} \cdot \frac{v^2}{2g}.$$

It is seen from this, that the resistance to the water is less, the wider the conduit pipe is. It is hence an important rule, to employ as wide pipes and hoses as possible, for leading water to *jets d'eau* and for fire-engines.

If, further, we represent the co-efficient of resistance for the mouth-piece by ζ_3, we have then the height due to the resistance for this $= \zeta_3 \frac{v^2}{2g}$, and the sum of all the heights due to the resistance is then:

$$= \left[\left(\zeta + \zeta_1 \frac{l}{d} + \zeta_2\right) \frac{d^4}{d_1^4} + \zeta_3\right] \frac{v^2}{2g}.$$

If, lastly, the height of pressure, *i. e.* the depth RO, Fig. 452, of the outer orifice O below the surface of water H in the reservoir $= h$, the formula

$$h = \left[1 + \zeta_3 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4}\right] \frac{v^2}{2g},$$

holds true, and hence the velocity of efflux is:

$$v = \frac{\sqrt{2 g h}}{\sqrt{1 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} + \zeta_3}}.$$

If the jet were to rise perpendicularly and in vacuo, the height of ascent would be:

$$s = \frac{v^2}{2g} = \frac{h}{1 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} + \zeta_3},$$

but because the air and the descending water offer impediments to the ascent, and to the direction of the jet, as is the case in fire-engines, the effective height of ascent is somewhat less. According to d'Aubuisson's conclusions from the experiments undertaken upon this subject by Mariotte and Bossut, the effective height of ascent is $s_1 = s - 0{,}01 \,.\, s^2 = s\,(1 - 0{,}01 \,.\, s)$ metres, or for English measure $= s\,(1 - 0{,}00305\, s)$ feet.

We see from this that in great ascents proportionately more height is lost than in small velocities. Thick jets ascend somewhat higher than thin ones. In order to diminish the resistance of the descending water, the jet must be directed a little inclined. As to the height and amplitude of oblique jets, see § 38.

Example. If the conduit pipe for a fountain be 250 feet long, and 2 inches diameter, if the co-efficient of resistance corresponding to the mouth-piece $= 0{,}32$, if the entrance orifice at the reservoir be sufficiently rounded, and the bends that occur have sufficient radii of curvature to allow of our neglecting the corresponding co-efficients of resistance, to what height will a jet ½ inch thick, under a bead of water of 30 feet, rise? If we take the co efficient ζ_1 of friction $= 0{,}025$, we shall then obtain the entire height due to the resistance:

$$h = \left(1 + 0{,}025 \,.\, \frac{250}{\frac{1}{6}} \,.\, \left(\frac{\frac{1}{2}}{4}\right)^4 + 0{,}32\right) \frac{v^2}{2g} = 1{,}47 \,.\, \frac{v^2}{2g};$$

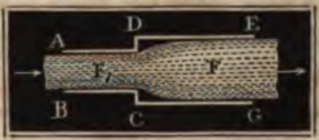
hence, the height due to the velocity $s = \frac{h}{1{,}47} = \frac{30}{1{,}47} = 20{,}41$ feet, and the effective height of ascent $s_1 = 20{,}41\,(1 - 0{,}00305 \,.\, 20{,}41) = 20{,}41 - 1{,}27 = 19{,}14$ feet.

CHAPTER IV.

ON THE RESISTANCES OF WATER IN PASSING THROUGH CONTRACTION.

§ 337. *Abrupt Widening.*—Changes in the transverse section of a tube, or of any other reservoir of efflux, produce changes in the velocity of the water. The velocity is inversely proportional to the transverse section of the stream. The wider the vessel is, the less is the velocity, and the narrower the vessel, the greater the velocity of the water flowing through it. If the transverse section of a vessel be suddenly altered, as, for example, in the tube *ACE*, Fig. 454, there then ensues a sudden alteration of the velocity, and this is accompanied by a loss of *vis viva*, or connected with a corresponding diminution of pressure. This loss may be as accurately measured as the mechanical effect in the impact of inelastic bodies (§ 258). Every particle of water which passes

Fig. 454.



from the narrower tube BD into the wider tube DG, strikes against the slowly moving mass of water in this tube, and, after impulse, joins itself to and proceeds onwards with it. It is exactly the same with the collision of solid and inelastic bodies; these bodies go on likewise after impact with a common velocity. Since we have already found that the loss of mechanical effect by the impact of these bodies is

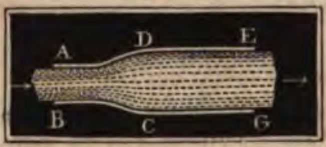
$$L = \frac{(v_1 - v_2)^2}{2g} \cdot \frac{G_1\, G_2}{G_1 + G_2},$$

so we may here, as the impinging particle of water G_1 is indefinitely small compared with the impinged mass of water G_2, put: $L = \frac{(v_1 - v_2)^2}{2g} G_1$, and, consequently, the corresponding loss of head $h = \frac{(v_1 - v_2)^2}{2g}$.

There arises, therefore, from a sudden change of velocity a loss of head, which is measured by the height due to the velocity corresponding to this change.

If now the transverse section of the one tube AC, $= F_1$, and that of the other CE, $= F$, the velocity of the water in the first tube $= v_1$, and that in the other $= v$, we then have $v_1 = \frac{Fv}{F_1}$, hence the loss in head of water in the passage from one tube to the other is $h_1 = \left(\frac{F}{F_1} - 1\right)^2 \cdot \frac{v^2}{2g}$, and the corresponding co-efficient of resistance $\zeta = \left(\frac{F}{F_1} - 1\right)^2$.

Fig. 455.



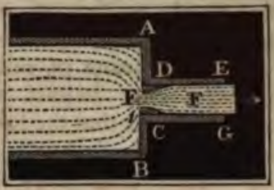
The experiments undertaken by the author on this subject accord well with theory. That the tube DG may be filled with water, it is requisite that it be not very short, nor much wider than the tube AC. This loss vanishes, when, as represented in Fig. 455, a gradual passing from one tube into the other is accomplished by the rounding of the edges.

Remark. The head of water found $h_1 = \left(\frac{F}{F_1} - 1\right)^2 \frac{v^2}{2\,g}$ cannot, of course, be utterly lost, we must rather assume that the mechanical effect produced by it is expended on the separation of the previous continuity of the particles of water.

Example. If the diameter of a tube, of the construction in Fig. 454, is as great again as that of another tube, then is $\frac{F}{F_1} = \left(\frac{2}{1}\right)^2 = 4$, hence the co-efficient of resistance $\zeta = (4-1)^2 = 9$, and the corresponding height due to the resistance on passing from the narrow into the wide tube $= 9 \cdot \frac{v^2}{2g}$. If the velocity of the water in the latter tube $=$ 10 feet, the height due to the resistance is then $= 9 \cdot 0{,}0155 \cdot 10^2 = 13{,}95$ feet.

§ 338. *Abrupt Contraction.*—A sudden change of velocity also

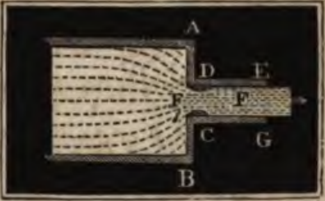
Fig. 456.



occurs when water passes from a cistern AB, Fig. 456, into a narrow tube DG, especially when there is a diaphragm at the place of entrance CD whose orifice is less than the transverse section of the tube DG. If the area of the contraction is $= F_1$, and α the co-efficient of contraction, we have then the transverse section F_2 of the contracted fluid vein $= \alpha F_1$; and if, on the other hand, F is the transverse section of the tube and v the velocity of efflux, we then find that the velocity at the contracted section F_2 is, $v_2 = \frac{F}{\alpha F_1} v$, and hence the loss of head in passing from F_2 into F, or from v_2 into v: $h = \frac{(v—v)^2}{2g} = \left(\frac{F}{\alpha F_1} - 1\right)^2 \frac{v^2}{2g}$, and the corresponding co-efficient due to the resistance: $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$.

Without the diaphragm, we have a mere short tube, Fig. 457; hence, $F = F_1$ and $\zeta = \left(\frac{1}{a} - 1\right)^2$.

Fig. 457.



If we assume $\alpha = 0{,}64$, we then obtain:

$$\zeta = \left(\frac{1 - 0{,}64}{0{,}64}\right)^2 = \left(\tfrac{9}{16}\right)^2 = 0{,}316.$$

But the co-efficient due to the resistance for the transit through an orifice in a thin plate is about 0,07; hence, here, where the water flows out $\frac{1}{\alpha}$ times as fast as from the contracted transverse section, the corresponding height due to the resistance $= 0{,}07 \,.\, \left(\frac{v}{\alpha}\right)^2 \cdot \frac{1}{2g} = 0{,}07 \,.\, \frac{1}{\alpha^2} \cdot \frac{v^2}{2g} \frac{0{,}07}{0{,}41} \cdot \frac{v^2}{2g} = 0{,}171 \,.\, \frac{v^2}{2g}$.

By combining these two resistances, we obtain the entire height due to the resistance for efflux through a short tube:

$$= 0{,}316 \frac{v^2}{2g} + 0{,}171 \frac{v^2}{2g} = 0{,}49 \,.\, \frac{v^2}{2g},$$

whilst we before found it $= 0{,}50 \frac{v^2}{2g}$.

Experiments on the efflux of water through an additional tube, with a narrow inner orifice, as in Fig. 456, have led the author to the following results. The co-efficient of resistance for transit through a diaphragm, and for a contraction at the wider tube, may be expressed by the formula $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$, but there must be put:

for $\frac{F_1}{F}$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
α	0,616	0,614	0,612	0,610	0,607	0,605	0,603	0,601	0,598	0,596

And it follows that:

ζ	231,7	50,99	19,78	9,612	5,256	3,077	1,876	1,169	0,734	0,480

From this, for example, the co-efficient of resistance in the case where the narrow transverse section is half as great as that of the tube, is $\zeta = 5,256$, *i. e.*, for transit through this contraction, a head of water is required which is $5\frac{1}{4}$ times as great as the height due to the velocity.

Example. What discharge will the apparatus delineated in Fig. 456 give, if the head of water is $1\frac{1}{2}$ feet, the width of the circular contraction $1\frac{1}{2}$, and that of tube 2 inches? We have here $\frac{F_1}{F} = \left(\frac{1\frac{1}{2}}{2}\right)^2 = (\frac{3}{4})^2 = \frac{9}{16} = 0,56$, hence $\alpha = 0,606$, and $\zeta = \left(\frac{16}{9 \,.\, 0,606} - 1\right)^2 = \left(\frac{16 - 5,454}{5,454}\right)^2 = \left(\frac{10,546}{5,454}\right)^2 = 3,74$. If now we put $h = (1 + \zeta)\,\frac{v^2}{2g}$, we shall then obtain the velocity of efflux:

$v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta}} = \frac{8,02\sqrt{1,5}}{\sqrt{4,74}} = 4,56$ feet, and consequently the quantity discharged:

$Q = \frac{\pi d^2}{4}\, v = \frac{\pi}{4} \,.\, 4 \,.\, 12 \,.\, 4,56 = 54,72 \,.\, \pi = 172$ cubic inches.

§ 339. *Effect of Imperfect Contraction.*—In the case considered in the last paragraph, the water issues from a large cistern, hence the contraction may be regarded as perfect; but if the transverse section of the cistern, or of the stream of fluid arriving at the narrow part, is not great with respect to the transverse section F_1, Fig. 458, of this contracted part; the contraction is then imperfect, and hence also the corresponding co-efficient of resistance is less than in the case above investigated. If we retain the former denominations, we have then also here the height due to the resistance, or the head of water expended by the transit through F_1, $h = \left(\frac{F}{\alpha F_1} - 1\right)^2 \frac{v^2}{2g}$, only, for α, we must substitute variable numbers which are greater the greater the ratio $\frac{G}{F_1}$ between the transverse section of the contraction and that of the conducting tube AB. If the diaphragm CD lies in a uniform tube AG, Fig. 459, then the same condition holds, only here the co-efficient α depends on $\frac{F_1}{F}$.

Fig. 458.

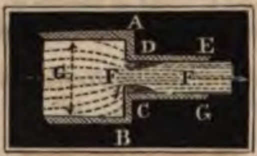
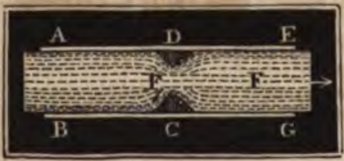


Fig. 459.



From experiments undertaken by the author, we must put into the formula $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$ for the co-efficient of resistance,

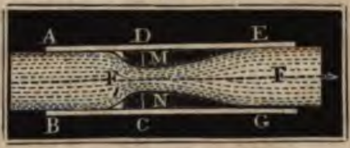
for $\frac{F_1}{F} =$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
α	0,624	0,632	0,643	0,659	0,681	0,712	0,755	0,813	0,892	1,000

And it follows:

ζ	225,9	47,77	17,50	7,801	3,753	1,796	0,797	0,290	0,060	0,000

These losses are diminished when, by rounding off the edges, the contraction is diminished or counteracted, and they may be entirely neglected, if, as is represented in Fig. 460, a gradually widening tube *MN* is put on.

Fig. 460.



Example. What head of water is requisite that the apparatus in Fig. 461 may deliver 8 cubic feet of water per minute? If the width of the diaphragm $F_1 = 1\frac{1}{2}$, the width of the efflux tube $DG = 2$ inches, and the lower width of the afflux tube $AC = 3$ inches, we shall then have $\frac{F_1}{G} = \left(\frac{1\frac{1}{2}}{3}\right)^2 = \frac{1}{4}$, hence $\alpha = 0{,}637$; further, $\frac{F}{F_1} = \left(\frac{2}{1\frac{1}{2}}\right)^2 = (\frac{4}{3})^2$ $= \frac{16}{9}$, and the co-efficient of resistance:

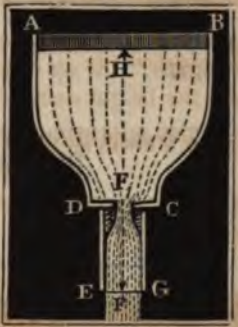
$$\zeta = \left(\frac{16}{9 \,.\, 0{,}637} - 1\right)^2 = \left(\frac{10{,}267}{5{,}733}\right)^2 = 3{,}207.$$ The velocity of efflux is now:

$$v = \frac{4\,Q}{\pi\, d^2} = \frac{4\,.\,8}{60\,.\,\pi\,(\frac{1}{6})^2} = \frac{19{,}2}{\pi} = 6{,}112 \text{ feet};$$ hence, the head of water in question is

$$h = (1 + \zeta)\,\frac{v^2}{2g} = 4{,}207\,.\,0{,}0155\,.\,6{,}112^2 = 2{,}43 \text{ feet.}$$

§ 340. *Slides, Cocks, Valves.*—For regulating the flow of water from pipes and cisterns, slides, cocks, valves, &c., are used, by which contractions are produced which offer obstacles to the passage of the water, and these may be determined in a manner similar to the loss estimated in the last paragraph. But since the water here undergoes further changes of direction, divisions, &c., the co-efficients α and ζ cannot be determined directly, but special experiments are necessary for this purpose. Such experiments have been also made,[*] and their principal results are communicated in the following tables.

Fig. 461.



---

[*] Experiments on the efflux of water through valves, slides, &c., undertaken and calculated by Julius Weisbach, under the title "Untersuchungen im Gebiete der Mechanik und Hydraulik," &c. Leipzig, 1842.

TABLE I.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDING VALVES IN RECTANGULAR TUBES.

Ratios of transverse section $\frac{F_1}{F}$	1,0	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1
Co-efficient of resistance ζ.	0,00	0,09	0,39	0,95	2,08	4,02	8,12	17,8	44,5	193

TABLE II.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDES IN CYLINDRICAL TUBES.

Height of place s .	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$
Ratio of transverse section.	1,000	0,948	0,856	0,740	0,609	0,466	0,315	0,159
Co-efficient of resistance ζ.	0,00	0,07	0,26	0,81	2,06	5,52	17,0	97,8

TABLE III.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A RECTANGULAR TUBE.

Angle of position.	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	66⅔
Ratio of transverse section.	0,926	0,849	0,769	0,687	0,604	0,520	0,436	0,352	0,269	0,188	0,110	0
Co-efficient of resistance.	0,05	0,31	0,88	1,84	3,45	6,15	11,2	20,7	41,0	95,3	275	∞

TABLE IV.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A CYLINDRICAL TUBE.

Angle of position.	5°	10°	15°	20°	25°	30°	35°
Ratio of transverse section.	0,926	0,850	0,772	0,692	0,613	0,535	0,458
Co-efficient of resistance.	0,05	0,29	0,75	1,56	3,10	5,47	9,68

Angle of position.	40°	45°	50°	55°	60°	65°	82⅛°
Ratio of transverse section.	0,385	0,315	0,250	0,190	0,137	0,091	0
Co-efficient of resistance.	17,3	31,2	52,6	106	206	486	∞

TABLE V.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN RECTANGULAR TUBES.

Angle of position.	5°	10°	15°	20°	25°	30°	35°
Ratio of transverse section.	0,913	0,826	0,741	0,658	0,577	0,500	0,426
Co-efficient of resistance.	0,28	0,45	0,77	1,34	2,16	3,54	5,72

Angle of position.	40°	45°	50°	55°	60°	65°	70°	90°
Ratio of transverse section.	0,357	0,293	0,234	0,181	0,134	0,094	0,060	0
Co-efficient of resistance.	9,27	15,07	24,9	42,7	77,4	158	368	∞

TABLE VI.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN CYLINDRICAL TUBES.

Angle of position.	5°	10°	15°	20°	25°	30°	35°
Ratio of transverse section.	0,913	0,826	0,741	0,658	0,577	0,500	0,426
Co-efficient of resistance.	0,24	0,52	0,90	1,54	2,51	3,91	6,22

Angle of position.	40°	45°	50°	55°	60°	65°	70°	90°
Ratio of transverse section.	0,357	0,293	0,234	0,181	0,134	0,094	0,060	0
Co-efficient of resistance.	10,8	18,7	32,6	58,8	118	256	751	∞

§ 341. By means of the co-efficients derived from the above tables, we may not only assign the loss of pressure corresponding to a certain slide, cock, or position of a valve, but also deduce what position is to be given to this apparatus that the velocity of efflux or the resistance may be of a certain amount. Such a determination is, of course, the more to be relied on, the more the regulating arrangements are like to those used in the experiments. The numerical values given in the tables are only true for the case where the water, after its transit through the contractions produced by means of this apparatus, again fills this tube. That this full flow may take place for small contractions, the tube should have a considerable length. The transverse sections of the rectangular tubes were 5 centimetres (1,97 inch) broad and $2\frac{1}{2}$ (,98 inch) deep. The transverse sections of the cylindrical tubes had, however, a width of 4 centimetres (1,57 inch). By the slide, Fig. 462, there is a simple contraction, whose transverse section forms in the one tube a mere rectangular F_1, Fig. 463; in the second, however, a lune, F_1, Fig. 464. In the case of cocks, two contractions present themselves, and also two changes of direction; on this account the resistances are also very considerable. The transverse sections of the greatest contractions have very peculiar figures. The stream in throttle valves, Fig. 466, divides itself into

Fig. 462.

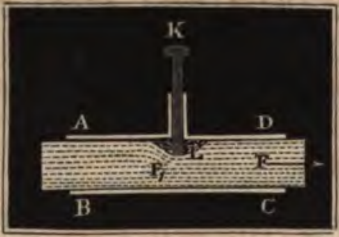


Fig. 463.

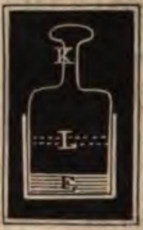


Fig. 464.

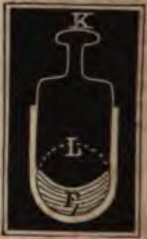


Fig. 465.

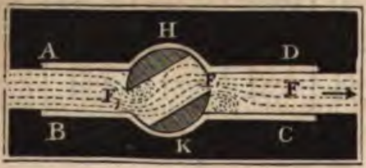
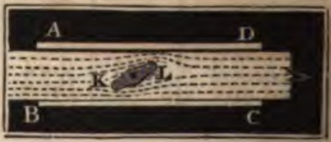


Fig. 466.



two portions, each of which passes through a contraction. The transverse sections of these contractions are, in the throttle valve of rectangular tubes, rectangular, and in cylindrical ones lunar-shaped. The following examples will suffice to show the application of the above tables.

Examples.—1. If a sliding valve is applied to a cylindrical conducting pipe 500 feet long and 3 inches wide, and this be drawn up to $\frac{3}{8}$ of its entire height, and therefore close $\frac{5}{8}$ of the pipe, what discharge will it deliver under a pressure of 4 feet? The co-efficient of resistance ζ for entrance into the pipe may be put from the above at 0,505, and that of the resistance of the slide from Table II. = 5,52, hence the velocity of the flow $v = \frac{8{,}02 \,.\, \sqrt{4}}{\sqrt{1{,}505 + 5{,}52 + \zeta_2 \frac{l}{d}}} = \frac{8{,}02 \,.\, 2}{\sqrt{7{,}025 + 500 \,.\, 4 \zeta_2}} = \frac{16{,}04}{\sqrt{7{,}025 + 2000\,\zeta_2}}$. If we put the co-efficient of friction $\zeta_2 = 0{,}025$, we shall then obtain $v = \frac{16{,}04}{\sqrt{57{,}025}} = 2{,}12$ feet. But the velocity $v = 2{,}12$ feet, gives more correctly $\zeta_2 = 0{,}026$; hence, more accurately, $v = \frac{16{,}04}{\sqrt{59{,}025}} = 2{,}08$ feet, and the discharge per second $= \frac{\pi}{4} \,.\, 9 \,.\, 12 \,.\, 2{,}08 = 56{,}16 \,.\, \pi = 177$ cubic inches.—2. A conducting pipe, 4 inches wide, delivers, under a head of water of 5 feet, 10 cubic feet of water per minute; what position must be given to the throttle-valve applied, that it may afterwards deliver only 8 cubic feet? The velocity at the beginning is $= \frac{10 \,.\, 4}{60 \,.\, \pi \,(\frac{1}{3})^2} = \frac{6}{\pi} = 1{,}91$ feet, and after putting on the valve $\frac{8}{10} \,.\, 1{,}91 = 1{,}528$ feet. The co-efficient of efflux is $\frac{v}{\sqrt{2gh}} = \frac{1{,}91}{8{,}02\sqrt{5}} = 0{,}106$, hence the co-efficient of resistance $= \frac{1}{\mu^2} - 1 = \frac{1}{0{,}106^2} - 1 = 88$; the co-efficient of efflux for the second case is $= \frac{8}{10} \,.\, 0{,}106 = 0{,}0848$; hence the co-efficient of resistance $= \frac{1}{0{,}0848^2} - 1 = 138{,}0$, and consequently to produce the co-efficient of resistance of the throttle-valve: $\zeta = 138 - 88 = 50$. But now, from Table VI., the angle of position $\alpha = 50^\circ$, $\zeta = 32{,}6$, and the angle of position $\alpha = 55^\circ$, $\zeta = 58{,}8$; hence we may be allowed to assume, for a position of $50^\circ + \frac{15{,}7}{26{,}2} \,.\, 5^\circ = 53^\circ$, the desired quantity of discharge may be obtained. If we consider, further, for a change of velocity of 1,91 feet to 1,528 feet, the co-efficient of resistance passes from 0,0266 into 0,0281; then, more correctly, $\zeta = 138{,}0 - 88 \,.\, \frac{281}{266} = 138{,}06 - 92{,}96 = 45{,}04$, and, accordingly, the angle of position $= 50^\circ + \frac{10{,}9}{26{,}2} \,.\, 5^\circ = 52^\circ$.

§ 342. *Valves.*—The knowledge of the resistance produced by valves is of great importance. Experiments have been made by the author on this subject. The *conical* and *clack*, or *flap-valves*, Figs. 467 and 468, are those which most frequently are met with in practice. In both, the water passes through the aperture formed by a

Fig. 467.

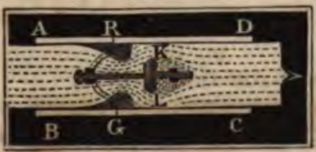
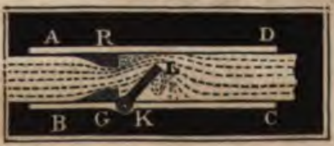


Fig. 468.



ring *RG*; the conical valve *KL* has a guide rod, by which it is fixed in guides, and admits of an outward push only in the direction of the axis; the clack valve *KL* opens by turning like a door. It is easily seen in both apparatuses that a resistance is opposed to the water, not only by the valve ring, but also by the valve plate.

For the conical valve with which the experiments were undertaken, the ratio of the aperture in the valve ring, to the transverse section of the whole tube was 0,356, and, on the other hand, the ratio of the surface of the ring for the opened valve to the transverse section of the tube 0,406; hence, for the mean, we may put $\frac{F_1}{F} = 0,381$. Whilst the efflux in different positions of the valve was observed, it was found that the co-efficient of resistance diminished when the valve slide was greater, and that this diminution was almost insignificant when it exceeded half the width of the aperture. Its amount was in this case = 11, therefore, the height due to the resistance or the loss of head of water $= 11 \cdot \frac{v^2}{2g}$, v being the velocity of the water in the full tube. This number may be also used for determining the co-efficients of resistance corresponding to the other ratios of the transverse sections. Let generally $\zeta = \left(\frac{F}{\alpha F_1} - 1\right)^2$, we then obtain for the observed case $\frac{F_1}{F} = 0,381$, $\zeta = 11$, and $11 = \left(\frac{1}{0,381\, \alpha} - 1\right)^2$, hence

$$\alpha = \frac{1}{0,381\,(1 + \sqrt{11})} = \frac{1}{4,317 \,.\, 0,381} = 0,608,$$

and, lastly, in general:

$$\zeta = \left(\frac{F}{0,608\, F_1} - 1\right)^2 = \left(1,645 \,.\, \frac{F}{F_1} - 1\right)^2.$$

If, for example, the transverse section of the aperture is one half that of the tube, the co-efficient of resistance will accordingly $= (1,645 \,.\, 2 - 1)^2 = 2,29^2 = 5,24$.

For the clack, or trap valve, the ratio of the transverse section of the aperture to the tube was $\frac{F_1}{F}$, = 0,535; but the following table shows in what degree the co-efficient of resistance diminishes with the size of the aperture.

TABLE

OF THE CO-EFFICIENTS OF RESISTANCE FOR TRAP VALVES.

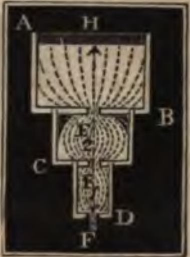
Angle of aperture.	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°
Co-efficient of resistance.	90	62	42	30	20	14	9,5	6,6	4,6	3,2	2,3	1,7

The co-efficients of resistance of these valves may be calculated approximately with the help of this table, even when the ratio of the transverse section is any other. The same method must be adopted as that followed for conical valves.

Example. A forcing-pump delivers by each descent of the piston, 5 cubic feet of water in 4 seconds, the width of the tube of ascent, in which lies the valve opening upwards, is 6 inches, the aperture of the valve-ring $3\frac{1}{2}$ inches, and the greatest diameter of the valve $4\frac{1}{2}$ inches; what resistance has the water in its passage through the valve to overcome? The ratio of the transverse section for the aperture is $\left(\frac{3,5}{6}\right)^2 = (\frac{7}{12})^2 = 0,34$, and that of the annular contraction to the transverse section of the tube $= 1 - \left(\frac{4,5}{6}\right)^2 = 1 - (\frac{3}{4})^2 = 0,44$, hence the mean ratio of section $= \frac{0,34 + 0,44}{2} = 0,39$ and the corresponding co-efficient of resistance $= \left(\frac{1,645\,.}{0,39} - 1\right)^2 = 3,22^2 = 10,4$. The velocity of the water is: $v = \frac{5}{4\,.\,\frac{\pi}{4}\,.\,(\frac{1}{2})^2} = \frac{20}{\pi} = 6,37$ feet, the height due to the velocity $= 0,629$ feet; and, consequently, the resistance due to the height $= 10,4\,.\,0,629 = 6,54$ feet. The quantity forced up in one second weighs $\frac{5}{4}\,.\,62,5 = 77,6$ lbs.; hence the mechanical effect which by the transit of the water through the valve is consumed in this time $= 77,6\,\cdot\,6.54 = 507,5$ ft. lbs.

§ 343. *Compound Vessels.*—The principles already laid down on the resistance of water in its passage through contractions, find their application in the efflux of water through compound vessels. The apparatus AD represented in Fig. 469, is divided by two partition walls containing the orifices F_1 and F_2, and on this account forms three vessels of communication. Were there no partition walls, and the edges, where one vessel passes into the other, rounded off, we should then have as for a single vessel the velocity of flow through F: $v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta}}$, h representing the depth FH below the surface of water, and ζ the co-efficient of resistance for the passage through the orifice F. But since obstacles are to be overcome on passing through F_1 and F_2, we then have to put $v = \sqrt{\frac{2gh}{1+\zeta+\zeta_1+\zeta_2}}$, and to substitute for ζ_1 and ζ_2, the co-efficients of resistance corresponding to the contractions F_1 and F_2. If we represent the transverse sections of the vessels CD, BC and AB, by G, G_1 and G_2, we may further put (§ 338):

Fig. 469.

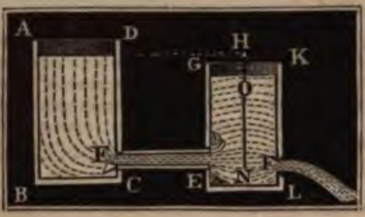


$$\zeta_1 = \left(\frac{G}{\alpha F_1} - 1\right)^2, \text{ and } \zeta_2 = \left(\frac{G_1}{\alpha F_2} - 1\right)^2,$$

hence also:

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \left(\frac{G}{\alpha F_1} - 1\right)^2 + \left(\frac{G_1}{\alpha F_2} - 1\right)^2}}.$$

Fig. 470.



Exactly the same relations take place in the compound apparatus of discharge represented in Fig. 470, except only that the friction of the water in the tube of communication *CE* has perhaps to be taken into account. If l is the length, and d the width of this tube, but ζ_1 the co-efficient of friction, and v_1 the velocity of the water in the tube of communication, we then have the height which the water loses in passing from *AC* to *GL*:

$$h_1 = \left[1 + \left(\frac{1}{\alpha} - 1\right)^2 + \zeta_1 \frac{l}{d}\right] \frac{v_1^2}{2g},$$

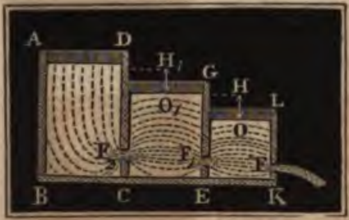
or, since the velocity is to be put:

$$v_1 = \frac{\alpha F}{F_1} v,\ h_1 = \left[1 + \left(\frac{1}{\alpha} - 1\right)^2 + \zeta_1 \frac{l}{d}\right] \left(\frac{\alpha F}{F_1}\right)^2 \frac{v^2}{2g}.$$

If this height be deducted from the whole head of water h, there will remain the head of water in the second vessel $h_2 = h - h_1$, and hence the velocity of efflux:

$$v = \frac{\sqrt{2gh_2}}{\sqrt{1+\zeta}} = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \left[1 + \left(\frac{1}{\alpha} - 1\right)^2 + \zeta_1 \frac{l}{d}\right]\left(\frac{\alpha F}{F_1}\right)^2}}.$$

Fig. 471.



This determination is very simple with the apparatus represented in Fig. 471, because the transverse sections G, G_1, G_2, of the cisterns may be made indefinitely great with respect to the transverse sections of the orifices F, F_1, F_2. Hence the first difference of level OH, or height due to the resistance in passing through:

$$F_1 \text{ is } h_1 = \frac{1}{2g}\left(\frac{v_1}{\alpha_1}\right)^2 = \left(\frac{\alpha F}{\alpha_1 F_1}\right)^2 \cdot \frac{v^2}{2g},$$

and likewise the second difference of level $O_1 H_1$, or the height due to the resistance in passing through

$$F_2 \text{ is } h_2 = \left(\frac{\alpha F}{\alpha_2 F_2}\right)^2 \cdot \frac{v^2}{2g},$$

where α, α_1, α_2, represent the co-efficients of contraction for the orifices F, F_1 and F_2. It accordingly follows that:

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \left(\frac{\alpha F}{\alpha_1 F_1}\right)^2 + \left(\frac{\alpha F}{\alpha_2 F_2}\right)^2}},$$

and the quantity of discharge:

$$Q = \frac{\alpha F \sqrt{2gh}}{\sqrt{1 + \left(\frac{\alpha F}{\alpha_1 F_1}\right)^2 + \left(\frac{\alpha F}{\alpha_2 F_2}\right)^2}}$$

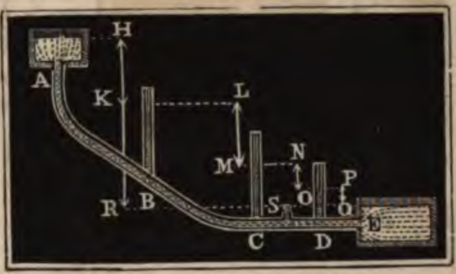
$$= \frac{\sqrt{2gh}}{\sqrt{\left(\frac{1}{\alpha F}\right)^2 + \left(\frac{1}{\alpha_1 F_1}\right)^2 + \left(\frac{1}{\alpha_2 F_2}\right)^2}}.$$

It is easy to perceive that compound reservoirs of efflux deliver less water, under otherwise similar circumstances, than simple ones.

Example. If in the apparatus, Fig. 470, the whole head of water or depth of the centre of the orifice F below the surface of water of the first cistern is 6 feet, the orifice 8 inches broad and 4 inches deep, the tube connecting both reservoirs 10 feet long, 12 inches broad, and 6 inches deep, what discharge will this reservoir give? The mean width of the tube $= \frac{4 \,.\, 1 \,.\, 0{,}5}{2 \,.\, 1{,}5} = \frac{2}{3}$ ft., hence $\frac{l}{d} = \frac{3 \,.\, 10}{2} = 15$; let us now put the co-efficient of friction $\zeta_1 = 0{,}025$, and it follows that $\zeta_1 \,.\, \frac{l}{d} = 0{,}025 \,.\, 15 = 0{,}375$; if the co-efficient of resistance for entrance into prismatic tubes be here put 0,505, we obtain $1 + \left(\frac{1}{\alpha_1} - 1\right)^2 + \zeta_1 \frac{l}{d} = 1 + 0{,}505 + 0{,}375 = 1{,}88$. As $\frac{\alpha F}{F_1} = \frac{0{,}64 \,.\, 8 \,.\, 4}{12 \,.\, 6} = 0{,}2845$, the co-efficient of resistance for the entire connecting tube $= 1{,}88 \,.\, 0{,}2845^2 = 0{,}152$, and the co-efficient of resistance for the transit through $F, = 0{,}07$, we then obtain the velocity of efflux: $v = \frac{8{,}02 \sqrt{6}}{\sqrt{1{,}07 + 0{,}152}} = \frac{8{,}02 \sqrt{6}}{\sqrt{1{,}222}} = 17{,}77$ feet. The contracted section is $0{,}64 \,.\, 1 \,.\, \frac{1}{2} = 0{,}32$ square feet, hence the discharge $= 0{,}32 \,.\, 17{,}77 = 5{,}68$ cubic feet.

§ 344. *Piëzometers.*—The loss of pressure which water suffers in conduit pipes from contractions, friction, &c., may be measured by columns of water, which are sustained in vertically placed tubes, which, when used for this purpose, are called *piëzometers*.

Fig. 472.



If v is the velocity of the water at a place B, Fig. 472, where a piëzometer is applied, l the length, d the width of the portion of tube AB, h the head of water or the depth of the point B below the surface of water; if, further, ζ is the co-efficient of resistance for entrance from the reservoir into the tube, and ζ_1 the co-efficient of friction, we then have for the height of the piëzometer measuring the pressure at B,

$$z = h - \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g}.$$

If the length of a portion of the tube $BC = l_1$, and its fall $= h_1$, we then have the height of the piëzometer at C:

$$z_1 = h + h_1 - \left(1 + \zeta + \zeta_1 \frac{l}{d} + \zeta_1 \frac{l_1}{d}\right) \frac{v^2}{2g},$$

and hence the difference of these two heights:

$$z_1 - z = h_1 - \zeta_1 \frac{l_1}{d} \cdot \frac{v^2}{2g},$$

hence, inversely, the height of the portion of tube BC, due to the resistance:

$\zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g} = h_1 + z - z_1 =$ *the fall of the portion of tube plus the difference of the heights of the piëzometers.*

From this it is seen that piëzometers are applicable to the measurement of the resistance which the water has to overcome in conduit pipes. If a particular impediment is found in the tubes; if, for instance, some small body is found fixed there, this will immediately be shown by the falling of the piëzometer, and the amount of the resistance produced, expressed. The resistances which are caused by regulating apparatus, such as cocks, slides, &c., may be likewise expressed by the height of the piëzometer. The piëzometer, for example, stands lower at D than at C, not only in consequence of the friction of the water in the portion of water CD, but also in consequence of the contraction which the slide S produces in this tube. If for a perfectly opened slide the difference NO of the height of the piëzometer $= h_1$, and for the slide partly closed $= h_2$, the new difference or depression $h_2 = h_1$ gives the height due to the resistance which corresponds to the passage of the water through the slide. Lastly, the velocity of efflux may be also estimated by the height of the piëzometer. If the height of the piëzometer $PQ = z$, the length of the last portion of tube $DE = l$, and its width $= d$, we then have:

$$z = \zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g}, \text{ and hence } v = \sqrt{\frac{2gz}{\zeta_1 \frac{l}{d}}} = \sqrt{\frac{d}{l} \cdot \frac{2gz}{\zeta_1}}.$$

Example. If the height of the piëzometer $PQ = z$, Fig. 471, $\frac{3}{4}$ foot, and the length of the tube DE, measured from the piëzometer to the discharging orifice, $= 150$ feet, the width of the tube $3\frac{1}{2}$ inches, the velocity of efflux then follows:

$v = 8{,}02 \,.\, \sqrt{\frac{3{,}5}{150 \,.\, 12} \cdot \frac{0{,}75}{0{,}025}} = 8{,}02 \,.\, 0{,}2415 = 1{,}937$ feet, and the discharge

$Q = \frac{\pi}{4} \cdot \left(\frac{3{,}5}{12}\right)^2 . \, 1{,}937 = 01{,}274$ cubic feet.

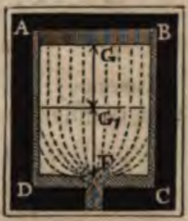
CHAPTER V.

ON THE EFFLUX OF WATER UNDER VARIABLE PRESSURE.

§ 345. *Prismatic Vessels.*—If a cistern from which water flows through an orifice at the side or bottom, has no influx to it from any other side, a gradual sinking of the surface of water will take place, and the cistern at last empty itself. If, further, the quantity of influx Q, be greater or less than the quantity of efflux $\mu F \sqrt{2gh}$, the surface of water will then rise or fall until the head of water $h = \frac{1}{2g}\left(\frac{Q}{\mu F}\right)^2$, and after this the head of water and the velocity of efflux will remain unaltered. Our problem, then, is to find how the time, the rise and fall of the water, and the emptying of vessels of given form and dimensions, depend on each other.

The efflux from a prismatic vessel presents the most simple case when it takes place through an opening in the bottom, and when there is no efflux from above or below. If x is the variable head of water FG_1, F the area of the orifice, and G the transverse section of the vessel AC, Fig. 473, we have then the theoretical velocity of efflux $v = \sqrt{2gx}$, the theoretical velocity of the falling surface of the water $= \frac{F}{G} v = \frac{F}{G}\sqrt{2gx}$, and the effective velocity $v_1 = \frac{\mu F}{G}\sqrt{2gx}$. At the commencement: $x = FG = h$, and at the end of the efflux $x = 0$, therefore, the initial velocity is: $c = \frac{\mu F}{G}\sqrt{2gh}$, and the final velocity $c_1 = 0$.

Fig. 473.



It is seen from the formula $v_1 = \sqrt{2\left(\frac{\mu F}{G}\right)^2 g\, x}$, that the motion of the surface is uniformly retarded, and the measure of the retardation $p = \left(\frac{\mu F}{G}\right)^2 g$, hence we also know (§ 14), that this velocity $=0$, and the discharge ceases, when

$$t = \frac{c}{p} = \frac{\mu F}{G}\sqrt{2gh} \div \left(\frac{\mu F}{G}\right)^2 g = \frac{G}{\mu F}\sqrt{\frac{2gh}{g^2}}, \text{ i.e. } t = \frac{2G\sqrt{h}}{\mu F\sqrt{2g}}.$$

We may also put:

$$t = \frac{2\,G\,h}{\mu\,F\sqrt{2gh}} = \frac{2\,G\,h}{Q}.$$

and, according to this, assume that double the time is required for the efflux of the discharge Gh through the orifice at the bottom F, under a head of water decreasing from h to 0, than under a uniform pressure.

As the co-efficient of efflux μ is not quite constant, but is greater for a diminution of pressure, we must, therefore, in calculations of this kind, substitute a mean value of this co-efficient.

Example. In what time will a rectangular cistern, of 14 square feet section, empty itself through a round orifice at the bottom, of 2 inches width, if the original head of water amount to 4 feet? The time of efflux would be theoretically:

$$t = \frac{2 \,.\, 14 \sqrt{4}}{8{,}02 \,.\, \frac{\pi}{4} (\frac{1}{6})^2} = \frac{2 \,.\, 14 \,.\, 144 \,.\, 2}{8{,}02 \,.\, \pi} = \frac{8064}{8{,}02 \,.\, \pi} = 320'' = 5 \text{ min. } 20 \text{ sec.}$$

At the end of half the time of efflux, the head of water will be $= (\frac{1}{2})^2 \,.\, h = \frac{1}{4} \,.\, 4 = 1$ ft. Now the co-efficient of efflux, which corresponds to the head of water $= 1$ foot is 0,613, hence the effective time of discharge will be $= \frac{320''}{0{,}613} = 521'' = 8$ minutes, 41 seconds.

§ 346. *Vessels of Communication.*—Since for an initial head of water h_1, the time of efflux $t_1 = \frac{2\,G\sqrt{h_1}}{\mu\,F\sqrt{2g}}$, and for an initial head of water h_2 this time $t_2 = \frac{2\,G\sqrt{h_2}}{\mu F \,.\, \sqrt{2g}}$, it then follows by subtraction, that the time within which the head of water passes from h_1 to h_2, and the surface of water sinks $h_1 - h_2$ is:

$$t = \frac{2\,G}{\mu\,F \,.\, \sqrt{2\,g}} (\sqrt{h_1} - \sqrt{h_2}), \text{ or for the English foot measure:}$$

$$t = 0{,}249\, \frac{G}{\mu F} (\sqrt{h_1} - \sqrt{h_2}).$$

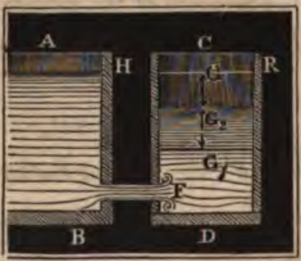
Inversely, the depression of the surface corresponding to a given time of efflux is $s = h_1 - h_2$, and is given by the formula:

$$h_2 = \left(\sqrt{h_1} - \frac{\mu\sqrt{2g} \,.\, F}{2G}\, t\right)^2, \text{ or,}$$

?/ $$s = \frac{\mu\sqrt{2g} \,.\, F t}{G}\left(\sqrt{h_1} \mp \frac{\mu\sqrt{2g}}{4\,G} \,.\, t\right).$$

The same formulæ are further applicable, when a vessel CD, Fig. 474, is filled by another AB in which the water maintains a uniform height. If the transverse section of the tube of communication, or of the orifice $= F$, the transverse section of the vessel to be filled $= G$, and the original level $G\,G_1$ of the two surfaces of water $= h$, we have then, since here the surface of water G_1 in the second vessel is uniformly retarded, the time of filling likewise, or the time within which the second surface of water comes to the level HR of the first:

Fig. 474.



$$t = \frac{2\,G\,\sqrt{h}}{\mu F . \sqrt{2g}},$$

and likewise the time in which the height of level h_1 passes into h_2, and, therefore, the surface of water ascends to:

$$GG_1 = s = h_1 - h_2.$$

$$t = \frac{2\,G}{\mu\,F . \sqrt{2g}}\,(\sqrt{h_1} - \sqrt{h_2}).$$

Examples. 1.—How much will the surface of water in the vessel of the last example sink in two minutes? $h_1 = 4$, $t = 2 . 60 = 120$, $\frac{F}{G} = \frac{\pi .}{14 . 144}$ and if we assume, further, $\mu = 0{,}605$, it follows then $h_2 = \left(\sqrt{h_1} - \mu . \sqrt{2\,g} . \frac{F\,t}{2\,G}\right)^2$ $= \left(2 - \frac{0{,}605 . 8{,}02 . \pi . 120}{2 . 14 . 144}\right)^2 = \left(2 - 0{,}605 . 8{,}02 . \frac{5 . \pi}{168}\right)^2 = 2{,}393$ feet, and the depression sought is $s = 4 - 2{,}393 = 1{,}607$ feet.

2. What time does the water in the 18 inch wide tube *CD*, Fig. 475, require to run over if it communicates with a vessel *AB* by a short 1½ inch wide tube, and the rising surface of water *G* stands, at the beginning, 6 feet below the uniform surface of water *A*, and 4½ feet below the head *C* of the tube. It is:

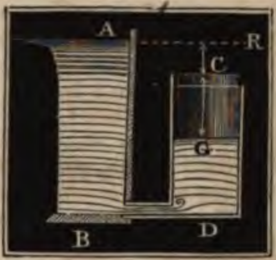
$$t = \frac{2\,G}{\mu\,\sqrt{2g} . F}\,(\sqrt{h_1} - \sqrt{h_2}),$$

$h_1 = 6$, $h_2 = 6 - 4{,}5 = 1{,}5$, $\frac{G}{F} = \left(\frac{18}{1{,}5}\right)^2 = 144$ and $\mu = 0.81$, whence it follows that:

$$t = \frac{2 . 144}{0{,}81 . 8{,}02}\,(\sqrt{6} - \sqrt{1{,}5}) = \frac{288 . 1{,}2248}{0{,}81 . 8{,}02}$$

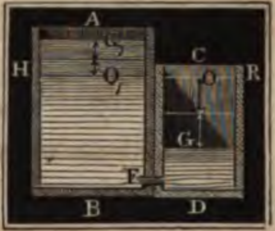
$= 54{,}3$ sec.

Fig. 475.



If the first vessel *AB*, Fig. 476, from which the water runs into the other, has no influx, and its section G_1 also not to be considered as indefinitely great compared with the section G of the subsequent vessel *CD*, we have then to modify the condition. If the variable distance $G_1 O_1$ of the first surface of water from the level *HR* at which both surfaces stand at the end of the efflux $= x$, and the distance GO of the second surface of water from this same plane $= y$, we have then the variable head of water $= x + y$, and the corresponding velocity of efflux: $v = \sqrt{2g\,(x + y)}$, and the quantity of water:

Fig. 476.



$$G_1 x = G y,\ v = \sqrt{2g\left(1 + \frac{G}{G_1}\right) y}.$$

The velocity with which the surface of water in the second vessel ascends is now:

$$v_1 = \frac{\mu F}{G}\,v = \frac{\mu F}{G}\,\sqrt{2g\left(1 + \frac{G}{G_1}\right) y},$$

consequently the retardation:

$$p = \left(\frac{\mu F}{G}\right)^2 \left(1 + \frac{G}{G_1}\right) g,$$

and the time of efflux:

$$t = \frac{\mu F}{G} \sqrt{2 g \left(1 + \frac{G}{G_1}\right) y \div \left(\frac{\mu F}{G}\right)^2 \left(1 + \frac{G}{G_1}\right) g}$$

$$= \frac{2 G \sqrt{y}}{\mu F \sqrt{2 g \left(1 + \frac{G}{G_1}\right)}}.$$

Let us substitute for x and y, the initial height of level h, and therefore put:

$$x + y = h, \text{ or } \left(1 + \frac{G}{G_1}\right) y = h,$$

and we then obtain:

$y = \dfrac{h}{1 + \dfrac{G}{G_1}}$, and the time in which the two surfaces of water come to a level:

$$t = \frac{2 G \sqrt{h}}{\mu F \left(1 + \frac{G}{G_1}\right) \sqrt{2 g}} = \frac{2 G G_1 \sqrt{h}}{\mu F (G + G_1) \sqrt{2 g}}.$$

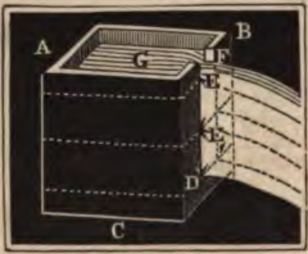
The time within which the level falls from h to h_1, is, on the other hand:

$$t = \frac{2 G G_1 (\sqrt{h} - \sqrt{h_1})}{\mu \sqrt{2} g F (G + G_1)}.$$

Example. If the section of a cistern from which water flows is 10 square feet, and the section G of the recipient cistern 4 square feet; if, further, the initial level h of the two surfaces amounts to 3 feet, and the cylindrical tube of communication is 1 inch wide, then the time in which the water comes in both vessels to the same level is:

$$t = \frac{2 \,.\, 10 \,.\, 4 \,.\, \sqrt{3}}{0{,}82 \,.\, 8{,}02 \,.\, \frac{\pi}{4} \,.\, \frac{14}{144}} = \frac{320 \,.\, 72 \,.\, \sqrt{3}}{0{,}82 \,.\, 8{,}02 \,.\, 7 \pi} = 275{,}9 \text{ sec.}$$

Fig. 477.



§ 348. *Notches in a Side.*—If water flows through the notch or cut DE of a prismatic cistern ABC, Fig. 477, to which there is no influx, the time of efflux may then be estimated in the following manner. Let us represent the transverse section of the cistern by G, the breadth EF of the notch by b, and the depth DE by h, and divide the whole aperture of efflux by horizontal lines into small slices, each of the breadth b and depth $\frac{h}{n}$. At a constant pressure the discharge per second will be, $Q = \frac{2}{3}\mu b\sqrt{2gh^3}$, if we divide this into the area $\frac{Gh}{n}$ of a stratum of water, we shall then obtain the time of efflux $\tau = \frac{Gh}{\frac{2}{3}\mu nb\sqrt{2gh^3}}$, which we may write:

$$\frac{3\,Gh}{2\,\mu\,n\,b\sqrt{2g}} \,.\, h^{-\frac{3}{2}}\,.$$

Now, to obtain the time of efflux t for a quantity of water $G(h - h_1)$, or to determine the time in which the head of water above the line $DE = h$ sinks to $DE_1 = h_1$, let us make $h_1 = \frac{m}{n}h$, and therefore h_1 to consist of m parts, and let us now substitute for $h^{-\frac{3}{2}}$, successively:

$$\left(\frac{m}{n}h\right)^{-\frac{3}{2}}, \left(\frac{m+1}{n}h\right)^{-\frac{3}{2}}, \left(\frac{m+2}{n}=\right)^{-\frac{3}{2}} \cdots \left(\frac{nh}{n}\right)^{-\frac{3}{2}},$$

and finally add the results obtained. In this manner we shall obtain the time required:

$$t = \frac{3\,G\,h}{2\,\mu n b\sqrt{2\,g}}\left[\left(\frac{mh}{n}\right)^{-\frac{3}{2}} + \left(\frac{m+1}{n}h\right)^{-\frac{3}{2}} + \ldots + \left(\frac{nh}{n}\right)^{-\frac{3}{2}}\right]$$

$$= \frac{3\,G\,h}{2\,\mu n b\sqrt{2\,g}} \,.\, \frac{h^{-\frac{3}{2}}}{n^{-\frac{3}{2}}}\left(m^{-\frac{3}{2}} + (m+1)^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right)$$

$$= \frac{3\,Gh^{-\frac{1}{2}}}{2\,\mu\,n^{-\frac{1}{2}}\,b\sqrt{2g}}\left[\left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right) - \left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots m^{-\frac{3}{2}}\right)\right],$$

or, from the "Ingenieur," Arithmetic, § 28:

$$t = \frac{3\,Gh^{-\frac{1}{2}}}{2\,\mu\,n^{-\frac{1}{2}}\,b\sqrt{2g}}\left(\frac{n^{-\frac{3}{2}-1}}{-\frac{3}{2}+1} - \frac{m^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right)$$

$$= \frac{3\,Gh^{\frac{1}{2}}}{2\,\mu\,b\sqrt{2\,gh}} \,.\, 2\left(m^{-\frac{1}{2}} - n^{-\frac{1}{2}}\right)$$

$$= \frac{3\,G}{\mu\,b\sqrt{2\,g\,h}}\left[\left(\frac{m}{n}\right)^{-\frac{1}{2}} - 1\right]$$

$$= \frac{3\,G}{\mu\,b\sqrt{2g}}\left[\left(\frac{m}{n}h\right)^{-\frac{1}{2}} - h^{-\frac{1}{2}}\right] = \frac{3\,G}{\mu b\sqrt{2g}}\left(\frac{1}{\sqrt{h_1}} - \frac{1}{\sqrt{h}}\right).$$

Let $h_1 = 0$, we have then $\frac{1}{\sqrt{h_1}}$, and therefore also $t = \infty$; an indefinite time, therefore, is required for the water to run down to the sill.

Example. If the water flows through a notch in a side, of 8 inches in breadth, from a reservoir 110 feet long and 40 feet broad, what time will it require to pass from a head of water of 15 inches to one of 6 inches?

$$t = \frac{3 \,.\, 110 \,.\, 40}{\mu \,.\, \frac{2}{3} \,.\, 8{,}02}\left(\frac{1}{\sqrt{0{,}5}} - \frac{1}{\sqrt{1{,}25}}\right) = \frac{19800}{\mu \,.\, 8{,}02}\left(\sqrt{2} - \sqrt{\tfrac{4}{5}}\right)$$

$$= \frac{19800}{8{,}02\,\mu}\,(1{,}4142 - 0{,}8944) = \frac{19800 \,.\, 0{,}5198}{8{,}02\,\mu} = \frac{1283}{\mu} \text{ sec.}$$

If we assume the co-efficient $\mu = 0{,}60$, the effective time of efflux will be $t = \frac{1283}{0{,}6} = 2138$ sec. $= 35$ min. 38 sec.

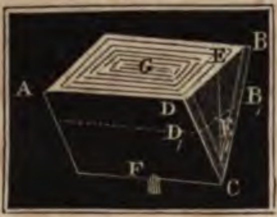
Remark. We may put for a rectangular lateral opening, approximatively:

$$t = \frac{2\,G}{\mu\,F\sqrt{2g}}\left[\left(\sqrt{h_1} - \sqrt{h_2}\right) - \frac{a^2}{288}\left(\sqrt{h_1^{-3}} - \sqrt{h_2^{-3}}\right)\right],$$

and F and G represent the transverse sections of the opening and of the vessel, a the depth of the opening, h_1 the head of water at the commencement, h_2 that at the end of the efflux. If $h_1 = \frac{a}{2}$, the opening becomes a notch, and we must then apply the proper formula.

§ 349. *Wedge and Pyramidal-shaped Vessels.*—If the cistern of discharge ABF, Fig. 478, forms a horizontal triangular prism, the time of efflux may be found in the following manner. Let us divide the height $CE = h$ into n equal parts, and carry horizontal planes through the points of division; let us then decompose the whole quantity of water into equally thick strata of equal length $AD = l$, and of breadths diminishing downwards. If the breadth of the upper stratum $BD = b$, we have then the breadth of another stratum D_1B_1, which stands about $CE_1 = x$ above the orifice F, lying at the lower edge, $b_1 = \frac{x}{h}\,b$, and its volume $= b_1\,l \,.\, \frac{h}{n} = \frac{b\,l\,x}{n}$. But now the discharge referred to a unit of time is: $Q = \mu\,F\sqrt{2\,g\,x}$, hence then the small time in which the surface of water sinks about $\frac{h}{n}$ is $\tau = \frac{bl}{n}x \div \mu\,F\sqrt{2\,g\,x} = \frac{b\,l}{n\,\mu\,F\sqrt{2g}}\,.\,x$. Finally, since the sum of all the $x^{\frac{1}{2}}$ from $x = \frac{h}{n}$ to $x = \frac{nh}{n}$ are $= \left(\frac{h}{n}\right)^{\frac{1}{2}} \,.\, \frac{n^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\,n\,h^{\frac{1}{2}}$, we have the time for the discharge of the entire prism of water:

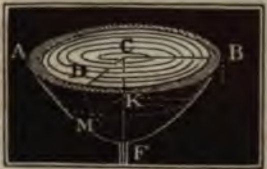
Fig. 478.



$$t = \frac{b\,l}{n\,\mu\,F\sqrt{2g}} \,.\, \frac{2}{3}\,n\,h^{\frac{1}{2}} = \frac{2}{3}\,\frac{b\,l}{\mu\,F\sqrt{2\,g}} \,.\, h^{\frac{1}{2}} = \frac{4}{3}\,\frac{\frac{1}{2}\,b\,l\,h}{\mu\,F\sqrt{2\,g\,h}}$$

$= \frac{4}{3} \,.\, \frac{V}{\mu\,F\,c}$, if V represents the whole quantity of water and c the initial velocity of efflux. Here the water, therefore, requires $\frac{1}{3}$ more time than if the velocity of efflux c were uniform.

Fig. 479.

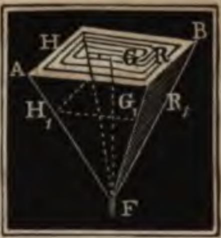


If the vessel ABF, Fig. 479, forms an erect paraboloid, we then have for the ratio of the radii $KM = y$ and $CD = b : \frac{y}{b} = \frac{\sqrt{x}}{\sqrt{h}}$, and hence the ratio of the principal sections $\frac{G_1}{G} = \frac{y^2}{b^2} = \frac{x}{h}$, consequently $G_1 = \frac{Gx}{h}$ and the contents of a stratum of water $= G_1 \cdot \frac{h}{n} = \frac{Gx}{n}$. The perfect accordance of this expression with that found for the triangular prism, admits of our here putting $t = \frac{4}{3} \cdot \frac{\frac{1}{2}\, Gh}{\mu F \sqrt{2\, gh}}$, or, as $V = \frac{1}{2}\, G\, h$ (§ 118), also

$$t = \tfrac{4}{3} \cdot \frac{V}{\mu F c}.$$

The formula may be used in many other cases for the approximative determination of the time of efflux, especially for that of the emptying of reservoirs. It is especially true in all cases where the horizontal sections increase as the distances from the bottom.

Fig. 480.



If, lastly, a vessel ABF be pyramidal, Fig. 480, then $G_1 : G = x^2 : h^2$, and hence $G_1 = \frac{Gx^2}{h^2}$ further the contents of the stratum $H_1\, R_1 : \frac{G_1 h}{n} = \frac{Gx^2}{nh}$, and the time for its discharge:

$$\tau = \frac{Gx^2}{nh} : \mu F \sqrt{2\, gx} = \frac{G}{n \mu F h \sqrt{2\, g}} \cdot x^{\frac{3}{2}}.$$

But as the sum of all the $x^{\frac{3}{2}}$ taken from $x = \frac{h}{n}$ to $x = \frac{nh}{n} = \left(\frac{h}{n}\right)^{\frac{3}{2}} \cdot \frac{n^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}\, n\, h^{\frac{3}{2}}$,

it follows that the time for the emptying of the whole pyramid is:

$$t = \frac{G}{n \mu F h \sqrt{2\, g}} \cdot \tfrac{2}{5}\, n\, h^{\frac{3}{2}} = \tfrac{2}{5} \cdot \frac{G h^{\frac{1}{2}}}{\mu F \sqrt{2 g}} = \tfrac{6}{5} \cdot \frac{\frac{1}{3}\, Gh}{\mu F \sqrt{2\, gh}},$$ /?

or if $\frac{1}{3}$ Gh be put $= V$, then will $t = \frac{6}{5} \cdot \frac{V}{\mu Fc}$.

As in this efflux the initial velocity of flow decreases gradually from c to zero, the time of efflux is then $\frac{1}{5}$th greater than if the velocity c remained uniform.

Example. In what time will a pond, whose surface has an area of 765000 square feet, empty itself, if there be a conduit 15 feet below the surface, and at the deepest place, which forms a channel 15 inches wide and 50 feet long? Theoretically, the time of efflux is $t = \frac{4}{3} \cdot \frac{V}{F\sqrt{2gh}} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{765000 \cdot 15}{\frac{\pi}{4} \cdot \left(\frac{5}{4}\right)^2 \cdot 8{,}02\sqrt{15}}$

$$= \frac{19584000}{\pi \cdot 8{,}02\sqrt{15}} = 200848 \text{ sec.}$$

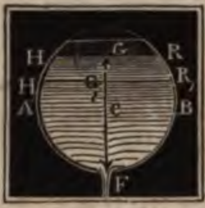
But now the co-efficient of resistance for entrance into the channel, inclined about 45°, is, $\zeta = 0{,}505 + 0{,}327$ (*see* § 323) $= 0{,}832$, and the resistance of the conduit due to friction $= 0{,}025\,\frac{l}{d} \cdot \frac{v^2}{2g} = 0{,}025 \cdot \frac{50}{\frac{5}{4}} \cdot \frac{v^2}{2g} = \frac{v^2}{2g}$; hence, the complete co-efficient of efflux for the channel is:

$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1}} = \frac{1}{\sqrt{2{,}832}} = 0{,}594$, and the time of efflux demanded:

$t = 200848 \div 0{,}594 = 338128 =$ 93 hours, 55 minutes, 28 seconds.

§ 350. *Spherical and Obelisk-shaped Vessels.*—By means of the formula of the last paragraph, we may now find the times of efflux for many other vessels, such as spherical, pontoon-shaped, pyramidal, &c. For the emptying of a spherical segment AB, Fig. 481, we obtain:

Fig. 481.



$$t = \frac{4}{3}\,\frac{\pi r h^2}{\mu F\sqrt{2gh}} - \frac{2}{5} \cdot \frac{\pi h^3}{\mu F\sqrt{2gh}}$$
$$= \frac{2}{15}\pi\,\frac{(10r - 3h)\,h^{\frac{3}{2}}}{\mu F\sqrt{2g}},$$

therefore, for the emptying of a full sphere, where $h = 2r$,

$$t = \frac{16\pi r^2\sqrt{2r}}{15\mu F\sqrt{2g}},$$

and for that of half a sphere where:

$$h = r\,t, = \frac{14\pi r^2\sqrt{r}}{15\mu F\sqrt{2g}}.$$

Here the horizontal stratum H_1R_1 corresponding to the depth FG_1 $= x = G_1 = \pi x(2r - x) \cdot \frac{h}{n} = \frac{2\pi r h x}{n} - \frac{\pi h x^2}{n}$, therefore:

$$\tau = \frac{2\pi r h}{n\mu F\sqrt{2g}} \cdot x^{\frac{1}{2}} - \frac{\pi h}{n\mu F\sqrt{2g}} \cdot x^{\frac{3}{2}};$$

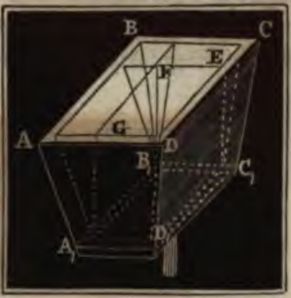
as the first part of this expression agrees with the formula for the emptying of a prismatic, and the second part for the emptying of a pyramidal vessel, if we put first $2\pi r h$ in place of bl, and secondly πh^2 in place of G, we shall obtain by means of the difference of the times of emptying of a prismatic and pyramidal vessel, found in the former paragraph:

$$t = \frac{2}{3} \cdot \frac{blh}{\mu F\sqrt{2gh}}, \text{ and } t = \frac{2}{5} \cdot \frac{Gh}{\mu F\sqrt{2gh}},$$

the time also of the emptying of a spherical segment.

The above formula may be likewise applied to the case of an obelisk or pontoon-shaped vessel ACD_1, Fig. 482, since this is composed of a parallelopiped, two prisms, and a pyramid. Let b be the

Fig. 482.



breadth at top AD, and b_1 the breadth $A_1 D_1$ at bottom, l the length at top AB, and l_1 the length at bottom A_1B_1, and lastly, h the height of the vessel, we have then for the area of the surface $AC: bl = b_1 l_1 + b_1(l - l_1) + l_1(b - b_1) + (l - l_1)(b - b_1)$, of which $b_1 l_1$ belongs to the parallelopiped A_1C_1EG, $b_1(l - l_1) + l_1(b - b_1)$ to the two prisms CFB_1C_1, and AFB_1A_1 and $(l - l_1)(b - b_1)$ to the pyramid BFB_1. But now the time of efflux for the parallelopiped, whose base is $b_1 l_1$, is $t_1 = \frac{2\, b_1 l_1 \sqrt{h}}{\mu F \sqrt{2\, g}}$; further, that for the two triangular prisms

$$t_2 = \tfrac{2}{3}\, \frac{[b_1(l - l_1) + l_1(b - b_1)]\sqrt{h}}{\mu F \sqrt{2g}}$$

and, lastly, for the pyramid:

$$t_3 = \tfrac{2}{5}\, \frac{(l - l_1)(b - b_1)\sqrt{h}}{\mu F \sqrt{2\, g}};$$

hence the time of discharge for the whole vessel is:

$$t = t_1 + t_2 + t_3$$

$$= [30\, b_1 l_1 + 10\, b_1 (l - l_1) + 10\, l_1 (b - b_1) + 6\,(l - l_1)(b - b_1)]\, \frac{\sqrt{h}}{15 \mu F \sqrt{2\, g}}$$

$$= [3\, bl + 8\, b_1 l_1 + 2\,(b l_1 + b_1 l)]\ \frac{2\sqrt{h}}{15\, \mu\, F\sqrt{2g}}.$$

If $\frac{b_1}{l_1} = \frac{b}{l}$, we have then a truncated pyramid to consider. Let the one base $bl = G$, and the other $b_1 l_1 = G_1$, we then obtain:

$$t = (3\, G + 8\, G_1 + 4\sqrt{G\, G_1})\ \frac{2\sqrt{h}}{15 \mu F \sqrt{2\, g}}.$$

It would be easy to show that this formula holds true also for every trilateral or multilateral pyramid.

Example. An obelisk-shaped water-cask is 5 feet long, and 3 feet broad at top, and at the depth of 4 feet, that is, at the level of a short horizontal discharge-tube, 1 inch in width, and 3 inches in length, it is 4 feet long and 2 feet broad, what time will be required for the water in the full cask to sink $2\frac{1}{2}$ feet? The time for emptying is, μ being taken $= 0{,}815$:

$$t = [8\,.\,4\,.\,2 + 3\,.\,5\,.\,3 + 2\,(3\,.\,4 + 5\,.\,2)]\, \frac{2\sqrt{4}}{15\,.\,0{,}815\,.\,\frac{\pi}{4}\,.\left(\frac{1}{12}\right)^2.\,8{,}02}$$

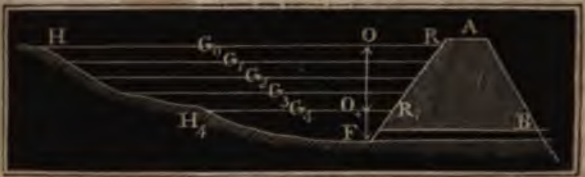
$= \frac{153\,.\,4\,.\,4\,.\,144}{15\,.\,0{,}815\,.\,8{,}02\,\pi} = 153\,.\,\frac{2304}{12{,}225\,.\,8{,}02\,\pi} = 153\,.\,7{,}481 = 1145$ sec. As the level $4 - 2\frac{1}{2} = 1\frac{1}{2}$ feet above the tube $l = l_1 + \frac{3}{8} = 4\frac{3}{8}$ and $b = b_1 + \frac{3}{8} = 2\frac{3}{8}$ feet, hence the time for emptying if the vessel be filled only up to this level, is:

$$t_1 = [8\,.\,4\,.\,2 + 3\,.\,\tfrac{35}{8}\,.\,\tfrac{19}{8} + 2\,(2\,.\,\tfrac{35}{8} + 4\,.\,\tfrac{19}{8})]\,.\,\frac{1152\sqrt{1{,}5}}{15\,.\,0{,}815\,.\,8{,}02\,\pi} = 131{,}672\,.$$

$4{,}5779 = 602{,}76$ sec. The difference of the times found gives the time in which the surface of water originally at the top of the vessel sinks $2\frac{1}{2}$ feet.

§ 351. *Irregular Vessels.*—When we have to find the time of efflux for an irregularly formed vessel *HFR*, Fig. 483, we must apply Simpson's rule as a method of approximation. If we divide the whole mass of water into four equally thick strata, and the heads of water G_0, G_1, G_2, G_3, G_4, corresponding to the horizontal slices, represented by h_0, h_1, h_2, h_3, h_4, the time of efflux will be given by Simpson's rule

Fig. 483.



$$t = \frac{h_0 - h_4}{12\,\mu\,F\sqrt{2g}}\left(\frac{G_0}{\sqrt{h_0}} + \frac{4G_1}{\sqrt{h_1}} + \frac{2G_2}{\sqrt{h_2}} + \frac{4G_3}{\sqrt{h_3}} + \frac{G_4}{\sqrt{h_4}}\right).$$

In assuming six strata:

$$t = \frac{h_0 - h_6}{18\mu F\sqrt{2g}}\left(\frac{G_0}{\sqrt{h_0}} + \frac{4G_1}{\sqrt{h_1}} + \frac{2G_2}{\sqrt{h_2}} + \frac{4G_3}{\sqrt{h_3}} + \frac{2G_4}{\sqrt{h_4}} + \frac{4G_5}{\sqrt{h_5}} + \frac{G_6}{\sqrt{h_6}}\right).$$

The discharge in the first case is:

$$Q = \frac{h_0 - h_4}{12}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + G_4),$$ in the second:

$$Q = \frac{h_0 - h_6}{18}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + 2\,G_4 + 4\,G_5 + G_6).$$

When the form and dimensions of the vessel of efflux are not known, we may then calculate very nearly the discharge by the heads of water noted in equal intervals of time. Let t be one such interval, we have then for apertures at the bottom and sides:

$$Q = \frac{\mu F t\sqrt{2g}}{3}\,(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}),$$

and for divisions or notches in a side:

$$Q = \tfrac{2}{9}\,\mu\,b\,t\sqrt{2g}\,(\sqrt{h^3_0} + 4\sqrt{h^3_1} + 2\sqrt{h^2_2} + 4\sqrt{h^3_3} + \sqrt{h^3_4}).$$

Example. In what time will the surface of water in a pond sink 6 feet, if the sluice forms a half cylinder, 18 inches wide, 9 inches deep, and 60 feet long, and the surfaces of water have the following areas?

G_0, at 20 feet head of water, $= 600000$ square feet.
G_1, " 18,5 " " $= 495000$ "
G_2, " 17,0 " " $= 410000$ "
G_3, " 15,5 " " $= 325000$ "
G_4, " 14,0 " " $= 265000$ "

$F = \frac{\pi}{8}\cdot(\frac{3}{2})^2 = \frac{9\pi}{32} = 0{,}8836$ square feet. Let the co-efficient of resistance for the entrance $= 0{,}832$, and that for the friction:

$= 0{,}025\,.\,\frac{l}{d} = 0{,}025\,.\,60\,.\,1{,}091 = 1{,}6356$, then is the co-efficient of efflux

$$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1{,}6365}} = \frac{1}{\sqrt{3{,}4685}} = 0{,}537,$$

and $\mu F \sqrt{2g} = 0{,}537 \,.\, 0{,}8836 \,.\, 8{,}02 = 3{,}8054$. Now

$$\frac{G_0}{\sqrt{h_0}} = \frac{600000}{\sqrt{20}} = 134170,\ \frac{G_1}{\sqrt{h_1}} = \frac{495000}{\sqrt{18{,}5}} = 115090,$$

$$\frac{G_2}{\sqrt{h_2}} = \frac{410000}{\sqrt{17}} = 99440,\ \frac{G_3}{\sqrt{h_3}} = \frac{325000}{\sqrt{15{,}5}} = 82550,$$

$\frac{G_4}{\sqrt{h_4}} = \frac{265000}{\sqrt{14}} = 70830$; hence, then, the time of efflux follows:

$$t = \frac{6}{12 \,.\, 3{,}8054}(134170 + 4 \,.\, 115090 + 2 \,.\, 99440 + 4 \,.\, 82550 + 70830)$$

$= \frac{1194440}{7{,}6108} = 156940$ sec., $=$ 43 hours, 35 min. 40 sec.

The discharge is:

$$Q = \tfrac{6}{12}(600000 + 4 \,.\, 495000 + 2 \,.\, 410000 + 4 \,.\, 325000 + 265000)$$

$= \frac{4965000}{2} = 2482500$ cubic feet.

§ 352. *Influx and Efflux.*—If the vessel during the efflux from below has an influx to it from above, the determination of the time in which the surface of water rises or falls a certain height becomes more complicated, so that we must be satisfied generally with but an approximate determination. If the discharge per second Q_1 is $> \mu F \sqrt{2gh}$, then there is a rise, and if $Q_1 < \mu F \sqrt{2gh}$, a fall of the surface. Moreover, a state of permanency occurs whenever the head of water is increased or decreased by $k = \frac{1}{2g}\left(\frac{Q_1}{\mu F}\right)^2$. The /? time τ, in which the variable head of water x increases by the small amount ξ, is given by the equation

$$G_1 \xi = Q_1 \tau - \mu F \sqrt{2gx} \,.\, \tau,$$

and, on the other hand, the time in which it sinks the height k, by /?

$$G_1 \xi = \mu F \sqrt{2gx} \,.\, \tau - Q_1 \tau.$$

Hence we have in the first case $\tau = \frac{G_1 \xi}{Q_1 - \mu F \sqrt{2gx}}$, and in the second $\tau = \frac{G_1 \xi}{\mu F \sqrt{2gx} - Q_1}$. By the application of Simpson's rule we then obtain the time of efflux, during which the lowering surface passes from G_0 to G_1, G_2 . . ., and the head of water from h_0 to h_1, h_2 . . .

$$t = \frac{h_0 - h_4}{12}\left[\frac{G_0}{\mu F\sqrt{2gh_0} - Q_1} + \frac{4\,G_1}{\mu F\sqrt{2gh_1} - Q_1} + \frac{2\,G_2}{\mu F\sqrt{2gh_2} - Q_1} + \right.$$
$$\left. + \frac{4\,G_3}{\mu F\sqrt{2gh_3} - Q_1} + \frac{2\,G_4}{\mu F\sqrt{2gh_4} - Q_1}\right],$$

or, more simply, if we represent $\frac{Q_1}{\mu F \sqrt{2g}}$ by $\sqrt{k}$,

$$= \frac{h_0 - h_4}{12 \mu F \sqrt{2g}}\left[\frac{G_0}{\sqrt{h_0} - \sqrt{k}} + \frac{4\,G_1}{\sqrt{h_1} - \sqrt{k}} + \frac{2\,G_2}{\sqrt{h_2} - \sqrt{k}} + \right.$$
$$\left. + \frac{4\,G_3}{\sqrt{h_3} - \sqrt{k}} + \frac{G_4}{\sqrt{h_4} - \sqrt{k}}\right].$$

If the vessel is prismatic, and has a uniform transverse section G, we then have:

$$t = \frac{2\,G}{\mu\,F\,\sqrt{2g}}\left(\sqrt{h} - \sqrt{h_1} + \sqrt{k}\,.\,hyp.\,log.\left(\frac{\sqrt{h} - \sqrt{k}}{\sqrt{h_1} - \sqrt{k}}\right)\right),$$

the time in which the head of water passes from h to h_1. Since for:

$$h_1 = k,\ \frac{\sqrt{h} - \sqrt{k}}{\sqrt{h_1} - \sqrt{h}} = \frac{\sqrt{h} - \sqrt{k}}{0} = \infty,$$

it follows that the condition of permanency takes place indefinitely late.

The following formula is the result of investigation for a wier or notch in a side.

$$t = \frac{G\,k}{3\,Q_1}\left[hyp.\,log.\,\frac{(\sqrt{h} - \sqrt{k})^2\,(h_1 + \sqrt{h_1 k} + k)}{(\sqrt{h_1} - \sqrt{k})^2\,(h + \sqrt{hk} + k)}\right.$$
$$\left. + \sqrt{12}\,.\,arc\left(tang. = \frac{(\sqrt{h} - \sqrt{h_1})\,\sqrt{12k}}{3\,k + (2\,\sqrt{h} + \sqrt{k})\,(2\,\sqrt{h_1} + \sqrt{k})}\right)\right],$$

where $k = \left(\frac{Q_1}{\frac{2}{3}\,\mu\,b\,\sqrt{2\,g}}\right)^{\frac{2}{3}}$, *hyp. log.* represents the hyperbolic logarithm, and *arc* (*tang.* $= y$) the arc whose tangent $= y$. According as k is $\lesseqgtr h$, and the inflowing quantity of water: $Q_1 \gtreqless \frac{2}{3}\,\mu\,b\,\sqrt{2\,g\,h^3}$, there is a rise or fall of the fluid surface. The condition of permanency occurs, when $h_1 = k$, and the time corresponding becomes ∞.

Example. In what time will the water in a rectangular tank 12 feet long and 6 feet broad rise from 0 to 2 feet above the edge of a notch $\frac{1}{2}$ foot broad, if 5 cubic feet of water flow in per second? We have here $h = 0$; hence, more simply:

$$t = \frac{G\,k}{3\,Q_1}\left[hyp.\,log.\,\frac{h_1 + \sqrt{h_1\,k} + k}{(\sqrt{h_1} - \sqrt{k})^2} + \sqrt{12}\ arc\left(tang. = \frac{-\sqrt{3\,h_1}}{2\,\sqrt{k} + \sqrt{h_1}}\right)\right].$$

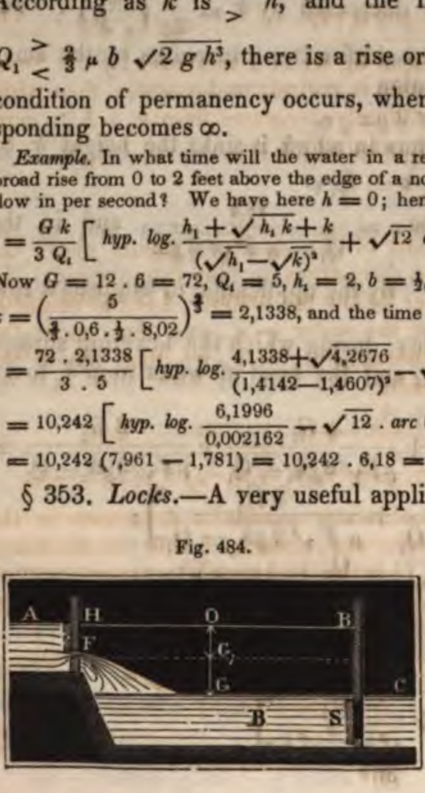
Now $G = 12\,.\,6 = 72$, $Q_1 = 5$, $h_1 = 2$, $b = \frac{1}{2}$, and $\mu = 0{,}6$,

$k = \left(\frac{5}{\frac{2}{3}\,.\,0{,}6\,.\,\frac{1}{2}\,.\,8{,}02}\right)^{\frac{2}{3}} = 2{,}1338$, and the time sought is:

$$t = \frac{72\,.\,2{,}1338}{3\,.\,5}\left[hyp.\,log.\,\frac{4{,}1338 + \sqrt{4{,}2676}}{(1{,}4142 - 1{,}4607)^2} - \sqrt{12}\,.\,arc\left(tang. = \frac{\sqrt{6}}{1{,}4142 + 2{,}9214}\right)\right]$$
$$= 10{,}242\left[hyp.\,log.\,\frac{6{,}1996}{0{,}002162} - \sqrt{12}\,.\,arc\left(tang. = \frac{\sqrt{6}}{4{,}3356}\right)\right]$$
$$= 10{,}242\,(7{,}961 - 1{,}781) = 10{,}242\,.\,6{,}18 = 63{,}29 \text{ sec.}$$

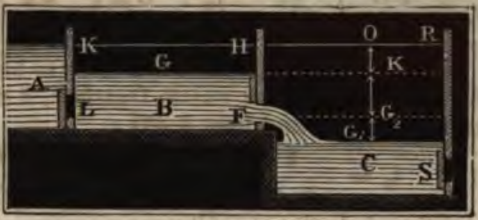
§ 353. *Locks.*—A very useful application of the doctrines hitherto treated of may be made to the filling and emptying of canal locks. We distinguish two kinds of navigation locks, single and double. The single lock, Fig. 484, consists of a chamber *B*, which is separated by the upper gate *HF* from the upper reach *A*, and by the

Fig. 484.



lower gate *RS* from the lower reach *C*. The double lock, Fig. 485, on the other hand, consists of two chambers, with the upper gate *KL*, the middle one *HF*, and the lower one *RS*.

Fig. 485.



Let the mean horizontal transverse section of a simple lock chamber $= G$, the distance of the middle of the sluice in the upper gate from the upper surface *HR* of the upper reach $= h_1$, and from that of the lower reach $= h_2$, and, lastly, the area of the aperture or sluice opening $= F$, we then obtain the time of filling up to the middle of the aperture $t_1 = \frac{G h_2}{\mu F \sqrt{2 g h_1}}$, and the time for filling the remaining space, where a gradual diminution of the head of water takes place, $t_2 = \frac{2\, G h_1}{\mu F \sqrt{2 g h_1}}$; consequently, the time for filling the single sluice is;

$$t = t_1 + t_2 = \frac{(h_2 + 2\, h_1)\, G}{\mu F \sqrt{2 g h_1}}.$$

If the aperture in the lower gate is entirely under water, then while emptying, the head of water gradually decreases from $h_1 + h_2$ to zero, hence the time for emptying or running off is:

$$t = \frac{2\, G \sqrt{h_1 + h_2}}{\mu F \sqrt{2 g}}.$$

If, on the other hand, a part of the aperture stands above the lower water, we then have two discharges to take into account; the one flowing above and the other below the water. Let the height of the part of the aperture above the water $= a_1$, and that under the water $= a_2$, the breadth of the aperture $= b$, we then obtain the time of efflux from the expression:

$$t = \frac{2\, G\, (h_1 + h_2)}{\mu\, b \sqrt{2g} \left(a_1 \sqrt{h_1 + h_2 - \frac{a_1}{2}} + a_2 \sqrt{h_1 + h_2} \right)}.$$

In double locks, the head of water gradually decreases in the chamber which is closed from the upper reach, during the discharge into the second chamber. If *G* is the horizontal transverse section of the first chamber, and the original head of water h_1 in this chamber sinks to x, whilst the water in the second chamber rises to the middle of the aperture of the sluice, we have then the corresponding time $t_1 = \frac{2\, G}{\mu F \sqrt{2g}} (\sqrt{h_1} - \sqrt{x})$. Now the quantity of water

$G\,(h_1 - x) = G_1 h_2$, hence $x = h_1 - \frac{G_1}{G}\,h_2$, and

$$t_1 = \frac{2\,G}{\mu F \sqrt{2g}}\left(\sqrt{h_1} - \sqrt{h_1 - \frac{G_1 h_2}{G}}\right) = \frac{2\sqrt{G}}{\mu F \sqrt{2g}}\left(\sqrt{G h_1} - \sqrt{G h_1 - G_1 h_2}\right).$$

The time in which the water rises as high in the second as in the first chamber, and in which, therefore, it comes to the same level in both, may be found from § 347:

$$t_2 = \frac{2\,G G_1 \sqrt{x}}{\mu F (G + G_1)\sqrt{2g}} = \frac{2\,G_1 \sqrt{G}\sqrt{G h_1 - G_1 h_2}}{\mu F (G + G_1)\sqrt{2g}},$$

and the whole time for filling:

$$t = t_1 + t_2 = \frac{2\sqrt{G}}{\mu F \sqrt{2g}}\left(\sqrt{G h_1} - \frac{G}{G + G_1}\sqrt{G h_1 - G_1 h_2}\right).$$

Example. What time is required for the filling and running off of the following single lock chamber? The mean length of the lock = 200 feet, mean breadth = 24 feet, therefore $G = 200 \,.\, 24 = 4800$ square feet, distance of the centre of the aperture of the sluice in the upper gate from the two surfaces of water 5 feet, breadth of both apertures $2\frac{1}{2}$ feet, height of the aperture in the upper gate 4 feet, and of that in the lower gate (entirely under water) 5 feet. Let $t = \frac{(2\,h_1 + h_2)\,G}{\mu\,F\sqrt{2\,g\,h_1}}$, $h_1 = 5$, $h_2 = 5$, $G = 4800$, $\mu = 0{,}615$, $F = 4 \,.\, 2\frac{1}{2} = 10$, $\sqrt{2g} = 8{,}02$, we then obtain the time of filling: $t = \frac{3 \,.\, 5 \,.\, 4800}{6{,}15 \,.\, 8{,}02\sqrt{5}} = \frac{14400}{1{,}23 \,.\, 8{,}02\sqrt{5}} = 652{,}85$ seconds. If we substitute in the formula $t = \frac{2\,G\sqrt{h_1 + h_2}}{\mu\,F\sqrt{2\,g}}$, $G = 4800$, $h_1 + h_2 = 10$, $F = 5 \,.\, 2\frac{1}{2} = 12{,}5$, we then obtain the time for emptying of the sluice:

$$t = \frac{2 \,.\, 4800\sqrt{10}}{0{,}615 \,.\, 12{,}5 \,.\, 8{,}02} = 491{,}78 \text{ sec.} = 8 \text{ min. } 21{,}78 \text{ sec.}$$

CHAPTER VI.

ON THE EFFLUX OF AIR FROM VESSELS AND TUBES.

§ 354. *Efflux of Still Air.*—Condensed air does not flow from vessels quite in accordance with the law which regulates the flow of water, because an expansion takes place during its discharge, which is not the case with water. But in order to discover a similar law for air and other gases, let us make the mechanical effect $Q\gamma\frac{v^2}{2g}$, which a quantity of air Q of the density γ requires to pass from a state of rest into that of the velocity v, equal to the mechanical effect $Q\,p$ *hyp. log.* $\left(\frac{p_1}{p}\right)$ found in § 298, which the same quantity of air produces when it passes from a greater pressure p_1 to a less p. If, therefore, p_1 be the elastic force of air enclosed in a vessel, v its velocity of efflux for the tension of the external air, and γ its density, then $Q\gamma \cdot \frac{v^2}{2g} = Qp \; hyp. \; log. \left(\frac{p_1}{p}\right)$, therefore, *the height due to the velocity:*

$$\frac{v^2}{2g} = \frac{p}{\gamma} \; hyp. \; log. \left(\frac{p_1}{p}\right) = 2{,}3026 \, \frac{p}{\gamma} \; log. \left(\frac{p_1}{p}\right);$$

and the velocity itself:

$$v = \sqrt{2g \frac{p}{\gamma} \; hyp. \; log. \left(\frac{p_1}{p}\right)}.$$

When the tensions p and p_1 differ little from each other, when $p_1 - p$ is $< \frac{1}{10} p$, then we may put:

$$hyp. \; log. \; \frac{p_1}{p} = hyp. \; log. \left(1 + \frac{p_1 - p}{p}\right) = \frac{p_1 - p}{p}, \text{ and hence}$$

$$v = \sqrt{2g\left(\frac{p_1 - p}{\gamma}\right)}.$$

But the height of an external column of air which is in equilibrium by its weight with the pressure $p_1 - p$ (§ 294), is $h = \frac{p_1 - p}{\gamma}$; hence we may put the velocity of efflux $v = \sqrt{2gh}$, and a perfect analogy with the efflux of water will hereby subsist. For high pressure this formula is not of course sufficient, for in this case:

$$hyp. \; log. \left(\frac{p_1}{p}\right) = \frac{p_1 - p}{p} - \tfrac{1}{2}\left(\frac{p_1 - p}{p}\right)^2 \text{ at least.}$$

Hence, then, more accurately:

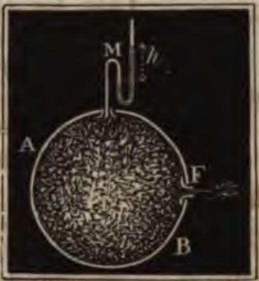
$$v = \sqrt{2g\left(\frac{p_1 - p}{\gamma} - \tfrac{1}{2}\frac{(p_1 - p)^2}{p\gamma}\right)} = \sqrt{2g\left(1 - \frac{p_1 - p}{2p}\right)h},$$

or if we represent the height of the barometer by b, $p = b\gamma$, and

$$v = \sqrt{2g\left(1 - \frac{h}{2b}\right)h} = \left(1 - \frac{h}{4b}\right)\sqrt{2gh}.$$

If the discharging orifice F of the vessel AB, Fig. 486, is accurately and smoothly rounded, the particles of air then flow in parallel lines, and hence the quantity of air flowing through the orifice in each second, and measured by the height of the external barometer, is:

Fig. 486.



$$Q = Fv = F\left(1 - \frac{h}{4b}\right)\sqrt{2gh},$$

or more accurately:

$$= F\sqrt{2g\,b \; hyp. \; log. \left(\frac{b + h}{b}\right)}.$$

§ 355. The above formulæ do not admit of direct application, because we cannot measure the internal or the external pressure by the length $b+h$, and b of the columns of air. These pressures are generally measured by columns of water or mercury. As regards the quotient $\frac{p_1}{p} = \frac{b+h}{b}$, it is immaterial whether b and h be expressed in columns of air, water, or mercury, because each reduction of b and h leaves the fraction $\frac{b+h}{b}$ constant, except that the quotient $\frac{p}{\gamma} = b$, is still dependent on the temperature of the effluent air, and varies for different kinds of gas. For atmospheric air (§ 301), if p represent the pressure of air on one square centimetre, and γ the weight of a cubic metre of air, and t the temperature in degrees centigrade, we have

$$\frac{p}{\gamma} = \frac{1+0{,}00367\,.\,t}{1{,}2572},$$ on the other hand, for steam,

$$\frac{p}{\gamma} = \frac{1+0{,}00367\,t}{0{,}7857}.$$

If we substitute these values in the general formula for v, we shall obtain for atmospheric air:

$$v = 395\sqrt{(1+0{,}00367\,.\,t)\ hyp.\ log.\left(\frac{b+h}{b}\right)}\ \text{metres},$$

or $\frac{h}{b}$ being small:

$$v = 395\sqrt{(1+0{,}00367\,.\,t)\frac{h}{b}}\ \text{metres, and for steam}$$

$$v = 500{,}6\sqrt{(1+0{,}00367\,.\,t)\ hyp.\ log.\left(\frac{b+h}{b}\right)}\ \text{metres.}$$

The theoretical discharge as estimated under the external pressure is $Q = Fv$, but if this is to be estimated at the internal pressure, we must then make $Q_1 p_1 = Q p$, hence $Q_1 = \frac{p}{p_1} Q = \frac{b\,Q}{b+h}$. Reduced to the temperature of zero, the quantity discharged is:

$$Q_2 = \frac{Q}{1+0{,}00367\,.\,t},\ \text{therefore, for atmospheric air}$$

$$= 395\,F\sqrt{\frac{hyp.\ log.\ (b+h) - hyp.\ log.\ b}{1+0{,}00367\,.\,t}}\ \text{cubic metres.}$$

If equal masses of air of different temperatures issue from different orifices F and F_1 at the same tension, we then have:

$$\frac{F_1}{F} = \sqrt{\frac{1+0{,}00367\,t_1}{1+0{,}00367\,t}}.$$

If, for example, $t = 0$ and $t_1 = 150°$ C., we then have:

$$F_1 = \sqrt{1{,}5505}\,.\,F = 1{,}245\,F.$$

If, therefore, a blast furnace is to be supplied with heated air of 150°, we must apply nozzle pipes, which have a one-fourth greater transverse section at the discharging orifice than if cold air were to be used.

For Prussian measure, and centigrade scale of temperature:

$$v = 1258 \,.\, \sqrt{(1 + 0{,}00367\, t)\; hyp.\; log. \left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v = 1595 \,.\, \sqrt{(1 + 0{,}00367\, t)\; hyp.\; log. \left(\frac{b+h}{b}\right)}.$$

For English measure, and Fahrenheit's scale of temperature:

$$v = 1295 \,.\, \sqrt{(1 + 0{,}00204\, t)\; hyp.\; log. \left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v = 1642 \,.\, \sqrt{(1 + 0{,}00204\, t)\; hyp.\; log. \left(\frac{b+h}{b}\right)}.$$

Example. In a large reservoir, air at 120° C., temperature is enclosed, which corresponds to the height of a mercurial manometer of 5 inches, whilst the external barometer stands at 27,2 inches; what quantity of air will flow from this through a round aperture 1½ inch wide? It is:

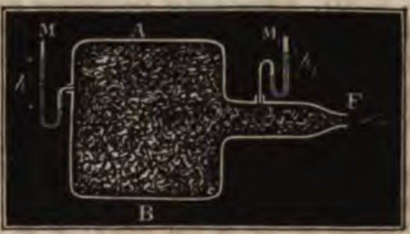
$hyp.\; log. \left(\frac{b+h}{b}\right) = hyp.\; log. \left(\frac{32{,}2}{27{,}2}\right) = hyp.\; log.\; 32{,}2 - hyp.\; log.\; 27{,}2 = 5{,}77455 - 5{,}60580 = 0{,}16875$, hence the velocity of efflux is:

$v = 1258 \,.\, \sqrt{(1 + 0{,}00367 \,.\, 120)\, 0{,}16875} = 1258 \,.\, \sqrt{1{,}4404 \,.\, 0{,}16875} = 620{,}2$ Prussian feet. Now the area of the orifice $= \frac{\pi}{4}\left(\frac{1}{8}\right)^2 = \frac{\pi}{256} = 0{,}01227$ square feet; hence it follows that the discharge $Q = 0{,}01227 \,.\, 620{,}2 = 7{,}61$ cubic feet. Estimated at the interior pressure, it is $= \frac{272}{322} \,.\, 7{,}61 = 6{,}43$ cubic feet, and reduced to the mean height of the barometer, 28 inches and 0° temperature (30 English inches and 32° temperature F.), the quantity discharged is:

$= 7{,}61 \,.\, \frac{272}{280} \,.\, \frac{1}{1{,}4404} = 5{,}13$ cubic feet.

§ 356. *Efflux of Air in Motion.*—The formula of efflux given: suppose the pressure p_1 or the height of the manometer h to be measured at a place where the air is at rest, or has a very slight motion, but if p_1 or h_1 is measured at a place where the air is in motion, if, for instance, the manometer M_1 communicates with the air in a conducting tube CF, Fig. 487, we shall then have to take into account the *vis viva* of the arriving air. If now c be the velocity of the air passing the orifice of the manometer we shall accordingly have to make:

Fig. 487.



$$Q\gamma \,.\, \frac{v^2}{2g} = Q\gamma \,.\, \frac{c^2}{2g} + Q\, p\; hyp.\; log. \left(\frac{p_1}{p}\right),$$

or if F be the transverse section of the orifice, and G that of the tube, or of the air passing the orifice of the manometer, according to the law of Mariotte, $\frac{Gc}{Fv} = \frac{p}{p_1}$, or $Gcp_1 = Fvp$, therefore,

$$c = \frac{F}{G} \,.\, \frac{p}{p_1}\, v,\; Q\gamma \left[1 - \left(\frac{F}{G}\right)^2 \left(\frac{p}{p_1}\right)^2\right] \frac{v^2}{2g} = Qp\; hyp.\; log. \left(\frac{p_1}{p}\right),$$

and the velocity of efflux in question:

$$v = \frac{\sqrt{2g\frac{p}{\gamma}\,hyp.\,log.\left(\frac{p_1}{p}\right)}}{\sqrt{1-\left(\frac{Fp}{Gp_1}\right)^2}}.$$

The velocity of efflux is, therefore, here exactly like that of water from vessels, the velocity is greater, the greater the ratio $\frac{F}{G}$ of the transverse section of the orifice to that of the tube or the arriving current of air. From this it is evident, that under otherwise similar circumstances, the height of the manometer p_1 is so much the less the narrower the conducting tube is, or the greater the velocity of the air issuing from it.

Examples.—1. A mercurial manometer, placed upon an air tube $3\frac{1}{2}$ inches wide, stands at $2\frac{1}{2}$ inches, while the air flows from its conical extremity through a round orifice 2 inches in diameter; with what velocity will the current move? If the external barometer stand at $27\frac{1}{2}$ inches, we shall then have $\frac{p_1}{p} = \frac{27\frac{1}{2} + 2\frac{1}{2}}{27\frac{1}{2}} = \frac{30}{27,5} = \frac{12}{11}$ and $\frac{F}{G}\frac{p}{p_1} = \left(\frac{2}{3,5}\right)^2 \cdot \frac{11}{12} = \frac{16\,.\,11}{49\,.\,12} = \frac{44}{147}$; hence the theoretical velocity of efflux at a temperature of the air 10° C.:

$$v = \frac{1258\,.\sqrt{1,0367\,.\,hyp.\,log.\,(\frac{12}{11})}}{\sqrt{1-(\frac{44}{147})^2}} = \frac{1258\sqrt{1,0367\,.\,0,087}}{\sqrt{0,9104}} = 396 \text{ Prussian feet.—}$$

2. The tension p_2 in the air regulator, where the air is without motion, is given by the formula,

$$hyp.\,log.\left(\frac{p_2}{p}\right) = \frac{v^2}{2g}\cdot\frac{\gamma}{p}, \text{ or } hyp.\,log.\,p_2 = hyp.\,log.\,p + \frac{hyp.\,log.\left(\frac{p_1}{p}\right)}{1-\left(\frac{Fp}{Gp_1}\right)^2}$$

therefore, in the present case, $= hyp.\,log.\,27,5 + \frac{0,087}{0,9104} = 3,3142 + 0,0965 = 3,4107$. Hence it follows that $p_2 = 30,3$ inches.

§ 357. *Efflux under Decreasing Pressure.*—If an air reservoir has no influx, whilst an uninterrupted efflux goes on, the density and tension gradually diminish, and hence the velocity of efflux becomes less and less. We may determine in the following manner in what ratio this diminution is to the time and to its discharge.

Let V be the volume of the reservoir, h_0 the initial height of the manometer, and h_n the height of the manometer at the end of a certain time t, b the height of the external barometer. Then the quantity of air in the reservoir at the commencement reduced to the external pressure $= \frac{V(b+h_0)}{b}$, and at the end of the time t, $= \frac{V(b+h_n)}{b}$, and, consequently, the quantity discharged in the time t, and at the external pressure is:

$$V_n = \frac{V(b+h_0)}{b} - \frac{V(b+h_n)}{b} = \frac{V(b_0-h_n)}{b};$$

and, inversely, the height of the manometer corresponding to the discharge V_n is:

$$h_n = h_0 - \frac{V_n}{V} . b.$$

If we take four intervals, and the initial height of the manometer h_0, and at the end of the time $t = h_4$, and

$$h_1 = h_0 - \frac{h_0 - h_4}{4},\ h_2 = h_0 - \tfrac{2}{4}(h_0 - h_4), \text{ and}$$

$h_3 = h_0 - \frac{3}{4}(h_0 - h_4)$, we shall then obtain by Simpson's rule the time

$$t = \frac{V(h_0 - h_4)}{12\, Fb \sqrt{2g \frac{p}{\gamma}}} \left(\frac{1}{\sqrt{hyp.\ log. \left(\frac{b+h_0}{b}\right)}} + \frac{4}{\sqrt{hyp.\ log. \left(\frac{b+h_1}{b}\right)}} + \frac{2}{\sqrt{hyp.\ log. \left(\frac{b+h_2}{b}\right)}} + \frac{4}{\sqrt{hyp.\ log. \left(\frac{b+h_3}{b}\right)}} + \frac{1}{\sqrt{hyp.\ log. \left(\frac{b+h_4}{b}\right)}} \right).$$

For moderate pressures or heights of the manometer:

$$hyp.\ log. \left(\frac{b+h}{b}\right) = \frac{h}{b}\left(1 - \frac{h}{2b}\right),$$

consequently $\sqrt{hyp.\ log. \left(\frac{b+h}{b}\right)} = \left(1 - \frac{h}{4b}\right)\sqrt{\frac{h}{b}}$ and

$$\frac{1}{\sqrt{hyp.\ log. \left(\frac{b+h}{b}\right)}} = \left(1 + \frac{h}{4b}\right)\sqrt{\frac{h}{b}}.$$

If we now take n intervals, and therefore the discharge for one interval: $\frac{V_1}{n} = \frac{V(h_0 - h_n)}{nb}$, we shall then obtain the corresponding element of time:

$$\tau = \frac{V(h_0 - h_n)}{nb} \div \mu F \sqrt{2g\frac{p}{\gamma}\, hyp.\ log. \left(\frac{b+h}{b}\right)}$$

$$= \frac{V(h_0 - h_n)}{nb} \frac{\left(1 + \frac{h}{4b}\right)\sqrt{\frac{b}{h}}}{F\sqrt{2g\frac{p}{\gamma}}}$$

$$= \frac{V(h_0 - h_n)\left(h^{-\frac{1}{2}} + \frac{h^{\frac{1}{2}}}{4b}\right)}{n . F\sqrt{2gb\frac{p}{\gamma}}}.$$

Now if we substitute for h; $h_0, h_1, h_2, \ldots h_n$, we shall then obtain the sum of all the

$$\left(\frac{h_0-h_n}{n}\right) h^{-\frac{1}{2}} = 2(h_0^{\frac{1}{2}}-h_n^{\frac{1}{2}}) = 2(\sqrt{h_0}-\sqrt{h_n}).$$

and the sum of all the

$$\left(\frac{h_0-h_n}{n}\right) h^{\frac{1}{2}} = \tfrac{2}{3}\left(h_0^{\frac{3}{2}}-h_n^{\frac{3}{2}}\right) = \tfrac{2}{3}(\sqrt{h_0^3}-\sqrt{h_n^3}),$$

whence the sum of all the small intervals of time, or the whole time in which h_n passes into h_0, and the quantity of air

$V_n = \frac{V(h_0-h_n)}{b}$ which flows out, is:

$$t = \frac{2\,V}{F\sqrt{2\,gb\frac{p}{\gamma}}}\left[(\sqrt{h_0}-\sqrt{h_n}) + \frac{1}{12b}(\sqrt[3]{h_0^3}-\sqrt[3]{h_n^3})\right], \text{ or}$$

$$= \frac{2\,V}{F\sqrt{2\,gb\frac{p}{\gamma}}}(\sqrt{h_0}-\sqrt{h_n})\left(1+\frac{h_0+\sqrt{h_0h_n}+h_n}{12\,b}\right),$$

approximately:

$$= \frac{2\,V}{F\sqrt{2\,gb\frac{p}{\gamma}}}(\sqrt{h_0}-\sqrt{h_n})\left(1+\frac{h_0+h_n}{8b}\right).$$

Example. A 50 feet long and 5 feet wide cylindrical air-regulator of a blowing machine is filled with air; the height of its manometer $h = 10$ inches, and the thermometer stands at 6° C. If now a flow of air takes place in a space where the height of the barometer is 27 inches, through a 1-inch wide circular orifice, then the question arises, in what time will the height of the manometer fall to 7 inches, and what will be the corresponding discharge? The volume of the chamber is for Prussian measures,—$= \frac{\pi}{4} \cdot 5^2 \cdot 50 = 1250 \cdot \frac{\pi}{4} = 981{,}75$ cubic feet, hence the discharge, measured at the external pressure, is $V_1 = \left(\frac{h_0-h_n}{b}\right) V = \left(\frac{10-7}{27}\right) \cdot 981{,}75 = 109{,}08$ cubic feet.

Now $\sqrt{2g\frac{p}{\gamma}} = 1258\sqrt{1+0{,}00367 \cdot t} = 1258\sqrt{1{,}02202} = 1272$, and

$F = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 = \frac{\pi}{576} = 0{,}005454$ square feet, hence the time of efflux in question is

$$t = \frac{2 \cdot 981{,}75}{0{,}005454 \cdot 1272}\left(\sqrt{\frac{10}{27}}-\sqrt{\frac{7}{27}}\right)\left(1+\frac{10+7}{8 \cdot 27}\right)$$

$$= \frac{1963{,}5}{5{,}454 \cdot 1{,}272} \cdot 0{,}0994 \cdot 1{,}079 = 30{,}3 \text{ seconds.}$$

§ 358. *Co-efficients of Efflux.*—The phenomena of contraction, which we have considered in the efflux of water from vessels, occur also in the efflux of air. If the orifice of efflux be cut in a thin plate, the air passing through it has a smaller transverse section than the orifice, and on this account the discharge is less than the product Fv of the transverse section F of the orifice and the theoretical velocity v. Let $\frac{F_1}{F}$ be the ratio of the transverse section F_1 of the blast to that of the orifice F, $= \mu$, we then have the effective discharge as for water:

$$Q_1 = \mu\, Q = F_1 v = \mu\, F v = \mu\, F \sqrt{2g \frac{p}{\gamma}\, hyp.\ log. \left(\frac{p_1}{p}\right)}.$$

From the author's reduction of Koch's experiments at pressures of the manometer of from $\frac{1}{200}$ to $\frac{1}{8}$ of an atmosphere, we may take the mean of $\mu = 0,58$.

The effective discharge in the issuing of air through short cylindrical adjutages, is likewise less than that determined theoretically; we have, therefore, to multiply this latter by a number deduced from experiment, the co-efficient of efflux, μ in order to obtain the former; only here μ is not the ratio of the transverse section $\frac{F_1}{F}$, but the ratio $\frac{v_1}{v}$ of the effective velocity of efflux v_1 to the theoretical v. Koch's experiments give for the above pressures, in the flow of air through cylindrical adjutages, which were nearly all six times as long as wide, as a mean $\mu = 0,74$.

Conically convergent adjutages, similar to the nozzles of bellows, give a still greater co-efficient of efflux; a tube of 6° lateral convergence in the experiments of Koch, gave when five times as long as wide, the mean co-efficient $\mu = 0,85$.

From this, therefore, the effective discharge for the flow of air through orifices in a thin plate, measured at the external pressure, is

$$Q_1 = 751,1\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0,00367\, t) \frac{b}{h}} \text{ cubic feet (Eng.),}$$

for efflux through short cylindrical adjutages:

$$Q_1 = 958,3\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0,00367\ t) \frac{h}{b}} \text{ cubic feet,}$$

and through conical adjutages of 6° convergence.

$$Q_1 = 1090,7\, F \left(1 - \frac{h}{4b}\right) \sqrt{(1 + 0,00367\ t) \frac{h}{b}} \text{ cubic feet.}[*]$$

---

[*] Experiments on the efflux of air have been undertaken by Young, Schmidt, Lagerhjelm, Koch, d'Aubuisson, Buff, and in later time, by Pecqueur, Saint-Venant, and Wantzel. For an account of the experiments of Young and Schmidt, we may refer to Gilbert's "Annalen," vol. 22, 1801, and vol. 6, 1820, and to Poggendorff's "Annalen," vol. 2, 1824; for those of Koch and Buff, to the "Studien des göttingschen Vereines bergmännischer Freunde," vol. 1, 1824; vol. 3, 1833; vol. 4, 1837, and vol. 5, 1838; also in Poggendorff's "Annalen," vol. 27, 1836, and vol. 40, 1837. The experiments of Lagerhjelm are described in the Swedish work, "Hydrauliska Försök of Lagerhjelm, Forselles och Kallstenius," 1 vol. Stockholm, 1818. D'Aubuisson's experiments are to be found in the "Annales des Mines," vol. 11, 1825; vol. 13, 1826; vol. 14, 1827, and likewise in his "Traité d' Hydraulique." The latest experiments instituted in France are reported in the "Polytechnischen Centralblatt," vol. 6, 1845. Most of these experiments were made with very narrow orifices, and, therefore, scarcely answer the purpose in practice. The experiments of d'Aubuisson and Koch deserve most consideration; and next to them, perhaps, those of Pecqueur; but the most extensive are those of Koch. The wished-for accordance is hardly to be met with in the results of all these experiments; the co-efficients of efflux found by d'Aubuisson vary considerably from those calculated by Koch. The grounds for my placing the most confidence in the co-efficients of Koch, are given in the "Allgemeinen Maschinenencyclopädie," under the article "Ausfluz," and in a Memoir of mine in Poggendorff's "Annalen," vol. 51, 1840. [For calculations of the above, and all similar cases, the co-efficient of t for the Fahrenheit's thermometer is 0,002039 instead of 0,00367; (see above, p. 346;) but the degrees computed are actually $t - 32$ on that scale.]—Am. Ed.

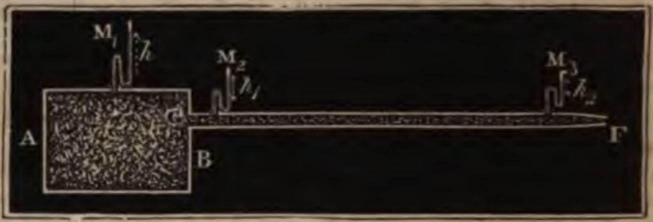
Example. If the two orifices of a bellows together possess an area of 3 square inches, if, further, the pressure of the manometer is 3 inches, the external barometer 27½ inches, and the temperature of the air 15°, then is the discharge:

$$Q_1 = 1069 \cdot \tfrac{3}{144} \left(1 - \frac{3}{4 \cdot 27,5}\right) \sqrt{(1 + 0,00367 \cdot 15)\,\frac{3}{27,5}}$$

$$= 22,27 \cdot \frac{107}{110} \sqrt{1,055 \cdot \tfrac{6}{55}} = 21,66 \sqrt{0,1151} = 7,34 \text{ cubic feet.}$$

§ 359. *Flow through Tubes.*—If the air issues through a long tube GF, Fig. 488, it has then the resistance of friction to overcome in the same manner as water; this resistance may also be measured by the height of a column of air, which has for expression

Fig. 488.



$h_n = \zeta \cdot \frac{l}{d} \cdot \frac{v^2}{2g}$, where, as in the conducting of water, v represents the velocity, l the length, d the width of the tube, and ζ a co-efficient of resistance to be determined by experiment.

Numerous experiments of Girard, d'Aubuisson, Buff and Pecqueur, lead to the mean value $\zeta = 0,024$. From this, therefore, the resistance generated by the friction of air in tubes may be measured by the height $h_n = 0,024\,\frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of air, or by the height $h_n = 0,0000023\,\frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of quicksilver, and the manometer will stand at this much less height at the end of the conducting tube than at the beginning.

If at the end of a conducting tube of the width d, the manometer stands at h_2, whilst the air flows through an orifice of the width d, then from what precedes, the velocity of discharge will be:

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\ hyp.\ log.\left(\frac{b + h_2}{b}\right)}}{\sqrt{1 - \left(\frac{b}{b + h_2}\right)^2 \left(\frac{d_1}{d}\right)^4}};$$

but if h_1 be the height of the manometer at the beginning of the conduit, we shall then have:

$$\frac{p}{\gamma}\ hyp.\ log.\left(\frac{b + h_1}{b}\right) = \left[1 \mp \left(\frac{b}{b + h_1}\right)^2 \left(\frac{d_1}{d}\right)^4 + 0,024\,\frac{l}{d}\left(\frac{d_1}{d}\right)^4\right] \frac{v^2}{2g},$$

because the velocity in the tube $= \frac{d_1^{\,2}}{d^2}\,v$; hence in this case

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\ hyp.\ log.\left(\frac{b + h_1}{b}\right)}}{\sqrt{1 + \left[0{,}024\frac{l}{d} \mp \left(\frac{b}{b + h_1}\right)^2\right]\left(\frac{d_1}{d}\right)^4}}.$$

If, lastly, the height of the manometer h is measured in the reservoir at the beginning of the conduit, where the air may be regarded as at rest, we then have:

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\ hyp.\ log.\left(\frac{b + h}{b}\right)}}{\sqrt{1 + 0{,}024\,\frac{l\,d_1^4}{d^5}}}.$$

If, further, we put the co-efficient of resistance ζ for entrance into the tube, which when $\mu_1 = 0{,}74$ amounts to 0,826, and, further, join to it the co-efficient of efflux μ for the outer adjutage, we then obtain for the velocity:

$$v = \frac{\mu\sqrt{2\,g\,\frac{p}{\gamma}\ hyp.\ log.\left(\frac{b + h}{b}\right)}}{\sqrt{1 + \zeta + 0{,}024\,\frac{l\,d_1^4}{d^5}}}$$

$$\text{or} = \frac{1294\,\mu\sqrt{(1 + 0{,}00367\,t)\ hyp.\ log.\left(\frac{b + h}{b}\right)}}{\sqrt{1 + \zeta + 0{,}024\,\frac{l\,d_1^4}{d^5}}}\ \text{feet (Pruss.)}$$

According as the point of the interior orifice lies s lower or higher than the point of the exterior orifice, we have to add $\pm s$ to the quantity under the radical in the denominator. Moreover, other hindrances may present themselves in the tube, such as curvatures, contractions, and widenings, &c. Satisfactory experiments on these obstacles do not exist, but we may assume with great probability that these resistances are not much different from what takes place in the case of water, because the co-efficients of efflux, and the co-efficient of friction are nearly the same for air as for water.

As long, therefore, as no further experiments are made on this subject, we may avail ourselves with tolerable safety of the co-efficient of resistance found for water in investigations on the motion and flow of air.

Example. In the regulator at the head of a 320 feet long and 4 inch wide air-conductor, the mercurial manometer stands at 3,1 inch, whilst the external barometer is at 27,2 inch; further, the width of the orifice of the conically contracted extremity of the conductor is 2 inches, and the temperature of the air 20° C., what quantity of air will this conductor deliver? It will be:

$1 + \zeta + 0{,}024\,\frac{l d_1^4}{d^5} = 1{,}826 + 0{,}024\,.\,\frac{320}{\frac{1}{3}}\,.\,(\frac{1}{2})^4 = 1{,}826 + 0{,}024\,.\,\frac{320\,.\,3}{16} = 1{,}826 + 1{,}44 = 3{,}266$; further, $(1 + 0{,}00367\,t)\ hyp.\ log.\left(\frac{b+h}{b}\right)$

$= (1 + 0,00367 \,.\, 20)\, hyp.log. \left(\frac{30,3}{27,2}\right) = 1,0734 \,.\, (5,7137 - 5,6058)$

$= 1,0734 \,.\, 0,1079 = 0,1158$; if now, further, we introduce the co-efficient of efflux, $\mu = 0,85$, we shall then obtain the velocity of flow:

$v = \frac{1258 \,.\, 0,85 \sqrt{0,1158}}{\sqrt{3,266}} = 201,3$ feet; and lastly, the discharge:

$Q = \frac{\pi d_1^2}{4} \,.\, v = \frac{\pi}{4} \cdot \frac{201,3}{36} = 4,39$ cubic feet (Prussian).

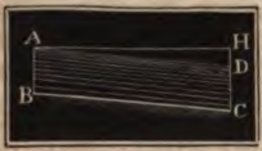
CHAPTER VII.

ON THE MOTION OF WATER IN CANALS AND RIVERS.

§ 360. *Running Water.*—The doctrine of the motion of water in canals and rivers, forms the second main division of hydraulics. Water flows either in a natural or in an artificial bed. In the first case, it forms streams, rivers, brooks; in the second, canals, cuts, drains, &c. In the theory of the motion of flowing water, this distinction is of little moment.

The bed of a river consists of the *bottom* and the two *banks* or *shores.* The *transverse section* is obtained by a plane at right angles to the direction of motion of the flowing water. Its *perimeter* is that of the transverse section, which again consists of the *air* and the *water section.* A vertical plane in the direction of the flowing water gives the *longitudinal section* or *profile.* By the *slope* or *declivity* of flowing water is understood the angle of inclination of its surface to the horizon. The *fall*, which is the vertical distance of the two extreme points of a definite length of the fluid surface, serves to assign the angle for a definite length of the flowing stream. For the length of course, $AD = l$, Fig. 489, BC is the bottom of the channel, $DH = h$ the fall, and the angle $DAH = \delta$, the slope $sin.\ \delta = \frac{h}{l} =$ absolute fall per unit of length.*

Fig. 489.



§ 361. *Different Velocities in the Transverse Section.*—The velocity of water in one and the same transverse section is very different at

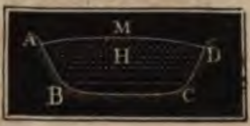
---

[*] The fall of brooks and rivers is very various. The Elbe, for example, for the extent of a German mile from the Upper Elbe to Podiebrad, has a fall of 57 feet, from thence to Leitmeritz 9 feet, from there to Mühlberg a mean of 5,8, and from thence to Magdeburg 2,5 feet. Mountain brooks have a fall of from 40 to 400 feet per German mile. For further particulars, *see* "Vergleichende hydrographische Tabellen," &c., von Stranz. Canals and other artificial water conduits have much smaller falls. Here the absolute fall, at most, is 0,001, often 0,0001, and even less. More on this subject will be given in the Second Part.

different points. The adhesion of the water to its bed, and the cohesion of the particles to each other, cause those lying nearer to the sides of the bed to suffer some constraint in their motion, and hence, to flow more slowly than the more remote. For this reason the velocity diminishes from the surface downwards to the bed, and is least near the side or at the bottom. The greatest velocity is found for straight rivers, generally in the middle, or at that part of the free surface of the water where there is the greatest depth. The place where the water attains its maximum velocity is called the *line of current*, and the deepest part of the bed, the *mid-channel*.

The upper surface does not form an exact horizontal line, because the elements lying on the surface of water, flow on with different velocities with respect to each other, they therefore exert on each other different pressures; the quicker ones a less, and the slower a greater pressure, and thus for the maintenance of relative equilibrium, the quicker elements superpose themselves on the slower. If v and v_1 are the velocities of two elements M and A, Fig. 490, then according to the doctrine of hydraulic pressure (§ 307) the difference of level of the two elements is:

Fig. 490.



$$MH = h = \frac{v^2}{2g} - \frac{v_1^2}{2g} = \frac{v^2 - v_1^2}{2g}.$$

This difference of level is always very small. If, for example, $v_1 = 0{,}9\, v$, and $v = 5$ feet, we then have this

$$= (1 - 0{,}81)\,\frac{v^2}{2g} = 0{,}19 \,.\, 0{,}0155 \,.\, 25 = 0{,}0736 \text{ feet} = 0{,}88 \text{ inches}$$

(Eng.). For this reason the water stands highest in the current, and lowest at the banks.

In bends, the current is generally near the concave bank.

§ 362. *Permanent Motion of Water.*—The mean velocity of water in a transverse section is, according to § 308:

$$c = \frac{Q}{F} = \frac{\text{quantity of water per second}}{\text{area of section}}.$$

The mean velocity besides may be further calculated from the velocities c_1, c_2, c_3, &c., of the separate portions of the section, and from the areas F_1, F_2, F_3, &c. It is namely:

$$Q = F_1 c_1 + F_2 c_2 + F_3 c_3 + \dots,$$

and hence also:

$$c = \frac{F_1 c_1 + F_2 c_2 + \dots}{F_1 + F_2 + \dots}.$$

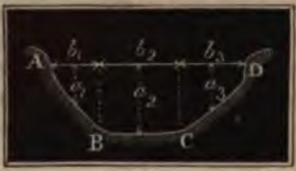
Besides the mean velocity, the mean depth of water has to be introduced, that is, the depth a which a section must have at all points that it may have the same area as it actually has with the variable depths a_1, a_2, a_3, &c. Hence, therefore,

$$a = \frac{F}{b} = \frac{\text{area of section}}{\text{breadth of section}}.$$

If the separate parts of the breadth b_1, b_2, b_3, have the corresponding mean depths a_1, a_2, a_3, &c., Fig. 491, we then have:

$$F = a_1 b_1 + a_2 b_2 + \ldots,$$

Fig. 491.



and hence also:

$$a = \frac{a_1 b_1 + a_2 b_2 + \ldots}{b_1 + b_2 + \ldots}.$$

Lastly:

$$c = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \ldots}{a_1 b_1 + a_2 b_2 + \ldots},$$

and if the portions b_1, b_2, &c., be of equal size,

$$c = \frac{a_1 c_1 + a_2 c_2 + \ldots}{a_1 + a_2 + \ldots}.$$

A river or brook is in a state of *permanency* when an equal quantity of water flows through each of its transverse sections in an equal time; when, therefore, Q or the product Fc of the area of the section and the mean velocity throughout the whole extent of the stream is a constant number. Hence this simple law comes out: *in the permanent motion of water, the mean velocities in two transverse sections are to each other inversely as the areas of these sections.*

Examples.—1. At the section of a canal, *ABCD*, Fig. 491, it was found that the

Portions of the breadth - - -	$b_1 = 3{,}1$ feet,	$b_2 = 5{,}4$ feet,	$b_3 = 4{,}3$ feet
Mean depth - - - - -	$a_1 = 2{,}5$ "	$a_2 = 4{,}5$ "	$a_3 = 3{,}0$ "
Corresponding mean velocities - -	$c_1 = 2{,}9$ "	$c_2 = 3{,}7$ "	$c_3 = 3{,}2$ "

Hence the area of these profiles $F = 3{,}1 \,.\, 2{,}5 + 5{,}4 \,.\, 4{,}5 + 4{,}3 \,.\, 3{,}0 = 44{,}95$ square feet, and the discharge:

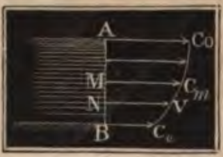
$Q = 3{,}1 \,.\, 2{,}5 \,.\, 2{,}9 + 5{,}4 \,.\, 4{,}5 \,.\, 3{,}7 + 4{,}3 \,.\, 3{,}0 \,.\, 3{,}2 = 153{,}665$ cubic feet, and the mean velocity $c = \frac{Q}{F} = \frac{153{,}665}{44{,}95} = 3{,}419$ feet.

2. When a cut is to conduct 4,5 cubic feet of water with a mean velocity c of 2 feet, we must then give to it a transverse section of $\frac{4{,}5}{2} = 2{,}25$ square foot area.—3. If one and the same stream has a mean velocity of $2\frac{1}{4}$ feet at a place 560 feet broad and 9 feet mean depth, it will then have, at a place 320 feet broad and 7,5 feet mean depth, the mean velocity

$$c = \frac{560 \,.\, 9}{320 \,.\, 7{,}5} \,.\, 2{,}25 = \frac{567}{120} = 4{,}725 \text{ feet.}$$

§ 363. *Mean Velocity.*—If we divide the depths of water at any point of a flowing stream into equal parts, and raise ordinates upon them corresponding to the velocities, we shall then obtain a scale of the velocity of the current *AB*, Fig. 492. Although it may be granted that the law of this scale, or of the difference of velocity is expressed by some curve, as according to Gerstner by an ellipse, yet it is allowable, without fear of any great error, to substitute for this a straight line, or assume that the velocity diminishes uniformly with the depth, because the diminution of velocity downwards is always very small. From the experiments of Ximenes, Brünnings, and Funk, the mean velocity in a perpendicular $c_m = 0{,}915\, c_0$, where c_0 represents

Fig. 492.



sents the velocity at the surface, or the maximum velocity. The velocity, therefore, diminishes from the surface to the middle M

$$\text{by } c_0 - c_m = (1 - 0{,}915)\, c_0 = 0{,}085\, c_0,$$

and, consequently, the velocity below or at the foot of the perpendicular may be put

$$c_u = c_0 - 2 \,.\, 0{,}085\, c_0 = (1 - 0{,}170)\, c_0 = 0{,}83\, c_0.$$

If, now, the whole depth $= a$, we then have, by assuming a straight line for the scale of the velocities, the corresponding velocity for a depth $AN = x$, below the water

$$v = c_0 - (c_0 - c_u)\frac{x}{a} = \left(1 - 0{,}17\,\frac{x}{a}\right) c_0.$$

Further, let $c_0, c_1, c_2, \ldots$ be the superficial velocities of a whole transverse profile of not very variable depth, we have then the corresponding velocities at a mean depth: $0{,}915\, c_0$, $0{,}915\, c_1$, $0{,}915\, c_2$, and hence the mean velocity in the whole profile:

$$c = 0{,}915\,\frac{(c_0 + c_1 + c_2 + \ldots c_n)}{n}.$$

Lastly, if we assume that the velocity diminishes from the line of current towards the banks, as it does according to the depth, we may then again put the mean superficial velocity

$$\frac{(c_0 + c_1 + \ldots + c_n)}{n} = 0{,}915\, c_0,$$

and so obtain the mean velocity in the whole profile:

$$c = 0{,}915 \,.\, 0{,}915 \,.\, c_0 = 0{,}837 \,.\, c_0,$$

i. e. from 83 to 84 per cent. of the maximum velocity, or of that of the line of current.

Prony deduced from Du Buat's experiments conducted with very small channels, and for these cases perhaps more correctly:

$$c_m = \left(\frac{2{,}372 + c_0}{3{,}153 + c_0}\right) c_0 \text{ metre} = \left(\frac{7{,}71 + c_0}{10{,}25 + c_0}\right) c_0 \text{ feet English.}$$

For medium velocities of 3 feet it hence follows that $c_m = 0{,}81\, c_0$.

Example. In the line of current of a brook the velocity of the water is 4 feet, and the depth 6 feet, we have then the mean velocity at a corresponding perpendicular $c_m = 0{,}915 \,.\, 4 = 3{,}66$ feet, and that at the bottom $= 0{,}83 \,.\, 4 = 3{,}32$ feet; further, the velocity 2 feet below the surface is $v = (1 - 0{,}17 \,.\, \frac{2}{6})\, 4 = (1 - 0{,}057)\, 4 = 3{,}772$ feet; lastly, the mean velocity throughout the profile is, $c = 0{,}837 \,.\, 4 = 3{,}348$ feet, and according to Prony, $c = \frac{11{,}50}{13{,}97} \,.\, 4 = \frac{46}{13{,}97} = 3{,}29$ feet.[*]

§ 364. *The Best Form of Transverse Sections.*—The resistance which the bed opposes to the motion of the water in virtue of its adhesion, viscosity, or friction, increases with the surface of contact between the bed and the water, and therefore with the perimeter p of the water profile, or of the portion of the transverse section which comprises the bed. But as more filaments of water pass through a profile, the greater its area is, so this resistance of a filament increases also inversely as the area, and hence on the whole as the quotient $\frac{p}{F}$ of the perimeter of the water profile, and the area of the whole transverse profile.

---

[*] This and the following subjects have been fully treated of under the article "Bewegung des Wassers," in the "Allgemeinen Maschinenencyclopädie." New experiments and new views may be found in the following writings: Lahmeyer's "Erfahrungsresultate über die Bewegung des Wassers in Fluszbetten und Kanälen." Brunswick, 1845.

That the resistance of friction of a running stream or river may be the smallest possible, we must give to its transverse section that form for which the perimeter p for a given area is a minimum, or the area for a given perimeter a maximum. In enclosed conduits, as, for example, pipes, p is the entire perimeter of the figure formed by the transverse profile. Now of all figures having an equal number of sides, the regular figure, and again, of all regular figures that which has the greater number of sides, has for the same area the least perimeter; hence for enclosed conduits, the co-efficient of friction comes out the less, the nearer its transverse profile approaches to a regular figure, and the greater its number of sides; and the circle, which is a regular figure of an infinite number of sides, is in this case the profile which corresponds to the minimum of friction. We must, therefore, in estimating this resistance of friction, leave out of our consideration in the quotient $\frac{p}{F}$ the upper side or surface in contact with the air. The rectangular and trapezoidal sections are those generally applied to canals, cuts, water-courses, &c. A horizontal line EF, Fig. 493, passing through the centre M of the square AC, divides as well the area as also the perimeter into two equal parts, hence it follows that what is true for the square is also correct for these halves, and, accordingly, of all rectangular transverse profiles, the half square AE, or that which is twice as broad as it is deep, corresponds to the least resistance of friction. The regular hexagon ACE, Fig. 494, may be likewise divided by a horizontal line CF into two equal trapeziums, each of which, like the entire hexagon, has the greatest relative area, and, consequently, of all trapezoidal profiles, half the regular hexagon or the trapezium $ABCF$ with the angle of slope $AFM = BCM$ of 60° is that which, when applied, gives the least resistance of friction. Half the regular octagon ADE, Fig. 495, half the regular decagon, and, lastly, the semi-circle ADB, Fig. 496, afford under given circumstances

Fig. 493.

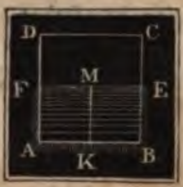


Fig. 494.

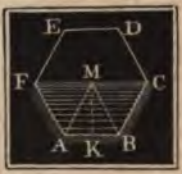


Fig. 495.

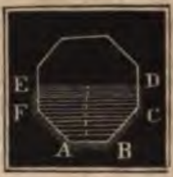
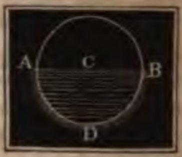


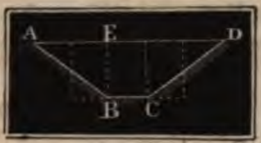
Fig. 496.



the most advantageous transverse profiles for canals. The trapezoidal, or half the regular hexagon, gives a still less resistance than half the square or rectangle, the ratio of whose sides is 1 to 2, because the hexagon has a less relative perimeter than the square. Half the regular decagon gives a still less friction, and, in general, the minimum of friction corresponds to the semi-circle. The profiles of channels of wood, stone or iron only, are made semi-circular and rectangular; the profiles of canals, on the other hand, which are cut and bricked, are constructed of the trapezoidal figure. Other figures, in consequence of difficulties in the execution, are not easily applicable.

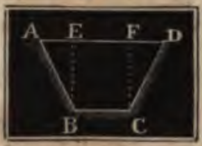
§ 365. In the case where a canal is not walled up, but dug out of loose earth or sand, the angle of 60° slope is too great, and the relative slope *cotg.* 60° = 0,57735 too small, because the banks would not have a sufficient stability; we are, therefore, under the necessity of applying the trapezoidal profile, for which the inclination of the sides to the base must be still less than 60°, perhaps scarcely 45°, or even less. For a trapezoidal profile $ABCD$, Fig. 497, which has a perimeter and area equal to that of half the square, the relative slope = $\frac{4}{3}$, and the angle of slope hardly 36° 52′. If the height BE be divided into three equal parts, the base BC will then have two of them, the parallel line AD ten, and each of the sides $AB = CD$ = five parts. In many cases the slope is made = 2, to which belongs an angle of 26° 34′, and sometimes it is even made still greater.

Fig. 497.



In every case the angle of slope $BAE = \theta$, Fig. 498, or the slope $n = \frac{AE}{BE} =$ *cotang.* θ may be regarded as a given quantity dependent on the nature of the ground in which the canal is dug, and hence the dimensions of the profile which offers the least resistance have only further to be determined. Let the lower breadth $BC = b$, the depth $BE = a$, and the slope $= n$, we then obtain for the perimeter:

Fig. 498.



$$AB + BC + CD = p = b + 2\sqrt{a^2 + n^2a^2} = b + 2a\sqrt{1 + n^2},$$

for the area:

$$F = ab + naa = a(b + na),$$

and hence, inversely, $b = \frac{F}{a} - na$, and the ratio:

$$\frac{p}{F} = \frac{1}{a} + \frac{a}{F}(2\sqrt{n^2 + 1} - n).$$

If we substitute for a, $a + x$, where x is a small number, we may then put:

$$\frac{p}{F} = \frac{1}{a+x} + \frac{(a+x)}{F}(2\sqrt{n^2 + 1} - n)$$

$$= \frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2}\right) + \frac{a+x}{F}(2\sqrt{n^2+1} - n)$$

$$= \frac{1}{a} + \frac{a}{F}(2\sqrt{n^2+1} - n) + \left(\frac{2\sqrt{n^2+1-n}}{F} - \frac{1}{a^2}\right)x + \frac{x^2}{a^3}.$$

Now that this value may be greater not only for a positive, but also for a negative value of x, than the first

$$\frac{1}{a} + \frac{a}{F}(2\sqrt{n^2+1} - n),$$

it is necessary that the member with the factor x should vanish, and therefore that $\frac{p}{F}$ may become a minimum, we must have

$$\frac{2\sqrt{n^2+1}-n}{F} - \frac{1}{a^2} = 0, \text{ i. e. } a^2 = \frac{F}{2\sqrt{n^2+1}-n},$$

or since:

$$n = cotang.\, \theta \text{ and } \sqrt{n^2+1} = \frac{1}{sin.\, \theta},\ a^2 = \frac{F\, sin.\, \theta}{2 - cos.\, \theta}.$$

Hence, therefore, the most appropriate form of profile corresponding to a given angle of slope θ, and a given area is determined by

$$a = \sqrt{\frac{F\, sin.\, \theta}{2 - cos.\, \theta}} \text{ and } b = \frac{F}{a} - a\, cotang.\, \theta.$$

Example. What dimensions must be given to the transverse profile of a canal, whose banks are to have 40° slope, and which is to conduct a quantity of water Q of 75 cubic feet, with a mean velocity of 3 feet? $F = \frac{Q}{c} = \frac{75}{3} = 25$ square feet, hence the depth $a = \sqrt{\frac{25\, sin.\, 40^\circ}{2 - cos.\, 40^\circ}} = 5\sqrt{\frac{0,64279}{1,23396}} = 3,609$ feet, the lower breadth $b = \frac{25}{3,609} - 3,609\, cotang.\, 40^\circ = 6,927 - 4,301 = 2,626$ feet the slope or cut of the banks $= 3,609 . cotang.\, 40^\circ = 4,301$, the upper breadth $= 6,927 + 4,301 = 11,228$ feet, the perimeter $p = b + \frac{2a}{sin.\, \theta} = 2,626 + \frac{7,218}{sin.\, 40^\circ} = 13,855$ feet, and the ratio determining the friction $\frac{p}{F} = \frac{13,855}{25} = 0,5542.$

§ 366. The dimensions of the most suitable profiles which correspond to different angles of slope and to a given profile are to be found in the following table.

Angle of slope. θ	Relative slope.	Dimensions of transverse profile.				Quotient
		Depth a.	Lower breadth b.	Absolute slope na.	Upper breadth $b + 2na$.	$\frac{p}{F}$
90°	0	$0{,}707 \sqrt{F}$	$1{,}414 \sqrt{F}$	0	$1{,}414 \sqrt{F}$	$\frac{2{,}828}{\sqrt{F}}$
60°	0,577	$0{,}760 \sqrt{F}$	$0{,}877 \sqrt{F}$	$0{,}439 \sqrt{F}$	$1{,}755 \sqrt{F}$	$\frac{2{,}632}{\sqrt{F}}$
45°	1,000	$0{,}740 \sqrt{F}$	$0{,}613 \sqrt{F}$	$0{,}740 \sqrt{F}$	$2{,}092 \sqrt{F}$	$\frac{2{,}704}{\sqrt{F}}$
40°	1,192	$0{,}722 \sqrt{F}$	$0{,}525 \sqrt{F}$	$0{,}860 \sqrt{F}$	$2{,}246 \sqrt{F}$	$\frac{2{,}771}{\sqrt{F}}$
36° 52′	1,333	$0{,}707 \sqrt{F}$	$0{,}471 \sqrt{F}$	$0{,}943 \sqrt{F}$	$2{,}357 \sqrt{F}$	$\frac{2{,}828}{\sqrt{F}}$
35°	1,402	$0{,}697 \sqrt{F}$	$0{,}439 \sqrt{F}$	$0{,}995 \sqrt{F}$	$2{,}430 \sqrt{F}$	$\frac{2{,}870}{\sqrt{F}}$
30°	1,732	$0{,}664 \sqrt{F}$	$0{,}356 \sqrt{F}$	$1{,}150 \sqrt{F}$	$2{,}656 \sqrt{F}$	$\frac{3{,}012}{\sqrt{F}}$
26° 34′	2,000	$0{,}636 \sqrt{F}$	$0{,}300 \sqrt{F}$	$1{,}272 \sqrt{F}$	$2{,}844 \sqrt{F}$	$\frac{3{,}144}{\sqrt{F}}$
Semicircle		$0{,}798 \sqrt{F}$			$1{,}596 \sqrt{F}$	$\frac{2{,}507}{\sqrt{F}}$

We see from this table that the quotient $\frac{p}{F}$ is least for the semicircle, namely, $= \frac{2{,}507}{\sqrt{F}}$; greater for the semi-hexagon, and greater still for the half square, and the trapezium of 36° 52′, &c.

Example. What dimensions must be given to a profile, which has for an area of 40 square feet, a slope of its banks of 35°? From the preceding table, the depth $a = 0{,}697 \sqrt{40} = 4{,}408$, the lower breadth $= 0{,}439 \sqrt{40} = 2{,}777$ feet, the absolute slope $= 0{,}995 \sqrt{40} = 6{,}293$ feet, the upper breadth $= 15{,}363$, and the quotient

$$\frac{p}{F} = \frac{2{,}870}{\sqrt{40}} = 0{,}4538.$$

§ 367. *Uniform Motion.*—The motion of water in beds is for a certain tract either *uniform* or *variable;* it is uniform when the mean velocity at all transverse sections of this length remains the same, and therefore, also, the areas of the sections equal; and variable, on the other hand, when the mean velocities, and therefore, also, the areas of the sections vary. We shall treat first of uniform motion.

In the uniform motion of water along the distance $AD = l$, Fig. 489, the whole fall $HD = h$ is expended in overcoming the friction of the water in the bed, because the water flows on with the same velocity with which it arrives, therefore, a height due to a velocity is neither taken up nor set free. If we measure this friction by the height of this column of water, we may then make the fall equal to this height. But the height due to the resistance of friction increases with the quotient $\frac{p}{F}$, with l and with the square of the mean velocity c (§ 329); hence then the formula holds good:

$$1.\ h = \zeta \cdot \frac{lp}{F} \cdot \frac{c^2}{2g},$$

in which ζ expresses a number deduced from experiment which may be called the *co-efficient of the resistance of friction.*

By inversion it follows:

$$2.\ c = \sqrt{\frac{F}{\zeta \cdot lp} \cdot 2gh}.$$

In determining the fall, therefore, when the length, the cross section and the velocity are given, and inversely, in deducing the velocity from the fall, the length and the cross section, we must know the co-efficient of friction ζ. According to Eytelwein's reduction of the ninety-one observations of Du Buat, Brünings, Funk and Waltmann, $\zeta = 0{,}007565$, and hence

$$h = 0{,}007565 \cdot \frac{lp}{F} \cdot \frac{c^2}{2g}.$$

If we put $g = 9{,}809$ metres or $31{,}25$ feet ($32{,}2$ feet English), we have for the metrical measure

$$h = 0{,}0003856\,\frac{lp}{F} \cdot c^2 \text{ and } c = 5{,}09\,\sqrt{\frac{Fh}{pl}},$$

and for the foot measure:

$$h = 0{,}00011726\,\frac{lp}{F} \cdot c^2 \text{ and } c = 92{,}35\,\sqrt{\frac{Fh}{pl}} \text{ English measure.}$$

For conduit pipes $\frac{lp}{F} = \frac{\pi\, l\, d}{\frac{1}{4}\pi d^2} = \frac{4l}{d}$, hence this formula gives for pipes $h = 0{,}03026\,\frac{l}{d} \cdot \frac{v^2}{2g}$, whilst we have found more correctly for these (§ 331) for mean velocities

$$h = 0{,}025\,\frac{l}{d} \cdot \frac{v^2}{2g}.$$

The friction, therefore, as might be expected, is greater in the beds of rivers than in metallic conducting pipes.

Examples. 1. What fall must be given to a canal of the length $l = 2600$ feet, lower breadth $b = 3$ feet, upper breadth $b_1 = 7$ feet, and depth $a = 3$ feet, if it is to conduct a quantity of water of 40 cubic feet per second? It is:

$p = 3 + 2\sqrt{2^2 + 3^2} = 10{,}211$, $F = \frac{(7+3)\,3}{2} = 15$ and $c = \frac{40}{15} = \frac{8}{3}$, hence the fall sought, $h = 0{,}0001173 \cdot \frac{2600 \cdot 10{,}211}{15} \cdot \left(\frac{8}{3}\right)^2 = \frac{0{,}305 \cdot 10{,}211 \cdot 64}{15 \cdot 9} = 1{,}476$ feet.—2. What quantity of water does a canal 5800 feet long, having a 3 feet fall, 5 feet deep, 4 feet lower and 12 feet upper breadth? Here:

$$\frac{p}{F} = \frac{4 + 2\sqrt{5^2 + 4^2}}{5 \cdot 8} = \frac{16{,}806}{40} = 0{,}42015;$$

hence the velocity

$$c = 92{,}35\,\sqrt{\frac{3}{0{,}42015 \cdot 5800}} = \frac{92{,}35}{\sqrt{0{,}14005 \cdot 5800}} = \frac{92{,}35}{\sqrt{812{,}29}} = \frac{92{,}35}{28{,}5} = 3{,}24 \text{ feet},$$

and the quantity of water $Q = Fc = 40 \cdot 3{,}24 = 129{,}6$ cubic feet, English measure.

§ 368. *Co-efficients of Friction.*—The co-efficient of friction for rivers, brooks, &c., the mean value of which, in the foregoing paragraphs, we have taken at 0,007565, is not constant, but, as in pipes, increases somewhat for small and diminishes for great velocities. We have, therefore, to put:

$$\zeta = \zeta_1 \left(1 + \frac{a}{c}\right) \text{ or } \zeta_1 \left(1 + \frac{a}{\sqrt{c}}\right).$$

The author of the work alluded to in § 363, finds from 255 experiments, the greater part of them undertaken by himself, for the Prussian measure $\zeta = 0{,}007409 \left(1 + \frac{0{,}0299}{c}\right)$, and hence it follows for the metre $\zeta = 0{,}007409 \left(1 + \frac{0{,}00939}{c}\right)$,

and for English measure $007409 \left(1 + \frac{0{,}0308}{c}\right)$.

It is manifest that these formulæ, for a velocity $c = 1\frac{1}{2}$ feet, give again the above assigned mean co-efficient of resistance $\zeta = 0{,}007565$. The following useful table of the co-efficients of resistance in the metrical measure serves for facilitating calculation.

Velocity c.	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	Meter.
Co-efficient of resistance $\zeta = 0{,}00$	811	776	764	758	755	753	751	750	749	

Velocity c.	1	1,2	1,5	2	3	Meter.
Co-efficient of resistance $\zeta = 0{,}00$	748	747	746	744	743	

The following table serves for the Prussian or English measure:

Velocity c.	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	$1\frac{1}{2}$	2	3	5	10 ft.
Co-efficient of resistance $\zeta = 0{,}00$	815	797	785	778	773	769	766	763	759	752	749	745	743

These tables find their direct application in all cases where the velocity c is given and the fall to be found, and where the formula No. 1 of the former paragraph is applicable. But if the velocity c is unknown, and its amount to be determined, these tables will then only admit of a direct application, when we have already an approximate value of c. We may set to work in the simplest manner by determining c approximately by the formula $c = 50{,}9 \sqrt{\frac{Fh}{pl}}$, and from this a value of ζ, taken from the table, and the value so obtained put into the formula

$$\frac{c^2}{2g} = \frac{h}{\zeta} \cdot \frac{F}{lp} \text{ or } c = \sqrt{\frac{F}{\zeta l p} \cdot 2gh}.$$

From the velocity c, the quantity of water is then given by the formula $Q = Fc$.

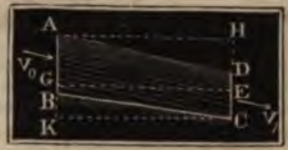
If, lastly, the quantity and the fall are given, and, as is often requisite in the construction of canals, it be required to determine the transverse section, we may put $\frac{p}{F} = \frac{m}{\sqrt{F}}$ (*see* Table, § 366) and $c = \frac{Q}{F}$ into the formula $h = 0{,}007565 \frac{lp}{F} \cdot \frac{c^2}{2g}$, and write, therefore, $h = 0{,}007565 \frac{m l Q^2}{2gF^{\frac{5}{2}}}$, and accordingly determine:

$F = \left(0{,}007565 \frac{m\, l\, Q^2}{2\, gh}\right)^{\frac{2}{5}}$, *i. e.*, for the metre $F = 0{,}0431 \left(\frac{m\, l\, Q^2}{h}\right)^{\frac{2}{5}}$ or the English foot measure $F = 0{,}0268 \left(\frac{m\, l\, Q^2}{h}\right)^{\frac{2}{5}}$. Hence it follows, approximately, that $c = \frac{Q}{F}$; if we take a correspondent value of ζ from one of the tables, more accurately $F = \left(\zeta \cdot \frac{m\, l\, Q^2}{2\, gh}\right)^{\frac{2}{5}}$; and hence, more exact values for $c = \frac{Q}{F}$, $p = m \sqrt{F}$, as also for a, b, &c.

Examples.—1. What fall does a canal 1500 feet long, 2 feet lower and 8 feet upper breadth, and 4 feet depth require to give a discharge of 70 cubic feet per second? It is $p = 2 + 2\sqrt{4^2 + 3^2} = 12$, $F = 5 \,.\, 4 = 20$, $c = \frac{70}{20} = 3{,}5$, hence $\zeta = 0{,}00748$, and $h = 0{,}00748 \,.\, \frac{1500 \,.\, 12}{20} \,.\, \frac{3{,}5^2}{2g} = 6{,}732 \,.\, 0{,}1902 = 1{,}28$ ft. (Eng.)—2. What discharge does a brook 40 feet broad, $4\frac{1}{2}$ feet mean depth, and 46 feet water profile, if it has a fall of 10 inches for a length of 750 feet? It is about $c = 92{,}35 \,.\, \sqrt{\frac{40 \,.\, 4{,}5 \,.\, 10}{46 \,.\, 750 \,.\, 12}} = \frac{92{,}35}{\sqrt{230}} = 6{,}089$ feet, and hence $\zeta = 0{,}00745$. Hence we obtain, more correctly: $\frac{c^2}{2g} = \frac{Fh}{\zeta lp} = \frac{4{,}5 \,.\, 40 \,.\, 10}{0{,}00745 \,.\, 46 \,.\, 750 \,.\, 12} = \frac{1}{1{,}7112} = 0{,}5844$, and $c = 6{,}119$ feet. Lastly, the corresponding discharge is $Q = 4{,}5 \,.\, 40 \,.\, 6{,}119 = 1101$ cubic feet, (Eng.)—3. A trench 3650 feet long is to be cut, which for a total fall of 1 foot is to carry off a discharge of 12 cubic feet per second, what dimensions are to be given to the transverse profile, if it is to preserve a regular semi-hexagonal figure? Here $m = 2{,}632$ (*see* Table, § 366), hence, approximately, $F = 0{,}0268\ (2{,}632 \,.\, 3650 \,.\, 144)^{\frac{2}{5}} = 7{,}665$ square feet, and $c = \frac{12}{7{,}665} = 1{,}539$ feet. Hence ζ is to be taken $= 0{,}00758$, and $F = \left(0{,}00758 \,.\, 2{,}632 \,.\, \frac{3650 \,.\, 144}{6{,}44}\right)^{\frac{2}{5}} = 7{,}67$ square feet. Therefore the depth must be made: $a = 0{,}760 \sqrt{F} = 2{,}104$ feet, the lower breadth $= 0{,}877 \sqrt{F} = 2{,}428$, and the upper breadth $= 2 \,.\, 2{,}428 = 4{,}846$ English feet.

§ 369. *Variable Motion.*—The theory of the variable motion of water in beds of rivers may be reduced to the theory of uniform motion, provided the resistance of friction for a short length of the river may be considered as constant, and the corresponding height, in like manner, as $= \zeta \cdot \frac{lp}{F} \cdot \frac{v^2}{2g}$. But, besides this, regard must be had to the *vis viva* of the water, which corresponds to a change of velocity.

Fig. 499.



Let $ABCD$, Fig. 499, be a short extent of river, of the length $AD = l$, the fall $DH = h$, and let v_0 be the velocity of the arriving, and v_1 that of the departing water. If we apply the rules of efflux to an element D of the surface, we shall obtain for its velocity v_1,

$$\frac{v_1^2}{2g} = h + \frac{v_0^2}{2g};$$

as regards an element E below the surface, it is true that on the one side it has a greater height of pressure $AG = EH$; but as the downstream water reacts with a pressure DE, there remains for it only the fall $DH = EH - ED$, as pressure inducing motion, and so, for this or any other element, the formula:

$$h = \frac{v_1^2 - v_0^2}{2g} \text{ answers};$$

and if, further, the resistance due to friction be added, we then obtain:

$$h = \frac{v_1^2 - v_0^2}{2g} + \zeta \cdot \frac{lp}{F} \cdot \frac{v^2}{2g},$$

where p, F and v are the mean values of the wetted perimeter, transverse section, and velocity. If F_0 is the area of the upper, and F_1 that of the lower section, we may then put:

$$F = \frac{F_0 + F_1}{2}, \text{ and } Q = F_0 v_0 = F_1 v_1,$$

whence it follows that:

$$\frac{v_1^2 - v_0^2}{2g} = \frac{1}{2g}\left[\left(\frac{Q}{F_1}\right)^2 - \left(\frac{Q}{F_0}\right)^2\right] = \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}, \text{ and}$$

$$\frac{v^2}{F} = \frac{v_0^2 + v_1^2}{F_0 + F_1} = \left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{F_0 + F_1}, \text{ we obtain:}$$

$$1.\ h = \left[\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{lp}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\right]\frac{Q^2}{2g}, \text{ as also}$$

$$2.\ Q = \frac{\sqrt{2gh}}{\sqrt{\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{lp}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)}}.$$

The corresponding fall h may be calculated by means of the formula 1, from the quantity of water, the length and transverse section of a river or canal; and, inversely, the quantity of water from the fall, the length and the transverse section, by formula 2. To obtain greater accuracy, we may make the calculation for several short portions of the river, and take the arithmetical mean. If the total fall only is known, we must substitute this at once for h in the last formula, and put

$$\frac{1}{F_1^2}-\frac{1}{F_0^2}=\frac{1}{F_n^2}-\frac{1}{F_0^2},$$

where F_n denotes the area of the last section, and in place of

$$\zeta \cdot \frac{l p}{F_0+F_1}\left(\frac{1}{F_0^2}+\frac{1}{F_1^2}\right)$$

the sum of all similar values of the separate lengths of the river.

Example. A brook has for a distance of 300 feet a fall of 9,6 inches, the mean perimeter of its water profile is 40 feet, the area of the upper transverse profile 70, that of the lower 60 square feet; what quantity of water does this brook discharge? It is

$$Q=\frac{8{,}02\sqrt{0{,}8}}{\sqrt{\frac{1}{60^2}-\frac{1}{70^2}+0{,}007567 \cdot \frac{300 \,.\, 40}{130}\left(\frac{1}{60^2}+\frac{1}{70^2}\right)}}=$$

$$=\frac{7{,}173}{\sqrt{0{,}0000731+0{,}0003365}}=\frac{7{,}173}{\sqrt{0{,}0004096}}=354{,}43$$ cubic feet. The mean velocity is $\frac{2\,Q}{F_0+F}=\frac{708{,}8}{130}=5{,}452$ feet; hence, more accurately, ζ must be taken $=0{,}00745$ in place of $0{,}007565$, and therefore more nearly:

$$Q=\frac{7{,}173}{\sqrt{0{,}0000731+0{,}0003314}}=357{,}5$$ cubic feet. If the same brook, with the same head of water, had for a length of 450 feet, a fall of 11 inches, and if its upper transverse profile had an area of 50 and its lower of 60 square feet, and the mean perimeter of the profile measured 36 feet, we should then have:

$$Q=\frac{8{,}02\sqrt{0{,}9167}}{\sqrt{\frac{1}{60^2}-\frac{1}{50^2}+0{,}00745 \,.\, \frac{450 \,.\, 36}{110}\left(\frac{1}{60^2}+\frac{1}{50^2}\right)}}$$

$$=8{,}02\sqrt{\frac{0{,}9167}{0{,}0001222+0{,}0007436}}=308$$ cubio feet.

The mean of these two values is $Q=\frac{357\,.\,5+308}{2}=332{,}75$ cubic feet.

§ 370. In order to obtain a formula for the depth of water, let the upper depth $=a_0$ and the lower $=a_1$, the slope of the bed $=\alpha$, consequently the fall of the bed $=l\ sin.\ \alpha$. We then obtain the fall of the water $h=a_0-a_1+l\ sin.\ \alpha$, and there results the equation:

$$a_0-a_1-\left(\frac{1}{F_1^2}-\frac{1}{F_0^2}\right)\frac{Q^2}{2g}=\left[\zeta\,\frac{p}{F_0+F_1}\left(\frac{1}{F_0^2}+\frac{1}{F_1^2}\right)\frac{Q^2}{2g}-sin.\ \alpha\right]l,$$

hence $$l=\frac{a_0-a_1-\left(\frac{1}{F_1^2}-\frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{\zeta\,\frac{p}{F_0+F_1}\left(\frac{1}{F_0^2}+\frac{1}{F_1^2}\right)\frac{Q^2}{2g}-sin.\ \alpha}.$$

The length l which corresponds to a difference a_0-a_1 of the depth of water, may be determined by this formula. But if the reverse problem is to be solved, we must do it by the method of approximation, and first determine the distances l_1 and l_2 corresponding to the assumed depressions a_0-a_1, and a_1-a_2, and from these calculate by a proportion, the depression corresponding to a given distance l.*

---

* See "Ingenieur," Arithmetik, § 16, v.

The formula is further capable of simplification when the breadth b of the running water is constant, or may be considered as such. In this case we put:

$$\left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \frac{F_0^2 - F_1^2}{F_0^2 F_1^2} \cdot \frac{Q^2}{2g} = \frac{(F_0 - F_1)(F_0 + F_1)}{F_1^2} \cdot \frac{v_0^2}{2g}$$

$$= \frac{(a_0 - a_1)(a_0 + a_1)}{a_1^2} \cdot \frac{v_0^2}{2g} \text{ approximately } = 2\,\frac{(a_0 - a_1)}{a_0} \cdot \frac{v_0^2}{2g},$$

and likewise:

$$\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} = \frac{p\,(F_0^2 + F_1^2)}{(F_0 + F_1)F_1^2} \cdot \frac{v_0^2}{2g} \text{ approximately}$$

$$= \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g}, \text{ hence } l = \frac{(a_0 - a_1)\left(1 - \frac{2}{a^0} \cdot \frac{v_0}{2g}\right)}{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - sin.\ \alpha}, \text{ and hence}$$

$$\frac{a_0 - a_1}{l} = \frac{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - sin.\ \alpha.}{1 - \frac{2}{a_0} \cdot \frac{v_0^2}{2g}}.$$

The difference $(a_0 - a_1)$ of the depth corresponding to a given extent l may be calculated directly by this formula.

Example. In a horizontal trench, 5 feet broad and 800 feet long, it is desired to carry off a 20 cubic feet discharge, and to let it flow in at a depth of 2 feet, what depth will the water at the end of the canal have? Let us divide the whole length into two equal portions, and determine from the last formula the fall for each of them.

Here the $sin.\ \alpha = 0$, $l = \frac{800}{2} = 400$, and $b = 5$; for the first portion $v = \frac{20}{2\,.\,5} = 2$, hence $\zeta = 0{,}00752$, also $a_0 = 2$; since $p = 8\frac{1}{2}$, it follows that $a_0 - a_1 = \left(\frac{0{,}00752 \,.\, \frac{8{,}5}{10} \cdot \frac{4}{2g}}{1 - \frac{2}{2} \cdot \frac{4}{2g}}\right) .\, 400 = \frac{0{,}15877}{0{,}9379} = 0{,}1692$ feet. Now, for the second half, $a_1 = 2 - 0{,}1692 = 1{,}8308$, and $p_1 = 8{,}2$, $v_1 = \frac{20}{9{,}154} = 2{,}1848$, and the depression of the second portion:

$$a_1 - a_2 = \left(\frac{0{,}00752 \,.\, \frac{8{,}2}{9{,}154} \cdot \frac{2{,}1848^2}{2g}}{1 - \frac{2}{1{,}8308} \cdot \frac{2{,}1848^2}{2g}}\right) .\, 400 = \frac{0{,}1997}{0{,}919} = 0{,}2173 \text{ feet, hence the}$$

whole depression $= 0{,}1692 + 0{,}2173 = 0{,}3865$, and the depth of water at the lower end $= 2 - 0{,}3865 = 1{,}6135$.

§ 371. *Floods.*—When the depth of water in rivers and canals varies, variations in the velocity and discharge take place likewise. A greater depth of water not only involves a greater section, but also a greater velocity, and hence, for two reasons, a greater quantity of water, and likewise a diminution of the depth of water, gives a diminution of the section and the velocity, and hence also a decrease of the discharge. If the original depth $= a$, and any increased depth

= a_1, the upper breadth of the canal = b, then the augmentation of the section may be put = $b\,(a_1 - a)$, and hence the section afterwards $a_1 - a$, $F_1 = F + b\,(a_1 - a)$, it also follows from this that

$$\frac{F_1}{F} = 1 + \frac{b\,(a_1 - a)}{F}, \text{ and}$$

$$\sqrt{\frac{F_1}{F}} \text{ approximately} = 1 + \frac{b\,(a_1 - a)}{2\,F}.$$

If further p be the original, p_1 the increased perimeter of the water profile, and θ the angle of slope of the banks, then

$$p_1 = p + \frac{2\,(a_1 - a)}{p\,sin.\,\theta}, \text{ hence } \frac{p_1}{p} = 1 + \frac{2\,(a_1 - a)}{p\,sin.\,\theta}, \text{ and}$$

$$\sqrt{\frac{p_1}{p}} = 1 + \frac{a_1 - a}{sin.\,\theta}, \text{ as also } \sqrt{\frac{p}{p_1}} = 1 - \frac{a_1 - a}{p\,sin.\,\theta}.$$

Now the velocity with the first depth of water is

$c = 92{,}35\sqrt{\frac{Fh}{pl}}$, and with the second $c_1 = 92{,}35\sqrt{\frac{F_1}{p_1} \cdot \frac{h}{l}}$,

hence we may put:

$$\frac{c_1}{c} = \sqrt{\frac{F_1}{F}} \cdot \sqrt{\frac{p}{p_1}} = \left(1 + \frac{b\,(a_1 - a)}{2\,F}\right)\left(1 - \frac{a_1 - a}{p\,sin.\,\theta}\right)$$

$$= 1 + (a_1 - a)\left(\frac{b}{2F} - \frac{1}{p\,sin.\,\theta}\right),$$

therefore the relative change of velocity:

$$1.\ \frac{c_1 - c}{c} = (a_1 - a)\left(\frac{b}{2\,F} - \frac{1}{p\,sin.\,\theta}\right).$$

On the other hand, the ratio of the discharge is:

$$\frac{Q_1}{Q} = \frac{F_1 c_1}{Fc} = \left(1 + \frac{b\,(a_1 - a)}{F}\right)\left[1 + (a_1 - a)\left(\frac{b}{2\,F} - \frac{1}{p\,sin.\,\theta}\right)\right]$$

$$= 1 + (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\,sin.\,\theta}\right),$$

and the relative increase:

$$2.\ \frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\,sin.\,\theta}\right).$$

Less accurately, but in many cases, especially in broad canals with little slope, we may put $F = ab$, and neglect $\frac{1}{p\,sin.\,\theta}$, whence it follows more simply that:

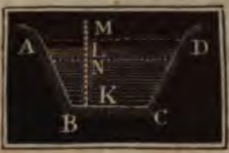
$$\frac{c_1 - c}{c} = \tfrac{1}{2}\,\frac{a_1 - a}{a}, \text{ and}$$

$$\frac{Q_1 - Q}{Q} = \tfrac{3}{2} \cdot \frac{a_1 - a}{a}.$$

From this, therefore, the relative change of velocity is ½, and the relative change in the quantity of water $\frac{3}{2}$, that of the relative change in the depth of water.

Examples.—1. When the head of water increases $\frac{1}{10}$ of its original amount, the velocity is then $\frac{1}{20}$, and the quantity $\frac{3}{20}$ greater than its original value.—2.—When the depth diminishes 8 per cent., the velocity then diminishes 4, and the quantity 12 per cent.—3. From the more correct formula:

Fig. 500.



$$\frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3\,b}{2F} - \frac{1}{p\ sin.\ \theta}\right)$$

a scale of the depth of water *KM*, Fig. 500, may be constructed, on which the discharge of a canal corresponding to any depth *KL*, may be read off, when the quantity of water for a certain mean depth is once known. If $b = 9$ feet, $b_1 = 3$, $a = 3$, and $\theta = 45°$, we then have $F = \frac{(9 + 3)\,3}{2} = 18$ square ft., $p = 3 + 2\,.\,3\sqrt{2} = 11{,}485$ and $sin.\ \theta = \sqrt{\tfrac{1}{2}} = 0{,}707$, hence:

$$\frac{Q_1 - Q}{Q} = \left(\frac{3\,.\,9}{2\,.\,18} - \frac{1}{11{,}485\,.\,0{,}707}\right)(a_1 - a) = (0{,}750 - 0{,}123)\,(a_1 - a) = 0{,}627$$

$(a_1 - a)$. If the quantity corresponding to a mean head of water $Q = 40$ cubic feet, we then have $Q_1 = 40 + 40\,.\,0{,}627\,(a_1 - a) = 40 + \frac{a_1 - a}{0{,}04}$. If $a_1 - a = 0{,}04$ feet $= 5{,}76$ lines, it follows that $Q_1 = 41$; $a_1 - a = 0{,}08$ feet $= 11{,}52$ lines, we then have $Q_1 = 42$ cubic feet; if, further, $a_1 - a = -0{,}04$, then is $Q_1 = 39$ cubic feet, &c. A scale, therefore, whose intervals are $LM = LN = 5{,}76$ lines, gives the discharge accurately to a cubic foot. Of course the accuracy is the less, the more the head of water differs from a mean value.

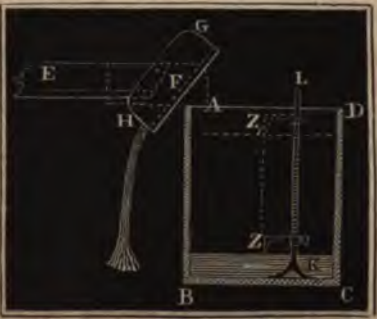
Remark. The conducting and carrying off of water in canals, as well as the subject of weirs and dams, will be fully treated of in the Second Part.

CHAPTER VIII.

HYDROMETRY, OR THE DOCTRINE OF THE MEASUREMENT OF WATER.

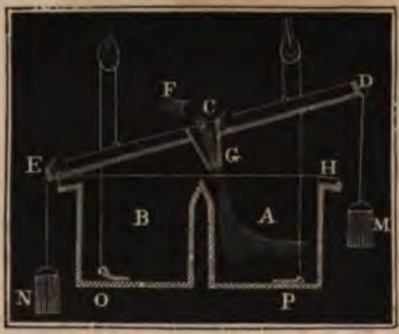
§ 372. *Gauges.*—The quantity, which a stream discharges in a certain time, is determined either by a gauge, by an apparatus of efflux, or by an hydrometer. The most simple way of measuring water is by the guage, *i. e.* by the use of a graduated vessel, but this method is only applicable to small discharges, carried off by pipes or small brooks, or drains. The gauge vessel is generally made of wood, and of a rectangular form, and to increase its strength is bound round with iron-hooping. The water is conducted into it by a trough *EF*, Fig. 501, at whose extremity there is a double valve *GH*, by which the water may be made to flow at will into the vessel *AC*, or by the side of it. To obtain the exact depth of the body of water in the vessel, a scale *KL* is fur-

Fig. 501.



ther applied. If before measurement, the index Z be moved down to the surface of the water, already in the vessel, and merely covering the bottom, and the head of water read off from the scale, we shall obtain the height ZZ_1 of the gauged water by subtraction of this from the head of water which the scale indicates when the index hand Z_1 is brought into contact with the surface of water at the end of the observation. Before measurement, the valve must be so placed that the water may flow off outside the vessel. When we are convinced that the efflux in the trough is in a state of permanency, and, watch in hand, have noted a certain moment, the valve must then be turned, so that the water may run into the gauge vessel, and after it is either partly or entirely filled, a second interval is noted by the watch, and the valve again brought into its first position. From the mean section F of the vessel, and the depth $ZZ_1 = a$ of the body of water, the whole quantity $= Fa$ may be estimated, and again from the time of filling t, given by the difference of the times observed, the quantity of water per second $Q = \frac{Fa}{t}$.

Fig. 502.



Remark. To determine a variable quantity of efflux at each period of the day, we may make use of the apparatus represented in Fig. 502, as applicable especially in salt works. There are here two gauge vessels, A and B, which alternately fill and empty themselves, and the water which is conducted by the pipe F passes through a short pipe CG, which is rigidly connected with a lever DE, revolving about C. When one vessel A becomes filled, the water then flows through a short tube H into the little vessel M, this draws the lever down again on one side, and the pipe CG comes into such a position that the water is conducted into B. The drawing up of the valves O and P takes place by means of strings passing over pulleys, whose extremities are connected with the lever, and sustained by iron balls, which impart a final impulse to the descent of the lever. The vessels M and N have small efflux orifices, by which they empty themselves after each reversion of the lever. An apparatus is besides applied, by which the number of strokes may be read off at any time.

§ 373. *Efflux Regulators.*—Small and medium discharges are very frequently determined by means of their flow through a definite orifice, and under a known head. From the area F of the orifice, the head of water h, and the efflux co-efficient μ, the discharge per second ?/ $Q = \mu F \sqrt{2gh}$ is given. The Poncelet orifices are those best adapted for this purpose, because the co-efficients of efflux of these under different heads of water are known with great accuracy (§ 316), still they are applicable only to certain medium discharges. The author availed himself of four such orifices for his measurements, one of five, one of ten, one of fifteen, one of twenty centimetres depth, but all of twenty centimetres width. These orifices are cut out of brass plate, and fixed to a wooden frame *AC*, Fig. 503, which is fastened by four strong iron screws to each wall. In many cases, indeed, greater orifices, the co-efficients of efflux for which are not so accurately determined, and sometimes wiers must be used, which admit generally of a still less accuracy. In all cases, however, the rule holds good, that we must endeavor to get as complete and perfect a contraction as possible, and for this reason must give to the orifice, if it is in a thick plate, a slope outward. The corrections which must be applied for incomplete and partial contraction, have been sufficiently distinguished in paragraphs 319, 320, &c. To measure the water of a brook we must set the frame with its orifice, and wait for the moment when the head of water is permanent. For the measurement of the head of water we must avail ourselves of the index scale, Fig. 500, or of the movable scale *EF*, Fig. 505. If we would note the efflux directly from the apertures of sluices, it is better to fix before hand a pair of brass scales *BC* and *DE*, Fig. 504, with their indices *F* and *G* to the slide, and to the sluice-board *A*, in order to read off more safely the height of the aperture. It is generally better for the purpose of measuring water, to put on a new sluice-board with its guide, and with the requisite slope outwards. The simplest means of measuring water in a channel, consists in putting in a board *CD*, with its upper edge sloped off, Fig. 505, and measuring the fall produced by it. If the channel is long, and there is little rise, it is generally some time before the condition of permanency takes place, and it is for this reason good, before measurement, to put on a second board, so as to impede the efflux of water for a long time, in order to accelerate the rise to a height corresponding to a state of permanency.

Fig. 503.

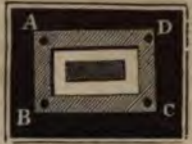


Fig. 504.

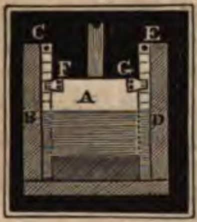
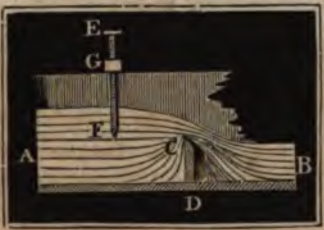
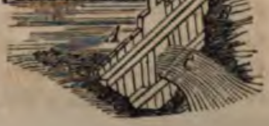


Fig. 505.



To measure the quantity of water of a brook, we may dam it up with posts and boards as in Fig. 506, and let the water *C* run off through an aperture, or we may avail ourselves of a simple overfall or wier, but of this we shall treat in the second part.

Fig. 506.



§ 374. But as it is often long before a state of permanency occurs in water dammed up by this construction, we may adopt with advantage the following method, first proposed by Prony. We may first close entirely the aperture by a sluice-board, and let the water rise to some height, or as high as circumstances will admit, then draw ?/ it so far up that more water may flow in than out, and measure the heads of water at equal and very short intervals; lastly, the aperture of the sluice must be again perfectly closed, and the time t in which the water rises to the first height, further noted. In each case, then, during the whole time of observation $t+t_1$, as much water flows in as out, and hence the quantity flowing in, in the time $t+t_1$, may be expressed by the quantity flowing out in the time t_1. If the heads of water during the depressions are h_0, h_1, h_2, h_3, and h_4, we have then the mean velocity of efflux:

$$v = \frac{\sqrt{2g}}{12}\left(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}\right) \text{ (see § 351)},$$

and if the area of the aperture $= F$, we have then the quantity of efflux in the time t:

$$V = \frac{\mu F t \sqrt{2g}}{12}\left(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}\right), \text{ and the}$$

quantity flowing in per second:

$$Q = \frac{V}{t+t_1} = \frac{\mu F t \sqrt{2g}}{12(t+t_1)}\left(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}\right).$$

Example. To measure the water of a brook used for the driving of a water-wheel, which has been dammed up by a sluice, Fig. 506, after opening the rectangular aperture, the following is observed: the original head of water is 2 feet, after 30″ 1,8 feet, after 60″ 1,55 feet, after 90″ 1,3 feet, after 120″ 1,15 feet, after 150″ 1,05 feet, and after 180″ 0,9 feet, breadth of the aperture 2 feet, depth $\frac{1}{2}$ foot, time of rising to the first height with closed aperture $= 110''$. The mean velocity of efflux is:

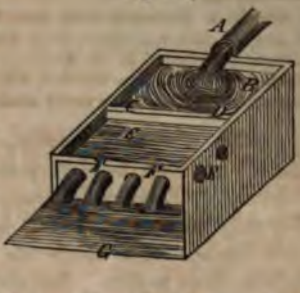
$$v = \frac{8{,}02}{18}\left(\sqrt{2} + 4\sqrt{1{,}8} + 2\sqrt{1{,}55} + 4\sqrt{1{,}3} + 2\sqrt{1{,}15} + 4\sqrt{1{,}05} + \sqrt{0{,}9}\right) =$$
$$0{,}440\,(1{,}414 + 5{,}364 + 2{,}490 + 4{,}561 + 2{,}145 + 4{,}099 + 0{,}949) = 0{,}440 \,.\, 21{,}022 =$$

9,248 feet; but now $F = 2 \,.\, \frac{1}{2} = 1$ square foot, hence it follows that the theoretical discharge is $= 9{,}248$ cubic feet. If the co-efficient of efflux is taken $= 0{,}61$, we finally obtain the quantity of water sought:

$$Q = \frac{0{,}61 \,.\, 180}{180 + 110} \,.\, 9{,}248 = 3{,}5015 \text{ cubic feet, (English.)}$$

§ 375. *The "Pouce d'Eau," or Water-Inch.*—To measure small discharges, we avail ourselves of the flow through round 1 inch wide orifices, in a thin plate, under a given pressure. The discharge given through such an aperture under the least pressure, or when the surface is only a line above the uppermost position of the orifice, is called an *inch of water*. The French assume for the water-inch (old Paris measure) 15 pints, or 19,1953 cubic metres of water in the 24 hours; therefore in 1 hour 0,7998, and in 1 minute 0,0133 cubic metres; yet older data, by Mariotte, Couplet, and Bossut, vary not a little from the above. According to Hagen, an inch of water (Prussian measure) delivers in 24 hours 520 cubic feet, therefore, in a minute, 0,3611 cubic feet. The double water modulus of Prony, which corresponds to an orifice of 2 centimetres diameter, with a pressure of 5 centi-

Fig. 507.



metres, and discharges 20 cubic metres of water in 24 hours, has not been adopted. The apparatus by which water is measured by the inch is represented in Fig. 507. The water to be measured flows through the tube *A* into a box; from this it passes through holes below in the partition *CD* into the box *E*, and from this through a horizontal row of round orifices *F*, of exactly 1 inch width, and cut in tin plate, into the reservoir *G*. That the surface of water may stand a line above the heights of these orifices, it is necessary that there be a sufficient number of them, and that a part of them be closed by stoppers. For great discharges the whole water is divided, and in this way a part, only one-tenth, is measured. This division may be accomplished easily, by first conducting the water into a reservoir, with a certain number of orifices at the same level, and only to receive the quantity delivered by one orifice in the apparatus represented above.

Remark 1. We may apply also cocks and other regulating apparatus to the measurement of water, if we know the co-efficient of resistance for each position. If h is the head of water, F the transverse section of the pipe, and μ the co-efficient of efflux, for a cock quite opened, we then have the discharge $Q = \mu F \sqrt{2 g h}$, as inversely, $\mu = \frac{Q}{F \sqrt{2 g h}}$ and $\frac{1}{\mu^2} = \left(\frac{F}{Q}\right)^2 . 2 g h$. If now we put the co-efficient of resistance corresponding to a position of the cock, and taken from the tables already given $= \zeta$, we then have the corresponding discharge:

$$Q_1 = F \sqrt{\frac{2 g h}{\frac{1}{\mu^2} + \zeta}} = \frac{\mu F \sqrt{2 g h}}{\sqrt{1 + \mu^2 \zeta}} = \frac{Q}{\sqrt{1 + \zeta \left(\frac{Q}{F}\right)^2 \cdot \frac{1}{2 g h}}}.$$

For convenience sake, we may construct for ourselves a table, so that we can find at a glance the discharge corresponding to a position of the cock, or the position of the cock corresponding to a given discharge. If, for example, $\mu = 0{,}7$ and $F = 5$ square inches, we have then:

$$Q_1 = \frac{0{,}7 \,.\, 4 \,.\, 12 \,.\, 8{,}02 \sqrt{h}}{\sqrt{1 + 0{,}49 \,.\, \zeta}} = 269{,}5 \sqrt{\frac{h}{1 + 0{,}49\, \zeta}} \text{ cubic inches,}$$

or if h is constantly 1 foot, $Q_1 = \frac{269{,}5}{\sqrt{1 + 0{,}49\, \zeta}}$. If now the positions of the cock are at 5°, 10°, 15°, 20°, 25°, &c., the co-efficients of resistance, 0,057; 0,293; 0,758; 1,559;

Fig. 508.

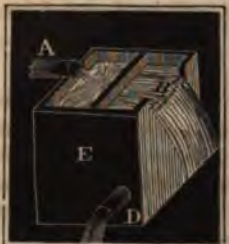
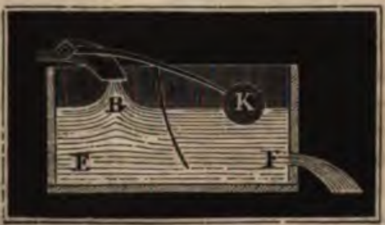


Fig. 509.



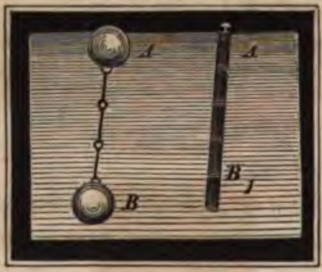
3,095, the discharges corresponding to these are: 265,8; 252; 230,1; 202,8; 169,9 cubic inches.

Remark 2. To regulate the flow through an orifice *D*, Fig. 508, we may apply a weir *B* that the excess of water from the pipe *A* may flow over, and that a constant pressure may be maintained in the reservoir *DE*. That there may be no loss of water, a cock or a valve *A*, Fig. 509, is applied, which is regulated by a float *K* acting upon a lever, so that as much water only flows in through *B* as flows out through *F*.

§ 376. *Floating Bodies.*—The discharge of large brooks, canals, and rivers, can be determined only by an hydrometer indicating the velocity. Of such instruments floating bodies are the most simple. We may use any floating body for this purpose, but it is better to have bodies of a moderate size, which are only a little specifically lighter than water. Substances of about $\frac{1}{10}$ of a cubic foot content are large enough. Very large ones do not easily assume the velocity of water, and very small ones again, especially when much above the water, are easily disturbed in their motion by accidental circumstances, sometimes by the air on the surface of the fluid. Often, plain pieces of wood are sufficient; it is better, however, if they have a coating of some bright varnish, and better still if the floats are hollow, such as glass flasks, tin balls, &c., because these may be filled at will with water. Swimming balls are most frequently used. They are from 4 to 12 inches diameter, and made of brass, and painted over with some light oil-color, to make them more visible to the eye, and have an opening with a neck, that they may be filled with water and stopped. A floating ball, such as *A*, Fig. 510, gives only the velocity at the surface, and often only that of the main current; but by suspending two balls one to the other, *A* and *B*, Fig. 511, we may determine the velocity at different depths. In this case, the one ball *B*, which swims under water, is quite filled with the fluid; the other, however, which swims on the surface, is only filled just enough to make it float a little above the surface. Both balls are connected with each other by a string or wire, or by a light wire chain. The velocity c_0 of the surface is first determined by the single ball, and then the mean velocity of the two observed by the connection of balls. If, now, the velocity at the depth of the second ball be denoted by c_1, we may then put $c = \frac{c_0 + c_1}{2}$, and hence, inversely, $c_1 = 2\,c - c_0$. Whilst now both balls are connected with one another by longer and longer wires, we may, in this manner, find the velocities at greater depths. The mean velocity c of a perpendicular is otherwise given if the second ball is allowed to swim a little above the bottom, and c is made $= 2\,c_1 - c_0$; still more accurately, however, if for c_1 the mean of all the velocities observed in a perpendicular be taken.

Fig. 510.

Fig. 511.



To find the mean velocity in a perpendicular, the floating staff A_1 B_1, represented in Fig. 512, is used. This is particularly convenient for measurements in canals and cuts when it is composed of short pieces screwed together. The floating staff which the author uses is composed of 15 hollow portions, each 1 decimetre in length. That this may swim pretty nearly upright, the lowermost piece is loaded with shot, so that the top just rises above the water. The number of pieces composing the staff, depends, of course, on the depth of the canal.

Both with the floating staff as well as the connection of balls, it may be observed that the velocity at the surface, when the motion of the water in beds is unimpeded, is greater than at the bottom, because the top of the staff swims in advance of the bottom, and the upper ball in advance of the lower. In contraction only, for example, when the water is dammed up by piles, &c., does the contrary take place.

Remark. As a rule, especially with large and floating bodies, as ships, &c., the velocity of the swimming body is somewhat greater than that of the water; not so much because these bodies in swimming float down an inclined plane formed by the surface of the water, but because they take none, or scarcely any, part in the irregular intimate motion of the water; still, the variation for small floating bodies is so slight that it may be neglected.

§ 377. The velocity of a floating ball is found by noting the time t with a good seconds watch, or a half-second pendulum (§ 247), which it takes while floating on the water to describe a measured distance s, marked out on the banks. Then the required velocity of the ball is $c = \frac{s}{t}$. That the time t corresponding to the space described along the bank may be accurately found, it is necessary, with the assistance of a cross line or lines, to erect at the opposite bank two signal staves C and D, perpendicular at A and B, Fig. 512. If we place ourselves behind A, we may then note the moment when the float K, dropped in a little above A, comes into the line AC, and if behind B, we may then also observe the time by a watch held in the hand, when the float reaches the line BD, and we then find by subtraction of the times of observation, the required time t corresponding to the describing of the space s. Besides the mean velocity c of the water, the area

Fig. 512.

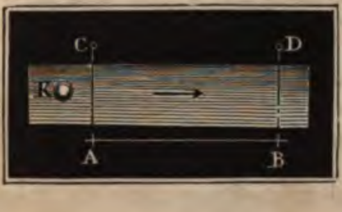


Fig. 513.

F of the transverse profile is further required for determining the quantity of water $Q = F c$. To find this area, it is necessary to know the breadth and the mean depth of the water. The depths are measured by a sounding rod AB, Fig. 513, having a rhomboidal section, and a board B at the foot; for greater depths we may also use a sounding chain, at whose extremity there is an iron plate, which, in sinking, rests on the bottom. The breadth and the abscissæ corresponding to the measured depths, or the distances from the banks in canals and small brooks EFG, Fig. 514, are found by stretching across a measuring chain AB, or the placing of a rod right across the running water. For broad rivers this is determined by a measure table M, which is placed at a proper distance AO, from the section EF, Fig. 515, which is to be measured. If ao is the distance AO between A and O, reduced to the table, and if ao is placed in the direction of AO, and thereby also the direction of the breadth af made parallel to the line of breadth AF marked out, then each line of vision will intersect in the direction of the points E, F, G, &c., in the profile, the corresponding points e, f, g on the table, and ae, af, ag, &c., are the distances AE, AF, AG, &c., in the reduced measure. It is not, therefore, necessary on putting in the sounding rod, and measuring the depths by it, to measure the distances of the corresponding points of the banks, if the engineer standing by the measure table looks at the sound on its being put in, in the line EF.

Fig. 514.

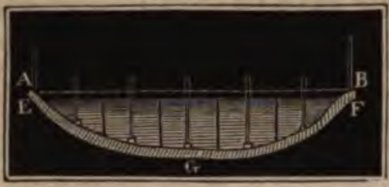
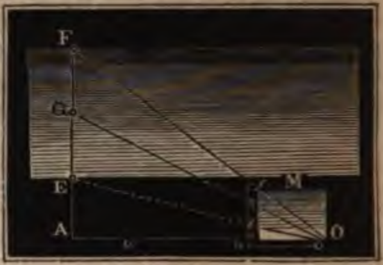


Fig. 515.



If, now, the breadth EF, Fig. 514, of a transverse profile, consist of parts b_1, b_2, b_3, &c., and the mean depths within those parts a_1, a_2, a_3, and the mean velocities c_1, c_2, c_3, &c., we have then the area of the profile:

$$F = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots,$$

the discharge:

$$Q = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \dots,$$

and, lastly, the mean velocity:

$$c = \frac{Q}{F} = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \dots}{a_1 b_1 + a_2 b_2 + \dots}.$$

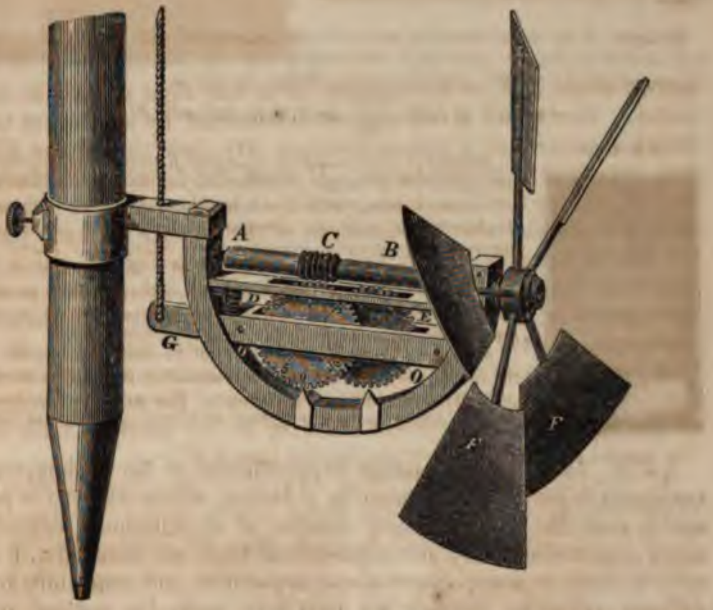
Example. In a tolerably straight and uniform extent of river, we have at the middle points of portions of the breadth:

	5 feet,	12 feet,	20 feet,	15 feet,	7 feet,
The depths - - - -	3 "	6 "	11 "	8 "	4 "
The mean velocities - -	1,9 "	2,3 "	2,8 "	2,4 "	2,1

Hence we may put:
The area of the profile $F = 5\,.\,3 + 12\,.\,6 + 20\,.\,11 + 15\,.\,8 + 7\,.\,4 = 455$ square feet.
The quantity of water $Q = 15\,.\,1{,}9 + 72\,.\,2{,}3 + 220\,.\,2{,}8 + 120\,.\,2{,}4 + 28\,.\,2{,}1$ $= 1156{,}9$ cubic feet. The mean velocity $c = \frac{1156{,}9}{455} = 2{,}54$ feet.

§ 378. *The Tachometer.*—The most eligible hydrometer is the tachometer of Woltmann, Fig. 516. It consists of a horizontal axle AB, with from two to five vanes F, placed at an inclination to the direction of the axis, and gives, when immersed in the water and held at right angles to the direction of motion, by the number of its revolutions in a certain time, the velocity of the running water. To read off the number of these revolutions, the axle has a few turns of a screw C, and these work into the teeth of a wheel D, upon whose lateral surfaces numbers are engraved, which give, by means of an index, the number of revolutions of the wheel. But to be able to register a great number of revolutions upon the axle of this toothed wheel, there is a pinion which works into the teeth of the wheel E, by which, like the hands of a watch, several multiple revolutions may be read off. If, for example, each of the two toothed wheels has fifty teeth, and the trundle ten, then the second wheel revolves one tooth whilst the first advances five teeth, or the vanes make five revolutions, if the index of the first wheel points to $27 = 25 + 2$, and that of the second to 32, the corresponding number of revolutions of the vanes is accordingly: $= 32\,.\,5 + 2 = 162$. The entire instrument is screwed to a staff having a tin vane attached, to admit of easy

Fig. 516.



immersion in the water, and of being kept opposed to the current. But that the wheelwork may only revolve during the time of observation, the axis is connected with a lever *GO*, which is pressed down by a spring, so that the teeth of the first wheel are thrown into gear with the screw only when the lever is drawn up by a string.

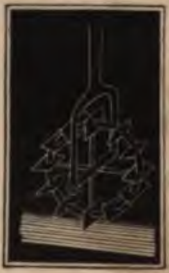
The number of revolutions of a wheel in a certain time, for example, in a second, is not exactly proportional to the velocity of the water, hence we cannot put $v = \alpha u$, where u is the number of revolutions, v the velocity, and α a number deduced from experiments; but rather: $v = v_0 + \alpha u$, or more correctly $v = v_0 + \alpha u + \beta u^2 \ldots$, or still more correctly:

$v = \alpha u + \sqrt{v_0^2 + \beta u^2}$, where v_0 is the velocity, at which the water is no longer able to turn the wheel, and α and β are co-efficients from experiment. The constants v_0, α and β, are to be determined for each instrument in particular. With their assistance the velocity is known from a single observation, nevertheless it is always safer to make at least two, and to substitute the mean value as the correct one.

Example. If for a sail-wheel $v_0 = 0{,}110$ feet, $\alpha = 0{,}480$, and $\beta = 0$, therefore $v = 0{,}11 + 0{,}48\, u$, and we have by an observation with this instrument found the number of revolutions 210 in 80″, then the corresponding velocity is:

$$v = 0{,}11 + 0{,}48 \,.\, \frac{210}{80} = 0{,}11 + 1{,}26 = 1{,}37 \text{ feet.}$$

Fig. 517.



Remark 1. The constants v_0, α and β depend principally on the magnitude of the angle of impact, *i. e.*, on the angle which the plane of the vane makes with the direction of motion of the water, and therefore, also, with the direction of the axis of the wheel. To observe with tolerable accuracy small velocities, it is well to have a large angle of impulse, *i. e.*, one of 70°. For the rest, it is desirable to have vanes of different sizes and with different angles of impulse, and to use the vane with small angles of impulse for great velocities, and a smaller one for shallow water.

Remark 2. To find the velocity of the surface of water, a small tin wheel may be used, as represented in Fig. 517, and its under part allowed to dip into the water. The number of its revolutions may be determined by a system of wheels, as in the tachometer.

§ 379. To find the constant or co-efficient of the tachometer, it is necessary to set this instrument in a stream, whose velocity is known, and to note the corresponding number of revolutions. Although as many observations only are required, as there are constants, it is still safer to have as many observations as possible, and especially for very different velocities, because we may then apply the method of least squares, and thereby eliminate the effect of accidental errors of observation. The velocity of the water may be found by the floating ball, or by receiving the water in a gauge vessel, and dividing the measured discharge by the transverse section. In using floating balls, the air should be still, and the tract of water straight and uniform. The tachometer is to be held at several places of the space described by the floating ball, and it is also requisite for accuracy, that the diameter of the ball should be equal to that of the tachometer.

The second method of determination has several advantages when the water in which the instrument is immersed is received into a gauge vessel. For this purpose, and especially for adjusting the hydrometer, it is well if the engineer can erect a proper hydraulic observatory, consisting of a vessel of efflux, a gauge reservoir, and a channel of communication between the two. With such an arrangement, we may impart to the water any arbitrary velocity, because we can not only regulate the entrance into the channel, but also the motion by means of boards placed in at pleasure. During observations we must keep the tachometer at different parts of the transverse section of the channel, measure the depth of this section by a scale, and, lastly, gauge the water running through in a definite time, in the lower reservoir (§ 372). We obtain the area F of the transverse profile by multiplication of the mean depth with the mean breadth, and the quantity of water Q is found from the mean transverse section G of the gauge measure, and the height (s) of the quantity which has flowed in during the time by the formula $Q = \frac{Gs}{t}$; but the mean velocity of the water: $v = \frac{Q}{F} = \frac{Gs}{Ft}$ follows from Q and F.

The corresponding number of revolutions u of the wheel is the mean of all the revolutions which are obtained when the instrument is immersed at different breadths and depths of the measured profile.

If from a series of experiments we have found the mean velocities v_1, v_2, v_3, &c., and the corresponding number of revolutions, we then obtain by substitution in the formula $v = v_0 + \alpha\, u$, or in the more correct one: $v = \alpha u + \sqrt{v_0^2 + \beta u^2}$ as many equations of condition for the constants v_0, α, β, as there have been observations made, and we may from these find the constants, if these equations are divided into as many groups as there are unknown constants, and these added together for as many equations of condition as are requisite for determining v_0, α, and also β when required.

Remark. If we adopt the more simple formula with 2 constants, we may then, after the method of least squares, put:

$$v_0 = \frac{\Sigma(y)^2\,\Sigma(x) - \Sigma(xy)\,\Sigma(y)}{\Sigma(x^2)\,\Sigma(y^2) - [\Sigma(xy)]^2} \text{ and } \alpha = \frac{\Sigma(x^2)\,\Sigma(y) - \Sigma(xy)\,\Sigma(x)}{\Sigma(x^2)\,\Sigma(y^2) - [\Sigma(xy)]^2},$$

where $x = \frac{1}{v}$ and $y = \frac{u}{v}$, and the sign Σ represents the sum of all successive similar values, for example, $\Sigma\,(x) = \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \cdots$,

$$\Sigma\,(xy) = \frac{1}{v_1} \cdot \frac{u_1}{v_1} + \frac{1}{v_2} \cdot \frac{u_2}{v_2} + \frac{1}{v_3} \cdot \frac{u_3}{v_3} + \cdots$$

Example. For a small tachometer the velocities are: 0,163; 0,205; 0,298; 0,366; 0,610 metres, the number of revolutions per second: 0,600; 0,835; 1,467; 1,805; 3,142 required to determine the constants corresponding to this wheel. From the formula given in the remark it follows, that:

$$\Sigma\,(x) = \frac{1}{0{,}163} + \frac{1}{0{,}205} + \ldots = 18{,}740,\ \Sigma\,(y) = \frac{0{,}600}{0{,}163} + \ldots = 22{,}759$$

$$\Sigma\,(x^2) = \left(\frac{1}{0{,}163}\right)^2 + \left(\frac{1}{0{,}205}\right)^2 + \ldots = 82{,}846,\ \Sigma\,(y^2) = 105{,}223, \text{ and}$$

$$\Sigma\,(xy) = \frac{0{,}600}{(0{,}163)^2} + \frac{0{,}835}{(0{,}205)^2} + \ldots = 80{,}961,$$

$$v_0 = \frac{105,223\,.\,18,740 - 80,961\,.\,22,759}{82,846\,.\,105,223 - (80,961)^2} = \frac{129,5}{2162} = 0,060 \text{ and}$$

$\alpha = \frac{368.3}{2162} = 0,1703$, hence for this instrument the formula $v = 0,060 + 0,1703\,u$.

If in this we put $u = 0,6$, we then obtain:

$v = 0,060 + 0,102 = 0,162$; further, $u = 0,835$,
$v = 0,060 + 0,142 = 0,202$; further, $u = 1,467$,
$v = 0,060 + 0,249 = 0,309$. $u = 1,805$,
$v = 0,060 + 0,307 = 0,367$; lastly $u = 3,142$,
$v = 0,060 + 0,535 = 0,595$;

therefore, the calculated values agree very well with the observed.

§ 380. *Pitot's Tube.*—Other hydrometers are not so satisfactory as the tachometer, for they either admit of less accuracy, or they are more complicated in their use. The most simple instrument of this kind is *Pitot's tube*. In its simplest form it consists of a bent glass tube *ABC*, Fig. 518, which is held in the water in such a manner that its lower part stands horizontally, and is opposed to the water. By the percussion of the water, a column of water is sustained in this tube, which stands above the level of the exterior fluid surface, and the elevation *DE* of this column is greater, the greater the percussion or the velocity of the water generating it; this elevation or difference of level may hence serve inversely for a measure of the velocity of the water. Let this elevation *DE* above the external surface of water $= h$, and the velocity $= v$, then $h = \frac{v^2}{2g\mu^2}$, where μ is a number derived from experiment, and we have inversely, $v = \mu\sqrt{2gh}$, or more simply: $v = \psi\sqrt{h}$. To find the constant ψ, the instrument is immersed at a place in the water where the velocity v_1 is known; if the elevation is here $= h_1$, we then have the constant $\psi = \frac{v_1}{\sqrt{h_1}}$, which is to be applied in other cases, where the velocity is to be determined with this instrument.

Fig. 518.

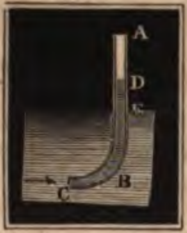
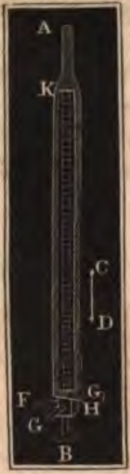


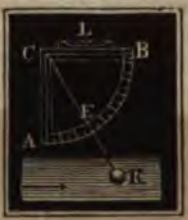
Fig. 519.



To facilitate the reading off of the height h, the instrument consists of two tubes, as shown in Fig. 519, and from the one a small tube F is directed against the stream, from the other two small tubes G and G_1 at right angles to the direction of the stream, both tubes are connected with a single cock H, by which the water can be retained in them. When the instrument is drawn out of the water, we may conveniently read off on a scale attached to both the tubes, the difference $CD = h$ of the two columns of water. That the water in the tube may not oscillate much, it is necessary to make the exterior orifices of the tubes narrow, and that the closing of them may take place quickly and safely; the cock is provided with an arm and an even rod *HK*, which terminates above, near the handle of the instrument.

§ 381. *Hydrometric Pendulum.*—The hydrometric pendulum has been used in preference by Ximenes, Michelotti, Gerstner and Eytelwein for the measurement of the velocity of running water. This instrument consists of a quadrant *AB*, Fig. 520, divided into degrees and smaller parts, and a metallic or ivory ball *K* of from two to three inches diameter, suspended by a thread from the centre *C*, the velocity of the water is given by the angle *ACE*, at which the thread when stretched by the ball deviates from the vertical, when the plane of the instrument is brought into the direction of the stream, and the ball submerged in the water. As the angle rarely amounts to forty or more degrees, this instrument has often the form of a right angled triangle given to it, and the divisions made on its horizontal arm. For the placing of the index or zero line in the vertical, it is best to use a spirit level on the horizontal arm of the instrument, or the ball itself may serve for this purpose, by letting it be suspended out of the water, and the instrument revolve until the thread coincides with the zero line of the division.

Fig. 520.



For velocities under four feet we may use the ivory ball, but for greater velocities the hollow metal ball. On account of the constant undulations of the ball in the direction of the motion of the water, as also at right angles to the direction of the current, the reading off is somewhat difficult, and leaves a good deal of uncertainty, for which reason this instrument cannot be relied upon for the more exact numbers.

The dependence between the angle of deviation and the velocity of the water may be determined in the following manner when the ball is not very deeply immersed. From the weight G of the ball and from the impulse of the water $P = \mu\, Fv^2$, increasing simultaneously with the square of the velocity v and the section of the ball F, the resultant R, whose direction the thread assumes, follows, and is determined by the angle of deviation β, for which the *tang.* $\beta = \frac{P}{G} = \frac{\mu\, Fv^2}{G}$, hence also inversely:

$$v^2 = \frac{G\, tang.\, \beta}{\mu\, F}, \text{ and } v = \sqrt{\frac{G}{\mu F}} \cdot \sqrt{tang.\, \beta}, \text{ i. e. } v = \psi \sqrt{tang.\, \beta},$$

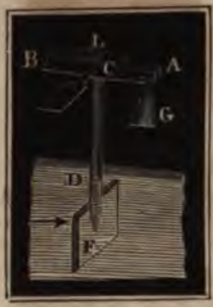
if ψ represents a co-efficient derived from experiment, which must be obtained before use, according to the above-mentioned instructions.

§ 382. *Rheometer.*—The remaining hydrometers, such as Lorgna's water lever, Ximenes's water vane, Michelotti's hydraulic balance, Brünning's tachometer, Poletti's rheometer, are more complicated in their use, and not altogether to be relied on. The principle of all these instruments is the same, they are composed of a surface of impulse and a balance, and the last serves for the purpose of giving the percussion P of the water against the former, but since this $= \mu Fv^2$, we then have inversely:

$v = \sqrt{\frac{P}{\mu F}} = \psi \sqrt{P}$, where ψ denotes a constant deduced from experiment dependent on the magnitude of the surface of impact F.

The *rheometer*, which was lately proposed by Poletti, and does not materially differ from the hydrometric balance of Michelotti, consists of a lever AB, Fig. 521, turning about a fixed axis C, and an arm CD to which the surface of impulse, or according to Poletti, a mere impulse-staff is screwed. To maintain equilibrium with the percussion of the water against the surface, the boxes suspended at the extremity A of the lever are loaded with weight or shot, and to put the empty balance in equilibrium in still water, a counterpoise is placed at B, which makes up the outermost end of the arm CB. From the weight put on G, the impulse P is found by means of the arm $CA = a$ and $CF = b$ from the formula $Pb = Ga$, whence, therefore,

Fig. 521.



$$P = \frac{a}{b} G, \text{ and } v = \sqrt{\frac{P}{\mu F}} = \sqrt{\frac{a\, G}{\mu\, b\, F}} = \psi \sqrt{G},$$

where ψ is a constant derived from experiment.

Remark. With respect to the last hydrometer, ample details will be found in Eytelwein's "Handbuch der Mechanik fester Korper und der Hydraulik;" further, in Gerstner's "Handbuch der Mechanik," vol. 2; in Brünning's "Treatise on the velocity of running water;" in Venturoli's "Elementi di Meccanica e d'Idraulica," vol. 2. Concerning Poletti's hydrometer, we must refer to Dingler's "Polytechn. Journal," vol. 20, 1826. The hydrometer described in Stevenson's treatise on Marine Surveying and Hydrometry is the tachometer of Woltmann, *see* Dingler's "Journal," vol. 65, 1842.

CHAPTER IX.

ON THE IMPULSE AND RESISTANCE OF FLUIDS.

§ 383. *Impulse and Resistance of Water.*—Water or any other fluid imparts a shock to a rigid body, when it meets it in such a manner that its condition of motion is thereby altered. The resistance which water opposes to the motion of a body, does not essentially differ from impulse. The investigation of these two forms the third principal division of hydraulics. We distinguish from each other:

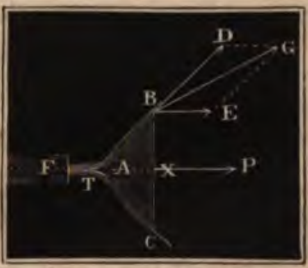
1. The impulse of an isolated stream.
2. The impulse of a limited stream.
3. The impulse of an unlimited stream.

An impulse of the first kind takes place when a body, for instance, the float board of an over-shot water-wheel, is opposed to a stream of water issuing from a reservoir; an impulse of the second kind occurs where water, in a canal or in a water-course, impinges against a body which entirely fills up its transverse section, as for instance, against the float board of an under-shot wheel; the third kind, lastly, presents itself, when running water strikes against a body immersed in it, whose transverse section is only a very small part of that of the current of water, as, for instance, against the float boards of a floating mill-wheel.

We must distinguish the impulse of water against a body at rest and against a body in motion, and further, the impulse against a curved and against a plane surface, and in this last again, between the perpendicular and the oblique impulse.

Let us consider at once the general case, namely, the impulse of an isolated stream against a surface of rotation which moves in its proper axis, and in the direction of motion of the stream.

Fig. 522.



§ 384. *Impact of Isolated Streams.*—Let BAC, Fig. 522, be a surface of rotation, AX its axis, and FA a fluid stream meeting it in this direction. Let the velocity of the water $= c$, that of the surface $= v$, and the angle BTX, which the tangent DT at the extremity B of the generating curve or of each of the filaments of water BT leaving the surface, includes with the direction of the axis $BE = \alpha$; lastly, let us further assume that the water in running off from the surface loses nothing in *vis viva* by friction. The water strikes against the surface with the relative velocity $c—v$, and hence leaves the surface with this, and therefore quits it in the tangential directions TB, TC, &c. From the tangential velocity $BD = c - v$, and the velocity of the axis $BE = v$, the absolute velocity $BG = c_1$ of the water after impinging against the surface is found by the known formula:

$$c_1 = \sqrt{(c-v)^2 + 2(c-v)\,v \cos.\alpha + v^2}.$$

But now a quantity of water Q is able to produce by virtue of its *vis viva* the mechanical effect $\frac{c^2}{2g} \cdot Q\gamma$, if its velocity c is fully imparted; accordingly the residuary effect of the water:

$= \frac{c_1^2}{2g} \cdot Q\gamma$; consequently the mechanical effect distributed over the surface is:

$$Pv = \frac{c^2}{2g} Q\gamma - \frac{c_1^2}{2g} Q\gamma = \frac{c^2 - c_1^2}{2g} \cdot Q\gamma.$$

$$= \frac{[c^2-(c-v)^2-2(c-v)\,v \,.\, \cos.\alpha-v^2]}{2g}\, Q\gamma$$

$$= \frac{2\,c\,v - 2\,v^2 - 2\,(c-v)\,v\,cos.\,\alpha}{2g}\,Q\gamma,\ i.e.$$

$$Pv = (1 - cos.\,\alpha)\,\frac{(c-v)\,v}{g}\,Q\gamma,$$

and the force or the impulse of the water in the direction of its axis is:

$$P = (1 - cos.\,\alpha)\,\frac{(c-v)}{g}\,Q\gamma.$$

If the surface meets the water with the velocity v, we then have:

$$P = (1 - cos.\,\alpha)\,.\,\frac{(c+v)}{g}\,Q\gamma,$$

and if this is without motion, therefore, $v = 0$, the impulse or hydraulic pressure of the axis comes out:

$$P = (1 - cos.\,\alpha)\,\frac{c}{g}\,.\,Q\gamma.$$

It follows from this, that the impulse of one and the same mass of water under otherwise similar circumstances is proportional to the relative velocity $c \mp v$ *of the water.*

From the area F of the transverse section of the fluid stream, it follows that the quantity discharged is $Q = Fc$; hence

$$P = (1 - cos.\,\alpha)\,\frac{(c \mp v)\,c}{g}\,F\gamma;$$

and for $v = 0$:

$$P = (1 - cos.\,\alpha)\,\frac{c^2}{g}\,F\gamma.$$

For an equal transverse section of the stream, the impulse against a surface at rest increases therefore as the square of the velocity of the water.

§ 385. *Impulse against Plane Surfaces.*—The impulse of one and the same fluid stream depends principally on the angle α, under which the water, after the impulse, leaves the axis; it is nothing if this angle $= 0$; and, on the other hand, a maximum, namely, $= 2\,\frac{(c \mp v)}{g}\,Q\,\gamma$, if this angle is 180°, therefore its cosine $= -1$, where the water, as represented in Fig. 523, leaves the surface in a direction opposite to that in which it impinges. This is generally greater for concave surfaces than for convex, because the angle is there

Fig. 523.

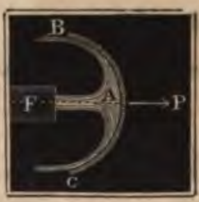
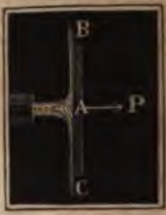


Fig. 524.



oblique, therefore the cosine negative and $1 - \cos. \alpha$ becomes $1 + \cos. \alpha$.

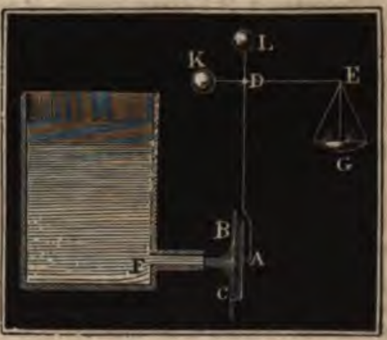
Most frequently the surface, as represented in Fig. 524, is plane, and hence $\alpha = 90°$, therefore $\cos. \alpha = 0$, and the impulse $P = \frac{(c \mp v)}{g} \,.\, Q\gamma$; for a surface at rest:

$$P = \frac{c}{g}\, Q\gamma = \frac{c^2}{2g}\, F\gamma = 2 \,.\, \frac{c^2}{2g}\, F\gamma = 2\, F\, h\, \gamma.$$

The normal impulse of water against a plane surface is therefore equivalent to the weight of a column of water which has for base the transverse section F of the stream, and for altitude, twice the height due to the velocity $2\, h = 2 \,.\, \frac{c^2}{2g}$.

The experiments made on this subject by Michelotti, Vince, Langsdorf, Bossut, Morosi, and Bidone, have nearly led to the same results when the transverse section of the impinged surface was at least six times as great as that of the stream, and when this surface was twice as far from the plane of the orifice as the thickness of the stream. The apparatus which was used for this purpose consisted of a lever, similar to that of Poletti's rheometer, which received upon one side the impulse of the water, and whilst its other side was kept in equilibrium by weights. The instrument which Bidone made use of is represented in Fig. 525. *BC* is the surface impinged on by the stream *FA*, *G* is the scale-pan for the reception of the weights, *D* the axis of rotation, *KL* counter-weights.*

Fig. 525.



§ 386. *Maximum Effect of Impulse.*—The mechanical effect of impulse:

$$Pv = (1 - \cos. \alpha)\, \frac{(c - v)\, v}{g}\, Q\, \gamma$$

depends principally on the velocity v of the impinged surface; it is, for example, nothing, not only for $v = c$, but also for $v = 0$; hence there is a velocity for which the effect of the impulse is a maximum. It is manifest that it only depends on $(c-v)\, v$ becoming a maximum. If we consider c as half the perimeter of a rectangle, and v as its base, we have then its height $= c-v$ and its area $= (c-v)\, v$. But of all rectangles the square is that which has for a given perimeter $2\, c$ the greatest area, hence also $(c-v)\, v$ is a maximum, when $c-v = v$, i. e. $v = \frac{c}{2}$, *and we therefore obtain the maximum value of the mechanical effect of the impulse when the surface moves from it with half the velocity of the water, and indeed*

---

* The latest and most extensive experiments on the percussion of water are those of Bidone. *See* "Memorie de la Reale Accademia delle Scienze di Torino," vol. 40, 1838. They were performed with a velocity of at least 27 feet, and on brass plates of from 2 to 9 inches diameter. In general, Bidone found that the normal impulse against a plane surface was somewhat greater than $2\, F\, h\, \gamma$, yet this variation is perhaps to be attributed to an augmentation of the leverage which is produced by the falling back of the water. *See* Duchemin's "Recherches expérim. sur les lois de la résistance des fluides." When the impinged surface was quite near the orifice, Bidone found that P was only $1{,}5\, F\, h\, \gamma$; when, further, the surface had a transverse section equal to that of the stream, in which case the water only deviated by an acute angle α, then, after Du Buat and Langsdorf, P was only $= F\, h\, \gamma$. Lastly, it has been deduced by Bidone and others that the impulse is in the first moment nearly as great again as the permanent impulse.

$$Pv = (1 - cos.\, \alpha) \cdot \tfrac{1}{2} \cdot \frac{c^2}{2g} \cdot Q\gamma = (1 - cos.\, \alpha) \cdot \tfrac{1}{2}\, Qh\gamma.$$

If now $\alpha = 180°$, and if, therefore, the motion of the water be reversed by the impulse, we then have the effect equal to $2 \cdot \frac{1}{2}\, Qh\gamma = Qh\gamma$. But if $\alpha = 90°$, *i. e.* if it impinges against a plane surface, this effect is then only $\frac{1}{2}\, Qh\gamma$, therefore, in the last case, the half only of the whole disposable effect, or that which corresponds to the ***vis viva*** of the water, is gained or brought to bear upon the surface.

Examples.—1. If a stream of water, of 40 square inches transverse section, delivers a quantity of 5 cubic feet per second, and strikes normally against a plane surface, and escapes with a 12 feet velocity, the effect of impulse is then:
$P = \frac{(c-v)}{g}\, Q\gamma = \left(\frac{5\, .\, 144}{40} - 12\right) .\, 0{,}031\, .\, 5\, .\, 62.5 = 6\, .\, 0{,}031\, .\, 312{,}5 = 58{,}12$ lbs., and the mechanical effect brought to bear upon the surface $Pv = 58{,}12 \times 12 = 697{,}44$ ft. lbs. The greatest effect is for $v = \frac{c}{2} = \frac{1}{2} \cdot \frac{5\, .\, 144}{40} = 9$ feet, and indeed:
$= \frac{1}{2} \cdot \frac{c^2}{2g} \cdot Q\,\gamma = \frac{1}{2}\, .\, 18^2\, .\, 0{,}0155\, .\, 5\, .\, 62{,}5 = 81\, .\, 0{,}0155\, .\, 62{,}5 = 784{,}68$ ft. lbs.; the corresponding impulse, or hydraulic pressure $= \frac{784{,}68}{9} = 87{,}18$ lbs.—2. If a stream FA, Fig. 526, of 64 square inches section, strikes with a 40 feet velocity against an immovable cone, having an angle of convergence $BAC = 100°$, then is the hydraulic pressure in the direction of the stream:

$$P = (1 - cos.\, \alpha)\, \frac{c}{g}\, Q\,\gamma = (1 - cos.\, 50°)\, 40\, .\, 0{,}031\, .\, \frac{64}{144}\, .\, 40\, .\, 62{,}5$$
$$= (1 - 0{,}64279)\, .\, 1\, .\, 24\, .\, \frac{10000}{9} = 0{,}35721\, .\, 1377{,}7 = 492\, .\, 13 \text{ lbs.}$$

Fig. 526.

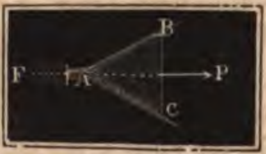
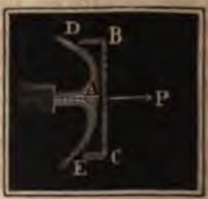


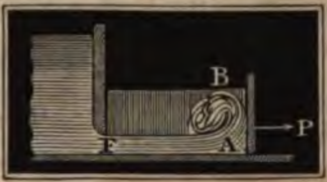
Fig. 527.



§ 387. *Impulse of a Limited Stream.*—If we add borders BD, CE, to the perimeter of a plane surface BE, Fig. 527, which project from the side impinged upon by the water, then will the water deviate from its direction at an obtuse angle, in a similar manner as from concave surfaces, and hence the impulse will be greater than for plane surfaces. The effect of this impulse depends principally on the height of the border and the ratio of the transverse section between the stream and the part confined. In an experiment, where the stream was 1 inch thick, the cylindrical enclosure 3 inches wide and 3½ lines deep, the water ran off almost in a reversed direction, and the impulse amounted to $3{,}93 \, \frac{c^2}{2g} F \gamma$; in every other case this force was less. In consequence of the friction of the water at the surface and the sides, the theoretical maximum value never reaches $4 \, \frac{c^2}{2g} F \gamma$.

In the impulse of a limited stream *FAB*, Fig. 528, a rising at the edges takes place; this rising occupies only a portion of the perimeter, and extends itself, on the other hand, simultaneously to the impinged surface and the fluid stream. The impinging water takes the direction of the unbordered portion of the perimeter, and here, therefore, becomes deflected 90 degrees, whence the formula above found for the isolated stream $P = \frac{(c-v)}{g} Q \gamma$ holds good; yet this may also be deduced in the following manner. If we assume that the velocity c of the arriving water by the impulse against its surface is changed into the velocity v of the surface, we may then also assume that a loss of mechanical effect $\frac{(c-v)^2}{2g} Q \gamma$ (similar to that in § 337), expended in the division of the water, is connected with it. But now the effect due to the *vis viva* of the arriving water $= \frac{c^2}{2g} Q \gamma$ and to that of the water going on $= \frac{v^2}{2g} Q \gamma$, hence it follows that the mechanical effect imparted to the surface is:

Fig. 528.



$$Pv = [c^2 - (c-v)^2 - v^2] \frac{1}{2g} Q\gamma = \frac{(c-v)\,v}{g} Q\gamma.^*$$

§ 388. *Oblique Impulse.*—In oblique impulse against a plane surface, we must distinguish whether the water flows away in one, two, or in all directions in the plane. If, as in the impact of limited water, the surface *AB*, Fig. 529, is confined at three sides, so that the water can run off only in one direction, we have then the hydraulic pressure against the surface in the direction of the stream $P = (1 - \cos. \alpha) \frac{(c-v)}{g} Q \gamma$.

---

* This formula will be found applicable hereafter, when we come to the theory of water-wheels.

Fig. 529.

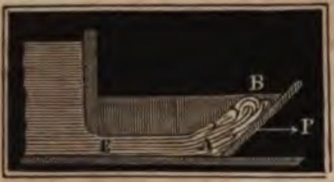
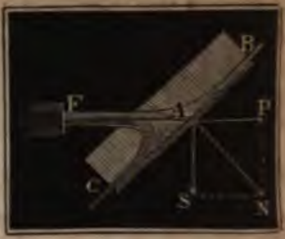


Fig. 530.



But if the impinged plane *BC*, Fig. 530, is only bordered on two oppositely situated sides, the stream then divides itself into two unequal portions; the greater portion Q_1 takes the small deflexion α, and the lesser Q_2, the greater deflexion $180 - \alpha$; hence, the whole impulse in the direction of the stream is:

$$P = (1 - cos.\, \alpha) \cdot \frac{c - v}{g} Q_1 \gamma + (1 + cos.\, \alpha) \cdot \frac{c - v}{g} Q_2 \gamma = \left(\frac{c - v}{g}\right) \gamma \left[(1 - cos.\, \alpha)\, Q_1 + (1 + cos.\, \alpha)\, Q_2\right].$$

Now the equilibrium of the two portions of the stream requires that the pressures

$$\frac{(c - v)}{g} \gamma (1 - cos.\, \alpha)\, Q_1 \text{ and } \frac{(c - v)}{g} \gamma (1 + cos.\, \alpha)\, Q_2$$

between them should be equal; hence, also:

$$(1 - cos.\, \alpha)\, Q_1 = (1 + cos.\, \alpha)\, Q_2, \text{ or since } Q_1 + Q_2 = Q,$$

$$(1 - cos.\, \alpha)\, Q_1 = (1 + cos.\, \alpha)\,(Q - Q_1), \text{ i. e.,}$$

$$Q_1 = \left(\frac{(1 + cos.\, \alpha)}{2}\right) Q, \text{ and } Q_2 = \left(\frac{1 - cos.\, \alpha}{2}\right) Q,$$

so that the whole impulse in the direction of the stream is finally:

$$P = \frac{(c - v)}{g} \gamma \,.\, 2\,(1 - cos.\, \alpha) \frac{(1 + cos.\, \alpha)\, Q}{2} = \frac{(c - v)\gamma}{g} (1 - cos.\, \alpha^2)\, Q,$$

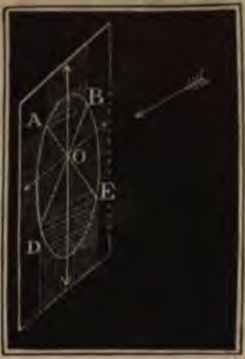
i. e., $P = \frac{c - v}{g}\, sin.\, \alpha^2 \,.\, Q \gamma.$

Besides the *parallel impulse* P, acting in the direction of the stream, we distinguish, further, the *lateral impulse* S, acting at right angles to the direction of the stream, and the *normal impulse* N, composed of these two, and at right angles to the surface. In every case $P = N\, sin.\, \alpha$, and $S = N\, cos.\, \alpha$; hence, inversely,

$$N = \frac{P}{sin.\, \alpha} = \frac{c - v}{2g}\, sin.\, \alpha\; Q\gamma \text{ and } S = \frac{c - v}{2g}\, sin.\, 2\,\alpha \,.\, Q\gamma.$$

The normal impulse, therefore, increases as the sine, the parallel impulse as the square of the sine of the angle of incidence, and the lateral impulse as double the same angle. Lastly, if the inclined surface impinged on is not bordered, then the water can spread over it in all directions; the impulse is then greater, because of all the angles by which the filaments of water are deflected, α is the least; and hence, each filament which does not move in the normal plane, exerts a greater pressure than the filament in this plane. Let us assume tha a portion Q_1 corresponding to the sectors *AOB* and *DOE*, Fig. 531, is deflected by the angles α and $180° - \alpha$, and another Q_2, corresponding to the sectors *AOD* and *BOE*, by 90°, and that both portions exert a parallel impulse, we may then put:

Fig. 531.



$P = \frac{c - v}{g} Q_1 \gamma \sin. \alpha^2 + \frac{c - v}{g} Q_2 \gamma$, $Q_1 \sin. \alpha^2 = Q_2$, and $Q_1 + Q_2 = Q$; hence it follows, that $Q_1 (1 + \sin. \alpha^2) = Q$, and the whole parallel impulse $P =$

$$\left(\frac{c - v}{g}\right) \frac{2 Q \gamma \sin. \alpha^2}{1 + \sin. \alpha^2} = \frac{2 \sin. \alpha^2}{1 + \sin. \alpha^2} \cdot \frac{c - v}{g} \cdot Q \gamma.$$

Although this hypothesis is only approximately correct, it tolerably well agrees, nevertheless, with the latest experiments of Bidone.

§ 389. *Action of an Unlimited Stream.*—If a body moves progressively in an unlimited fluid, or if a body is put into a fluid which is in motion, it then suffers a pressure which is dependent on the form and dimensions of this body, as well as on the density and on the velocity of the one or the other mass, and in the one case is called the *resistance*, and in the other the *impulse* of the fluid. This hydraulic pressure arises principally from the inertia of the water, whose condition of motion is altered by striking against the solid body, and also, further, from the force of cohesion of the particles of water, which are hereby partially separated from one another, or pushed aside. If a body *AC* moves against running water, Fig. 532, it pushes away before it a certain quantity with an augmented pressure. Whilst this mass of water, by the further advance of the body, always increases on the one side, on the other a constant flowing away takes place, while the particles lying near the anterior surface assume a motion in the direction of this surface. If the moving mass of water strikes against a body at rest, Fig. 533, then is there likewise an increased pressure produced in front of it, which causes the particles before the body to deviate from their original direction, and to run off at the surface *AB*. When these particles have reached the limits of the surface, they then turn and flow away by the lateral surfaces until they come to the back, when they then again immediately unite, but

Fig. 532.

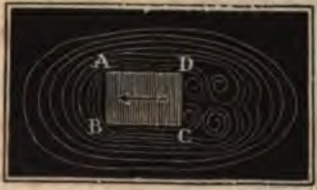
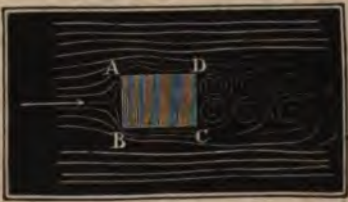


Fig. 533.



assume an eddying motion. It is manifest that the general circumstances of motion of the particles surrounding the body are the same in the impact of moving water as in the resistance of a body moving in water, except that in the eddies a difference so far takes place, that with short bodies the eddy in the latter case occupies a less space than in the former. In both cases the velocity of the particles increases more and more from the middle of the anterior surface to its limits, attains its maximum at the commencement of the lateral surfaces, where, for the most part, a contraction takes place, gradually diminishes in the water which passes away at the sides, and lastly, attains its minimum when the water reaches the back and passes into a whirling motion.

§ 390. *Theory of Impulse and Resistance.*—The normal pressure varies at different points of the body; it is greatest at the middle of the anterior, and least at the middle of the posterior surface, and, next to that, at the parts of the sides nearest this; because, in respect to the body, there is at the one place rather a flow to, and at the other a flow from these surfaces. If the body be symmetrical, as we shall suppose it to be, with respect to the direction of motion, then the aggregate pressures in this direction counteract each other, and hence only the pressures in the direction of motion are to be taken into account. But now the pressures on the posterior surface are opposed to those on the anterior; hence the *resultant impulse or resistance of the water may be equated to the difference of pressure of the anterior and posterior surfaces.*

If we cannot assign the amount of these pressures *à priori*, we may, nevertheless, from the great similarity of the circumstances to the impulse of isolated streams, assume that at least the general law for the impulse of unlimited water does not differ from that of the impulse of isolated streams. If, therefore, F is the area of a surface, which is impinged on by an unlimited current whose density is γ, with a velocity v, then the corresponding impulse or hydraulic pressure may be put $P = \zeta \, \frac{v^2}{2g} F\gamma$, where ζ represents a number deduced from experiment, dependent on the form of the surface. But this expression is not only applicable to action against the anterior, but also to that against the posterior surface, only that in this last, when the water has a tendency to flow away, it consists of a draught or negative pressure. If now $Fh\gamma$ is the hydrostatic pressure (§ 276) against the front and back surface of a body, the whole pressure against the front is: $P_1 = Fh\gamma + \zeta_1 \cdot \frac{v^2}{2g} F\gamma$, and that against the back: $P_2 = Fh\gamma - \zeta_2 \cdot \frac{v^2}{2g} F\gamma$, and the resultant impulse or resistance of the water is then found:

$P = P_1 - P_2 = (\zeta_1 + \zeta_2) \cdot \frac{v^2}{2g} F\gamma = \zeta \cdot \frac{v^2}{2g} F\gamma$, if $\zeta_1 + \zeta_2 = \zeta$. This general formula for the impulse of unlimited water is applicable to the percussion of the wind or to the resistance of the air. Besides the difference of aërodynamic pressure at the front and back, there is further a difference of aërostatic pressure, because the air in front, in consequence of its greater elasticity, has a greater density (γ) than that at the back. For this reason, in high velocities, as those of cannon-balls, the co-efficient of the resistance of air is greater than that of water.

Remark.—The adhesion of a certain quantity of air or water to the body, is a peculiar phenomenon of the impulse or resistance of an unlimited medium (water or air), whose influence is particularly remarkable in the variable motion of bodies, as, for example, in the oscillations of the pendulum. For a ball, the air or water adhering to the moving body is equal to 0,6 of the volume of the ball. For a prismatic body moved in the direction of its axis, the ratio of this volume $= 0{,}13 + 0{,}705 \frac{\sqrt{F}}{l}$, where l is the length, and F the transverse section of the body. These relations, discovered by Du Buat, have been fully confirmed by the later observations of Bessel, Sabine, and Baily.

§ 391. *Impulse and Resistance against Surfaces.*—The co-efficient of resistance ζ, or the number with which the height due to the velocity is to be multiplied to obtain the height of a column of water measuring this hydraulic pressure, varies for bodies of different figures, and only for plates which are at right angles to the direction of motion is it nearly a definite quantity. According to the experiments of Du Buat and those of Thibault, we may put $\zeta = 1{,}85$ for the impulse of air or water against a plane surface at rest, and, on the other hand, assume, but with less accuracy, for the resistance of air or water against a surface in motion $\zeta = 1{,}40$. In both cases, about two-thirds of the whole effect are expended on the front, and one-third on the back. The resistance which the air opposes to a surface revolving in a circle, has been found by Borda, Hutton, and Thibault to vary a good deal, but may be expressed by a mean of $\zeta = 1{,}5$. If the surface does not stand at right angles to the direction of the motion, but makes with it an acute angle α, we may then, with Duchemin, substitute for ζ, $\frac{2\,\zeta\, sin.\, \alpha^2}{1 + sin.\, \alpha^2}$ with tolerable correctness.

The impulse and resistance of unlimited media are also augmented when the surfaces are hollowed out or have projecting edges at their perimeters, but we have arrived at no general results on this subject.

Example. If the wind impinges with a 20 feet velocity against a firmly fixed wind-mill wheel, which consists of four wings, of which each has an area of 200 square feet and 75° inclination to the direction of the wind, then is the impinging force of the wind in its direction, or in that of the axis of the wheel:

$$P = 1{,}85 \,.\, \frac{2\,(sin.\, 75)^2}{1 + (sin.\, 75)^2} \,.\, \frac{20^2}{2g} \,.\, 4 \,.\, 200 \,.\, 0{,}081 = 1{,}85 \,.\, 0{,}965 \,.\, 6{,}21 \,.\, 800 \,.\, 0{,}081$$

$= 718{,}4$ ft. lbs., when the density of the wind is (from § 301) taken at 0,081 lbs.

Remark. Views, with respect to the impulse and resistance of unlimited fluids, entirely at variance with these, are put forward in the above-mentioned work of Duchemin. It is there maintained, for instance, that the impulse and resistance against the front surface of a thin plate amounts to $2 \,.\, \frac{v^2}{2g} F\, h$, and is not negative at the back, that the impulse $= 0{,}136 \frac{v^2}{2g} F\gamma$, and the resistance $= 0{,}746 \frac{v^2}{2g} F\gamma$. It would be too circumstantial here to give a detail of the reasons why the author cannot agree with the views of Duchemin, but more with reference to this will be found in Poncelet's "Introduction à la mécanique industrielle," 2d edition, 1841.

§ 392. *Impulse and Resistance to Bodies.*—The impulse and resistance of water to prismatic bodies, whose axis coincides with the direction of motion, diminishes when the length of the body is considerable. From the experiments of Du Buat and Duchemin, the impulse of the front surface is invariable, and only the effect against the back surface variable. To this corresponds the co-efficient $\zeta_1 = 1{,}186$, for the total effect, however, with the relative lengths

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$$\zeta = 1{,}86;\ 1{,}47;\ 1{,}35;\ 1{,}33.$$

For still greater ratios between the length l and the mean breadth $\sqrt{F}$ of the body, ζ diminishes, owing to the friction of the water at the lateral surfaces of the body. From the resistance of the water, reverse relations take place. Here, from Du Buat, for the effect on the front surface, $\zeta_1 = 1$ invariably; for the total effect, however, with

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$\zeta = 1{,}25;\ 1{,}28;\ 1{,}31;\ 1{,}33$, so that, for a prism which is 3 times as long as broad, the impulse is the same as the resistance.

The experiments undertaken by Borda, Hutton, Vince, Desaguilliers, Newton, and others, with angular and with round bodies, leave still much uncertainty. In what relates to spheres, it appears that for moderate velocities the mean co-efficient for motion in air or water = 0,6. For a greater velocity and for motion in air, according to Robins and Hutton, for the velocities

$v = 1, \quad 5, \quad 25, \quad 100, \quad 200, \quad 300, \quad 400, \quad 500, \quad 600$ metr.
$\zeta = 0{,}59;\ 0{,}63;\ 0{,}67;\ 0{,}71;\ 0{,}77;\ 0{,}88;\ 0{,}99;\ 1{,}04;\ 1{,}10.$

Duchemin and Piobert have given particular formulæ for the rate of increase of these co-efficients.

For the impulse of water against a sphere, Eytelwein found $\zeta = 0{,}7886$.[*]

Example. If, according to Borda, we put the resistance and impact at right angles to the axis of a cylinder at half as great as that against a parallelopiped which has the same dimensions, we then obtain for the resistance $\zeta = \frac{1}{2} \,.\, 1{,}28 = 0{,}64$ and the impact $= \frac{1}{2} \,.\, 1{,}47 = 0{,}735$. If we apply these values to the human body, whose section has an area of some 7 square feet, we then find for the resistance and impulse of air against it, the values:

$P = 0{,}64 \,.\, 0{,}0155 \,.\, 7 \,.\, 0{,}081\, v^2 = 0{,}00562\, v^2$, and

$P = 0{,}735 \,.\, 0{,}0155 \,.\, 7 \,.\, 0{,}081\, v^2 = 0{,}00646\, v^2$. Hence the resistance of air for a velocity of 5 feet is only $0{,}00562 \,.\, 25 = 0{,}1405$ lbs.; and the corresponding mechanical effect per second $= 5 \,.\, 0{,}1405 = 0{,}70$ ft. lbs.; for a velocity of 10 feet this resistance is four times, and the effect expended eight times as great, and for a velocity of 15 feet, the resistance is 9 times and the effect 27 times as great as for a 5 feet velocity. If a man, with a 5 feet velocity, moves against wind having a 50 feet velocity, he has then a resistance $0{,}00646 \,.\, 55^2 = 19{,}54$ lbs. to overcome, corresponding to the relative velocity $50 + 5 = 55$ feet, and thereby to produce the mechanical effect of $19{,}54 \,.\, 5 = 97{,}7$ ft. lbs. (English.)

---

[*] Poncelet, in his work above cited, and Duchemin and Thibault in their "Recherches expérimentales," have treated very fully of these circumstances. In the Second Part we shall treat of the resistance to floating bodies, especially to ships, &c., as also the impact of the wind on wheels, &c.

§ 393. ***Motion in Resisting Media.***—The laws of the motion of a body in a resisting medium are rather complex, because we have here to deal with a variable force, *i. e.*, one increasing with the square of the velocity. From the force P_1 which urges the body forward, and from the resistance $P_2 = \zeta \cdot \frac{v^2}{2g} F\gamma$, which the medium opposes to the motion, the motive force is:

$$P = P_1 - P_2 = P_1 - \zeta \cdot \frac{v^2}{2g} F\gamma,$$

but since the mass of the body $= M = \frac{G}{g}$, the accelerating force is:

$$p = \frac{P}{M} = \left(P_1 - \zeta \frac{v^2}{2g} F\gamma\right) + M = \left(\frac{P_1 - \zeta \frac{v^2}{2g} F\gamma}{G}\right) \cdot g,$$

or if we represent $\frac{F\gamma}{2gP_1}$ by $\frac{1}{w^2}$.

$p = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g$. But the velocity v is accelerated in the instant of time τ by $\varkappa = p\,\tau$, hence:

$\varkappa = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g\,\tau$, and inversely:

$$\tau = \frac{G_1}{P_1} \cdot \frac{\varkappa}{g\left[1 - \zeta \left(\frac{v}{w}\right)^2\right]}.$$

Now to find the time corresponding to a given change of velocity, let us divide the difference $v_n - v_0$, of the final and initial velocity into n parts, let any such part $\frac{v_n - v_0}{n} = \varkappa$, and let us calculate the velocities:

$$v_1 = v_0 + \varkappa,\ v_2 = v_0 + 2\,\varkappa,\ v_3 = v_0 + 3\,\varkappa,\ \&c.,$$

and substitute these values in the formula of Simpson. In this manner, by taking four parts we shall obtain the time sought

$$1.\ t = \frac{G}{P_1} \cdot \frac{v_n - v_0}{12g} \left(\frac{1}{1 - \zeta\left(\frac{v_0}{w}\right)^2} + \frac{4}{1 - \zeta\left(\frac{v_1}{w}\right)^2} + \right.$$

$$\left. + \frac{2}{1 - \zeta\left(\frac{v_2}{w}\right)^2} + \frac{4}{1 - \zeta\left(\frac{v_3}{w}\right)^2} + \frac{1}{1 - \zeta\left(\frac{v_4}{w}\right)^2}\right).$$

Further, the small space described in any instant τ (§ 19), is $\sigma = v\,\tau$, or since $\tau = \frac{\varkappa}{p}$, $\sigma = \frac{v\varkappa}{p}$, therefore,

$\sigma = \frac{v\varkappa}{1 - \zeta\left(\frac{v}{w}\right)^2} \cdot \frac{G}{P_1 g}$. By the application of Simpson's rule, we shall now find the space which is described while the velocity v passes into that of v_n.

$$2.\ s = \frac{G}{P_1} \cdot \frac{v_n - v_0}{12\,g} \left(\frac{v_0}{1 - \zeta \left(\frac{v_0}{w}\right)^2} + \frac{4v_1}{1 - \zeta \left(\frac{v_1}{w}\right)^2} + \right.$$

$$\left. + \frac{2v_2}{1 - \zeta \left(\frac{v_2}{w}\right)^2} + \frac{4v_3}{1 - \zeta \left(\frac{v_3}{w}\right)^2} + \frac{v_4}{1 - \zeta \left(\frac{v_4}{w}\right)^2} \right).$$

Of course the accuracy is greater, when we take six, eight, or more parts. This formula takes into account the variability of the co-efficients of resistance, which in considerable velocities is necessary. For the free descent of bodies in air or water $P_1 = G$, and for motion on a horizontal plane $P_1 = 0$, is more correctly equal to the friction $f\,G$. Since this is a resistance, we have then to introduce it as negative into the calculation, whence

$$P = -(P_1 + P_2),\ \text{and}\ p = -\left[1 + \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G} g.$$

As it cannot be a question here of an increase, but only of a diminution of velocity, we have then to substitute in the above formula $v_0 - v_n$ for $v_n - v_0$.

In the case, where the body is urged by a force, by its weight for instance, the motion approximates more and more to a uniform one, so that after the lapse of a certain time, it may be considered as such, although not so in reality. The accelerating force $p = 0$, when

$$\zeta \cdot \frac{v^2}{2g} F \gamma = P_1,\ \text{when, therefore,}\ v = \sqrt{\frac{2g\,P_1}{\zeta\,F\gamma}}.$$

The velocity of a falling body approximates, therefore, to this limit more and more, without ever actually attaining it.

Example. Piobert, Morin, and Didion found, for a parachute whose depth was 0,31 that of the diameter of its opening $\zeta = 1{,}94 \,.\, 1{,}37 = 2{,}66$. Hence, from what height in Prussian feet will a man, of 150 lbs. weight, be able to descend with a similar parachute, of 10 lbs. weight and 60 square feet transverse section, without acquiring a greater velocity than that which he would have acquired by jumping from a 10 feet height, without a parachute? The last velocity is $v = 7{,}906 \sqrt{10} = 25$ feet, the force is $P_1 = G = 150 + 10 = 160$ lbs., the surface $F = 60$ square feet, the density $\gamma = 0{,}0859$, and the co-efficient of resistance $\zeta = 2{,}66$, hence:

$\frac{1}{w^2} = \frac{60 \,.\, 0{,}0859}{62{,}5 \,.\, 160} = 0{,}000515$, and $\zeta \cdot \frac{v^2}{w^2} = 2{,}66 \,.\, 0{,}000515 \,.\, 25^2 = 0{,}85625$. If, therefore, we take 6 parts, we then obtain for these:

$1 - \zeta \cdot \frac{v^2}{w^2} = 0{,}97621;\ 0{,}90486;\ 0{,}78593;\ 0{,}61944;\ 0{,}40537;\ 0{,}14375$, and for

$\frac{v}{1 - \zeta \frac{v^2}{w^2}} = 0;\ 4{,}268;\ 9{,}210;\ 15{,}905;\ 26{,}910;\ 51{,}393$, and $173{,}913$; from Simpson's rule the mean value is:

$= (1 \,.\, 0 + 4 \,.\, 4{,}268 + 2 \,.\, 9{,}210 + 4 \,.\, 15{,}905 + 2 \,.\, 26{,}910 + 4 \,.\, 51{,}393 + 1 \,.\, 173{,}913) \div 3 \,.\, 6$

$= \frac{532{,}42}{18} = 29{,}58$; and from this the space of descent sought:

$$s = \frac{v_n - v_0}{g}\ \text{times the mean value of}\ \frac{v}{1 - \zeta \cdot \frac{v^2}{w^2}} = \frac{25 - 0}{31{,}25} \,.\, 29{,}58 = 23{,}6\ \text{feet.}$$

The corresponding time of descent is, since the mean value of $\frac{1}{1 - \zeta \frac{v^2}{w^2}}$

$= (1.0 + 4 . 1{,}024 + 2 . 1{,}105 + 4 . 1{,}272 + 2 . 1{,}614 + 4 . 2{,}467 + 1 . 6{,}957) \div 18$
$= 1{,}747,\ t = \frac{25}{31.25} . 1{,}747 = 1{,}4$ sec.

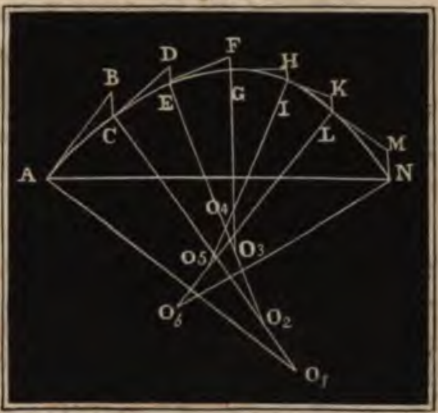
Remark. For a constant co-efficient of resistance, the higher calculus gives us:

$$v = \left(\frac{e^{\mu t} - 1}{e^{\mu t} + 1}\right) \sqrt{2g \frac{P}{\zeta F \gamma}} \text{ and } s = \frac{G}{\zeta F \gamma} L n \div \left(\frac{e^{\mu t} + 1)^2}{4 e^{\mu t}}\right),$$

where $\mu = \sqrt{2 g \zeta \frac{P F \gamma}{G^2}}$, e being the base of the hyperbolic system of powers, and Ln the hyperbolic logarithm.

§ 394. *Projectiles.*—We have already investigated the motion of projectiles in vacuo (§ 38), and found this motion to be parabolic; we may now obtain a more exact knowledge of motion in a resisting medium, and consider that, for instance, of a shot. In no case is the path *AGN*, Fig. 534, of a body passing through the air a symmetric curve; the portion *GN* in which the body descends is rather shorter, and, therefore, less inclined than the portion *AG* in which the body ascends, because the resistance of the air operating in the direction of motion tends always to shorten the portions of its path *AC*, *CE*, *EG*, &c., more and more; if, therefore, the first portion of the path *AC*, for motion in the air is only a little shorter than it would be in vacuo, the last portion *LN* is considerably shorter in the first motion than it is in the last. The construction of the path in a resisting medium by means of circles of curvature may be accomplished in the following manner.

Fig. 534.



From the initial velocity v_1, and the angle of elevation $BAN = \alpha_1$ it follows that the $\angle ABC = 90 - \alpha_1$, and *sin.* $ABC =$ *cos.* α_1, from § 40 the radius of curvature

$$O_1A = O_1C = r_1 = \frac{v_1^2}{g \cos. \alpha_1},$$

hence with this we may approximately describe the portion of arc *AC*. If now we assume the angle subtended at the centre $AO_1\ C = \phi^0_1$, therefore $AC = s_1 = r_1 \phi_1$, we then obtain for the succeeding particle of space *CE* the angle of inclination $\alpha^0_2 = \alpha^0_1 - \phi^0_1$. Let further, the height of fall $BC = h_1$, and the measure of the retardation due to the air's resistance $\zeta . \frac{v_1^2}{2g} F \gamma$ being

$$\zeta \cdot \frac{v_1^2}{2g} \cdot \frac{F\gamma}{G} = \mu\, v_1^2, \text{ therefore } \zeta \cdot \frac{F\gamma}{2\,G} = \mu,$$

from the principle of *vires vivæ*, we then obtain for the velocity v_2 at the initial point of the second portion of arc:

$$\frac{v_2^2}{2g} = \frac{v_1^2}{2g} - h_1 - \mu \left(\frac{v_1^2 + v_2^2}{2g}\right) s_1, \text{ or } (1 + \mu s_1) \frac{v_2^2}{2g}$$

$$= (1 - \mu s_1) \frac{v_1^2}{2g} - h_1, \text{ and hence } v_2 = \sqrt{\frac{(1 - \mu s_1)\, v_1^2 - 2\,g\,h_1}{1 + \mu s_1}}.$$

Since now the height of fall $h_1 = \frac{1}{2} g \tau^2 = \frac{1}{2} g \left(\frac{s_1}{s_1}\right)^2$, it follows that:

$$v_2 = \sqrt{\frac{(1 - \mu s_1)\, v_1^2 - \left(\frac{g s_1}{v_1}\right)^2}{1 + \mu s_1}} = v_1 \sqrt{\frac{1 - \mu r_1 \phi_1 - \frac{\phi_1^2}{\cos.\ a_1^2}}{1 + \mu r_1 \phi_1}}.$$

If we substitute these values of α_2 and v_2 in the equation: $r_2 = \frac{v_2^2}{g \cos.\ \alpha_2}$, we then obtain the radius of curvature $O_2C = O_2E$ of the succeeding portion of arc CE, and if we assume an angle of revolution $CO_2E = \phi_2$, it again follows from this that the angle of inclination in the vicinity of E; $\alpha_3 = \alpha_2\ \phi_2$, and the velocity at this point

$$v_3 = v_2 \sqrt{\frac{1 - \mu r_2 \phi_2 - \frac{\phi_2^2}{\cos.\ \alpha_2^2}}{1 + \mu r_2 \phi_2}}.$$

It is therefore easy to see how the entire path of the projectile may be successively composed of circular arcs.

Example. A cast iron ball, of 4 inches diameter, is shot off at an angle of elevation of 50° with a velocity of 1000 feet, required its path, if only approximately, according to Prussian weights and measures. The radius of curvature of the first portion of arc is $r_1 = \frac{v_1^2}{g\ \cos.\ \alpha} = \frac{1000000}{31{,}25 \cos.\ 50^\circ} = 49783$ ft. As the density of the air $= 0{,}0859$, and that of cast-iron $= 470$ lbs., we have then $\mu = \zeta \cdot \frac{F\gamma}{2G} = \zeta \cdot \frac{3 \,.\, 3 \,.\, 0{,}0859}{4 \,.\, 470} = 0{,}00041122 \,.\, \zeta$; now for $v = 1000$, $\zeta = 0{,}90$, hence $\mu = 0{,}0003701$. If we take an arc of 1° only, we then obtain the velocity at the end of it:

$$v_2 = 1000 \sqrt{\frac{1 - 0{,}0003701 \,.\, 49783 \,.\, 0{,}017453 - (0{,}017453 \div \cos.\ 50^\circ)^2}{1 + 0{,}0003701 \,.\, 49783 \,.\, 0{,}017453}}$$

$= 7697$ feet.

and the radius of curvature for a second portion of arc:

$$r_2 = \frac{(769{,}7)^2}{31{,}25 \cos.\ 49^\circ} = 28897 \text{ feet.}$$

For $v_2 = 769{,}7$ feet, $\zeta = 0{,}81$, therefore $\mu = 0{,}0003331$. If, therefore, we describe with the last radius, an arc $\phi_2 = 2^\circ$, the velocity at its ending point will be

$$v_3 = 769{,}7 \sqrt{\frac{1 - 0{,}33598 - 0{,}002831}{1{,}33598}} = 541{,}47 \text{ feet.}$$

For a third arc Q_3, the radius of curvature $r_3 = 13757$ feet, and if, therefore, we assume $\zeta = 0{,}75$, we shall then obtain at the end of a length of arc of 4°, the velocity $v_4 = 398{,}85$ feet. The radius of curvature for a fourth arc may be likewise found $r_4 = 6960{,}5$ by assuming $\zeta = 0{,}72$, and we shall then obtain the velocity $v_5 = 288{,}85$ feet, at the end of an arc of 8°, from which a fifth radius of curvature $r_5 = 3259$ feet may be calculated. Proceeding in this manner, we shall obtain, by degrees, the collective elements for the construction of the line of projection in question.

INDEX.

A.

Abrupt widening, 402
Absolute magnitude, 57
—— resistance, 183
Acceleration and combination of velocity, 41
——, normal, 47
Action and re-action, 57
—— of an unlimited stream, 480
Adhesion, 57
Aëriform bodies, 58
Aërodynamics, 58
Aërostatics, 58
Aggregation, state of, 56
Air, density of the, 346
——, efflux of still, 428
—— in motion, efflux of, 431
——, strata of, 345
Angle of friction and the cone of friction, the, 148
Angular tubes, 396
Areometer, 335
Attwood's machine, 241
Axes, free, 254
Axis, equilibrium of forces about an, 112
——, pressure on the, 111
Axle and wheel, 238
—— friction, 156
——, pointed, 165
——, the thickness of, 213
——, wheel of an, 142

B.

Balance, hydrostatic, 334
Ballistic pendulum, 298
Beams, hollow, 196
——, hollow and elliptical, 205
——, rectangular, 193
——, strongest, 203
——, the strongest form of, 210
Bodies, aëriform, 58
——, centre of gravity of, 98
——, dynamics of solid, 58
——, flexure of, 188
——, floating, 458
——, free descent of, 30
Bodies, impact of free, 279
——, mechanics of solid, 58
———————— fluid, 58
——, perfect elastic, 279
——, rotary, 295
——, statics of aëriform, 58
——, statics of solid, 58
——, strongest form of, 185
Breaking twist, 223
Buoyancy, 322
———— centre of, 332

C.

Catenary, 135
Centre of buoyancy, 332
—— of gravity, 88
———————— of bodies, 98
———————— of curved surfaces, 97
———————— of plane figures, 92
———————— of lines, 90
—— of parallel of forces, 83
—— of percussion, 299
—— of pressure, 311
Centrifugal force, 245, 246
———————— of extended masses, 248
Centripetal force, 246
Chains, rigidity of, 178
Circular lateral orifices, 360
—— pendulum, 265
Co-efficient of efflux, 367, 434
———————— through rectangular orifices, Tables I. and II. from Poncelet and Lesbros, 370, 371
———————— for circular orifices, table of corrections of the, 378
—— of contraction, 365
Co-efficient of velocities, 364
———— of discharge by wiers, 373
———— of resistance, 384
———— table of efflux and velocity, 390
———— of friction, 147, 447
———— of friction, Table of the, 394
———— of the friction of repose and motion, Table of, 151, 152
———— of resistance of curvature in tubes, 398
Cohesive force, 57

Columns, 218
Combination of velocity, 42
Communication, vessels of, 416
Components, 64
Composition and resolution of velocities, 41
——— of forces, 64
————— in a plane, 79
Compound motion, 37
——— pendulum, 273
——— vessels, 411
Compression, rupture by, 214
————— under, 216
Cone of friction, the, 148
Conical tubes, 389
Constrained paths, 259
Cords, friction of, 170
—— rigidity of, 179
Contraction, effect of imperfect, 408
———, co-efficient of, 365
———, imperfect, 377, 387
———, maximum and minimum of, 374
——— of the fluid vein, 366
———, partial, 375
Corrections for the Poncelet wiers, Table of, 381
——— for wiers over the entire side, or without any lateral contraction, 381
——— of the co-efficients of efflux for circular and rectangular orifices, 378
Coulomb's experiments, 180
Couples, 82
Crushing, Table of the modulus of resistance to, 215
Curve, elastic, 190
Curved motions in general, 46
——— tubes, 397
Curvilinear motion, 74
Cycloid, 270
Cycloidal pendulum, 271
Cylinders, 197
——— and prisms, 232
Cylindrical tubes, 383

D.

D'eau, jets, 399
Declivity of water, 438
Density, 54
——— and pressure, velocity of efflux, 353
——— of the air, 347
Depth of floatation, 325
Descent, free, of bodies, 30
Determination of the centre of gravity, 89
Different velocities in the transverse section, 438
Discharging vessels in motion, 362
Division of forces, 56
——— of mechanics, 57
Docks, floating, 330
Dynamical stability, 120
Dynamics of aëriform bodies, 58
——— of fluids, 58, 350
——— of rigid bodies, 225
——— of solid bodies, 58

E.

Edges, points and knife, 166
Effect of imperfect contraction, 404
Efflux, 350, 425
——, co-efficient of, 367, 434
—— of air in motion, 431
—— of still air, 428
—— of water in motion, 379
————— through tubes, 382
——, regulators of, 454
—— under decreasing pressure, 432
——, velocity of, 351
——, velocity of pressure and density, 353
Elastic body, perfect and imperfect, 279
—— curve, 190
—— impact, 281, 288
—— pendulum, 276
Elasticity and rigidity, 182
——— and strength, 183
———, modulus of, 183
———, or spring force, 57
Elliptical beams, hollow and, 205
English, French, and German measures and weights, comparative tables of, xv.
Equality of forces, 51
Equilibrium, kinds of, 110
——— of bodies rigidly connected, 109
——— of forces about an axis, 112
——— in funicular machines, 127
——— of water in vessels, 309
————— with other bodies, 322
——— and pressure of air, 338
Excentric impact, 302
Experiments on beams, 200
——— on friction, 149
——— of Rennie, 150
Eytelwein, 201

F.

Fall of water, 438
Flexure of bodies, 188
———, reduction of the moment of, 194
Floatation, depth of, 325
——— oblique, 332
Floating bodies, 458
——— docks, 330
Floods, 451
Flow through tubes, 436
Fluidity, 304
Fluid surface, the, 306
—— vein, contraction of, 364
Force, 50
—— about an axis, equilibrium of, 112
——, centre of parallel of, 83
——, centripetal and centrifugal of extended masses, 246
——, cohesive, 57
——, composition of, 64
——, direction of a, 57
——, division of a, 51, 57
——, equality of, 51
——, in a plane, 67

Force, in a plane, composition of, 79
——, in space, 69, 83
——, living, 62
——, magnetic, 57
——, measure of, 53
——, muscular, 57
——, normal, 245
——, of inertia, 57
——, of heat, 57
——, parallel of, 81
——, parallelogram of, 65
——, resolution of, 66
——, simple constant, 58
Formulæ of stability, 118
Free axes, 254
—— descent of bodies, Table of, 30, 31
French, English, and German measures and weights, comparative tables of, xv.
Friction, axle, 156
——, Table of co efficients of, (from Morin,) 156
——, co-efficient of, 147, 447
——, experiments on, 149
——, kinds of, 146
——, laws of, 147
——, of cords, 170
——, of motion, 152
——, of repose, Table of the co-efficients of the, 151
——, pivot, 164
——, resistance of, 391
——, rolling, 168
——, the angle and cone of, 148
Funicular machines, 127
——, polygon, 130

G.

Gases, tension of, 338
Gay-Lussac's law, 346
Geodynamics, 58
Geomechanics, 58
Geostatics, 58
German, English, and French measures and weights, comparative tables of, xv.
Gerstner, 201
Graphical representation, 34
Gravity, specific, 55, 334
Gravity, 57
——, action of, along constrained paths, 259
——, centre of, 88
——, determination of the centre of, 89
—— of lines, centre of, 90
—— of plane figures, centre of, 92
—— of curved surfaces, centre of, 97
Gauges, 453
Guldinus's properties, 107

H.

Hardness, 287
Hollow beams, 196
—— and elliptical beams, 205
Horizontal and vertical pressure, 319
Hydraulic pressure, 355
Hydrostatics, 58
Hydrodynamics, 58
—— balance, 334
Hydrometers, 335
Hydrometry, 453
Hydrometric sail wheel, 461
Hydrometrical pendulum, 464

I.

Impact, doctrine of, 228
—— of isolated streams, 467
——, elastic, 281
——, excentric, 302
——, inelastic, 279, 288
——, in general, 278
——, imperfectly elastic, 290
——, oblique, 291
Imperfectly elastic impact, 290
Imperfect contraction, 371, 387
——, effect of, 404
Impulse, maximum effect of, 469
—— and resistance, theory of, 466, 474
—— oblique, 471
—— of a limited stream, 470
—— and resistance against surfaces, 475
—— to bodies, 576
—— of water, 466
Inclined plane, 152, 259
——, theory of the, 122
Inelastic impact, 279, 288
Inert masses, reduction of, 228
Inertia, 52
——, force of, 57
——, moment of, 227
——, radius of, 230
——, reduction of the moment of, 229
Influx, velocity of, 351
—— and efflux, 425
Intensity of a force, 57
Irregular vessels, 424

J.

Jets d'Eau, 399

K.

Kinds of support, 109
—— of equilibrium, 110
—— of friction, 146
—— of motion, 225
Knife, edges of, 166
Knots, 127

L.

Lateral pressure, 310
Laws of friction, 147
—— of Mariotte, 341
—— of Gay-Lussac, 346
—— of statics of rigid bodies, 76

Lever, mathematical, 113
———, equilibrating, 332
———, material or physical, 113
———, theory of the equilibrium of the, 113, 163
Living forces, 62
Liquids of different densities, 337
Loading beyond the middle, 208
Locks, 426
Long tubes, 394

M.

Machines, Attwood's, 241
———, funicular, 127
Magnetic force, 57
Manometer, 339
Mariotte's law, 341
Mass, 53
——, reduction of inert, 228
Masses, centrifugal forces of extended, 248
Mathematical pendulum, 266
——— lever, 113
Material pendulum, 266
Matter, 51
Maximum and minimum of contraction, 374
——— effect of impulse, 469
Mean velocity, 440
Measure of forces, 53
Mechanics, 50
——— fundamental laws of, 50
——— division of, 57
——— of solid bodies, 58
——— of fluid bodies, 58
——— of air, 58
——— of a material point, 58
Mechanical effect, 60
———, transmission of, 73
Media, motion in resisting, 477
Metacentre, 328, 332
Modulus of elasticity, 183
——— of working load and strength, 184
——— of elasticity and strength, 186
——— of relative strength, 201
Moment of inertia, rotation and mass, 225, 227
Morin's experiments, 151
Motion in resisting media, 477
——, accelerated, 27
—— and rest, 25
——, compound, 37
——, efflux of air in, 431
——, in general, curved, 46
——, kind of, 25, 26, 225
—— of water, permanent, 439
——, of rotation, 226
——, mean velocity of a variable, 35
——, parabolic, 43
——, rectilinear, 225
——, rolling, 263
——, simple, 25
——, uniform, 26, 445
——, uniformly variable, 27
——, variable, in particular, 33
Motion, variable, in water, 448
——, curvilinear, 74
Mouth pieces, 382
Muscular force, 57

N.

Nodes, 127
Normal acceleration, 47
——— force, 245
Notches in a side, 418
Numerical values, 187

O.

Obelisk shaped vessels, spherical and, 422
Oblique impact, 291
——— pressure, 207
——— floatation, 332
——— additional tubes, 386
——— impulse, 471
Orifice, triangular lateral, 359
——— rectangular lateral, 368
——— circular lateral, 360
Oscillation, time of, 266

P.

Parallelogram of the velocities, 38
——— of accelerations, 42
——— of forces, 65
Parallelopiped of velocities, 41, 119
———, rectangle of, 231
Parallel forces, 81
———, centre of, 83
Parabola, 43
Parabolic motion, 43
Particular cases of impact, 282
Partial contraction, 375
Pendulum, hydrometrical, 465
Pendulum, circular, 266
———, ballistic, 298
———, elastic, 276
Percussion, centre of, 299
Permanent motion of water, 439
Perimeter, 438
Permanency, 439
Phoronomy, 25
Physical lever, 113
Piëzometers, 413
Pile driving, 285
Pipes, thickness of, 320
Pitot's tube, 464
Pivot friction, 164
Plane of rupture, 209
——, composition of forces in a, 79
——, forces in a, 67
——, inclined, 153, 259
——, theory of the inclined, 122
Pneumatics, 58
Points and knife edges, 166
Pointed axles, 165
Polygon, funicular, 130
Poncelet and Lesbros, two tables of co-effi-

cients of efflux, through rectangular orifices, 370, 371
Pontoon, a, 102
Pouce d'Eau, 456
Pressure, 51
———, horizontal and vertical, 319
——— in a definite direction, 315
——— on the axis, 111
——— of bodies on one another, 115
——— of oblique, 207
——— of efflux under decreasing, 432
——— on curved surfaces, 317
Pressures, principles of equality of, 305
——— and density, velocities of efflux, 353
———, centre of, 311
——— of water in vessels, 309
———, hydraulic, 355
———, lateral, 310
——— on the bottom, 308
Principle of equality of pressures, 305
——— of the *vis viva*, 62
——— of virtual velocities, 85, 124
Prismatic vessels, 415
Prisms and cylinders, 232
Profile in river, 438
Projectiles, 479
Properties of Guldinus, 107
Pulley, the, 140
Pyramidal-shaped vessels, wedge, &c., 420

R.

Radius of gyration or inertia, 230
Rectangle and parallelopiped, 231
Rectangular beams, 193
——— lateral orifices, 368
Rectilinear motion, 225
Reduction of the moment of flexure, 194
——— of inert masses, 228
——— of the moment of inertia, 229
Regulators of efflux, 454
Relative strength, 183, 199
Rennie's experiments, 151
Rest and motion, 25
Resistance and impulse against surfaces, 475
——— to compression, 183
——— to torsion, 183
———, impulse and, to bodies, 476
———, co-efficient of, 384
——— of friction, 391
——————— and rigidity, 145
——— of water, 466
———, theory of impulse, &c., 474
Resolution of velocities, 41
——— of forces, 66
Restoring power of floating docks, 331
Resultant, 64
Rheometer, 465
Rigid bodies, statics of, 76
Rigidity of chains, 178
——— and friction, 145
——— of cords, 179
Rod, the moment of inertia of, 230
Rolling motion, 263
——— friction, 168
——— and dragging friction, 169
Rotary bodies, 295
Running water, 438
Rupture, plane of, 209
——— by compression, 214
——— under compression, 216

S.

Sail wheel, hydrometric, 461
Section, transverse, different velocities in the, 438
Segments, 236
Short tubes, 382
Simple constant force, 58
Slope of water, 438
Space, forces in, 69, 84
Specific gravity, 55, 333
Sphere and cone, 333
Spherical and obelisk-shaped vessels, 422
Spring force, 57
State of aggregation, 56
Stability, 116, 328
———, formulæ of, 118
———, dynamical, 120
——— of floating docks, 331
Statical moment, 78
Statics of solid bodies, 58
——— of rigid bodies, 75
——— of fluid, 58, 309
——— of aëriform bodies, 58
Strata of air, 345
Stream, action of an unlimited, 473
Streams, impact and isolated, 467
———, impulse of a limited, 470
Strength and elasticity, 183
——— modulus of working load and, 184
——— the moduli of elasticity and table, 187.
——— relative, 199
——— modulus of relative, 201
Strongest form of body, 185
——— beams, 203
——— form, beams of the, 210
Support of kinds, 109

T.

Table of co-efficients of the friction of repose, 151
——— comparative, of English, French and German measures and weights, 15
——————————— of motion, 152
——— of co-efficients of axle friction from Morin, 156
——— I., the moduli of elasticity and strength, 187
——— II., the modulus of strength for the flexure of bodies, 201
——— of the modulus of resistance to crushing, 215

Table of co-efficients of efflux through rectangular orifices, 370
—— of co-efficients of efflux for wiers, 372, 373
—— of the co-efficients of resistance for trap valves, 410
—— of the co-efficients of friction, 394
—— of the co-efficients of the resistance of curvature in tubes, 398
—— of the co-efficients of resistance for the passage of water through a cock in a rectangular tube, 406
———— in a cylindrical tube, 406
———— through throttle-valves in rectangular and cylindrical tubes, 407
—— showing the relations of the motion to the time in the free descent of bodies, 31
—— of corrections of the co-efficients of efflux for circular and rectangular orifices, 378
———— for the Poncelet wiers, 381
———— for wiers over the entire side, or without any lateral contraction, 381
—— of correction for imperfect contraction by efflux through short cylindrical tubes, 388
—— of the co-efficients of efflux, 390
Tachometer, 461
Tension of gases, 338
Theory of the inclined plane, 122
—— of impulse and resistance, 474
———— wedge, 125
Thickness of axles, the, 213
—— of pipes, 320
Time of oscillation, 266
Toggle joint, note on, 129
Torsion, 219
Traction, 51
Transference of the point of application, 76
Transmission of mechanical effect, 73
Transverse section, different velocities in the, 438
———— the best form of, 441
Tredgold, 213
Triangular lateral orifice, 359
Tricardo, 129
Trigonometric expression, 41
Tubes, angular, 396
——, conical, 389
——, curved, 397
——, cylindrical, 383
——, flow through, 436
——, long, 394
——, oblique additional, 386
——, of Pitot, 464
——, short, 382
——, table of co-efficients of the resistance of curvature in, 398
Twist, breaking, 223

U.

Uniform motion, 26, 445
Uniformly accelerated motion, 28, 428
—— variable motion, 27
Unit of weight, 52

V.

Valves, 408
——, table of the co-efficients of resistance of traps, 410
Variable motions in particular, 33
—— motion, mean velocity of a, 35, 448
Velocities, co-efficient of, 364
——, combination of, 42
——, composition and resolution of, 41
——, in the transverse section different, 438
——, mean, 440
——, parallelogram of the, 38
——, parallelopipedon of, 41
——, principle of virtual, 85, 124
—— of efflux and influx, 351, 352
——, of efflux, pressure and density, 353
——, virtual, 85
Vertical and horizontal pressure, 319
Vessels, compound, 411
—— in motion, discharging, 362
——, irregular, 424
—— of communication, 416
——, prismatic, 415
——, spherical and obelisk-shaped, 422
——, wedge and pyramidal-shaped, 428
Virtual velocities, 85
Vis viva, 62

W.

Water-inch, 456
——, flow of, through wiers, 372, 373
—— in motion, efflux of, 379
——, permanent motion of, 439
—— running, 438
——, slope of, 438
Wedge and pyramidal-shaped vessels, 420
——, the, 154
——, theory of the, 125
Wheel, and axle, the, 238
——, hydrometric sail, 461
—— and axle, the, 142
—— carriages, note on, 174
Widening, abrupt, 401
Wiers, 372
——, table of the co-efficients of efflux, for 372, 373
——, table of corrections for the Poncelet, 381
——, over the entire side, table of corrections for, 381

END OF VOL. I.

ERRATA.

Page 30, line 5 from bottom, *for* 32,22 *read* 32,2.

“ 31 “ 6 *for* 15,625 *read* 16,1, and *for* 250 *read* 257,6.

“ “ “ 8 *for* 0,016 *read* 0,0155.

“ “ “ 10 from bottom, *for* 241 *read* 241½.

“ 32 “ 9 from bottom, *for* 0,480 *read* 0,465, *for* 0,320 *read* 0,310, and *for* 0,160 *read* 0,155.

“ “ “ 11 from bottom, *for* 0,032 *read* 0,031, and *for* 0,480 *read* 0,465.

“ 38 “ 9 *for* 15⅜ feet *read* 15⅜ feet Prussian measure.

“ “ “ 23 from top, *for* $c_1\, t_1$, $c_2\, t_1$ *read* $c_1\, t$, $c_2\, t$.

“ 55 “ 5 *for* $\frac{205}{502}$ *read* $\frac{205}{485{,}8}$, *for* 0,4083 *read* 0,4219, also *for* 0,4083 *read* 0,4219, and *for* 705,54 *read* 729,04.

“ 73 “ 4 from top, *for* Fig. 40 *read* Fig. 41, and in line 5, *for* Fig. 41 *read* Fig. 40.

“ 75 “ 3 *for* $(h - h_1)\, M$ *read* $(h - h_1)\, G$.

“ “ “ 6 *for mass read* weight.

“ 100 “ 17 from bottom, *for* $x_3\, y_2\, z_1$ *read* $x_2\, y_3\, z_1$.

“ 153 “ 9 from bottom, *for* GOK *read* GOQ.

“ 163 “ 14 from bottom, *for* $\mp\, Q \cos.\, \beta$ *read* $\mp\, v\, Q \cos.\, \beta$.

“ 184 “ 10 from bottom, *for* 1 *read* l, and *for* $\lambda\, E$ *read* $\frac{\lambda}{l}\, E$.

“ 189 “ 17 from top, *for* $z_1 \,.\, F_1\, S\, z$, *read* $z_1 \,.\, F\, S\, z_1$.

“ 257 “ 13 from top, *for* $\cos.\, \alpha$ *read* $\mathrm{cotg.}\, \alpha$.

“ 264 “ 2 from top, *for* G *read* g.

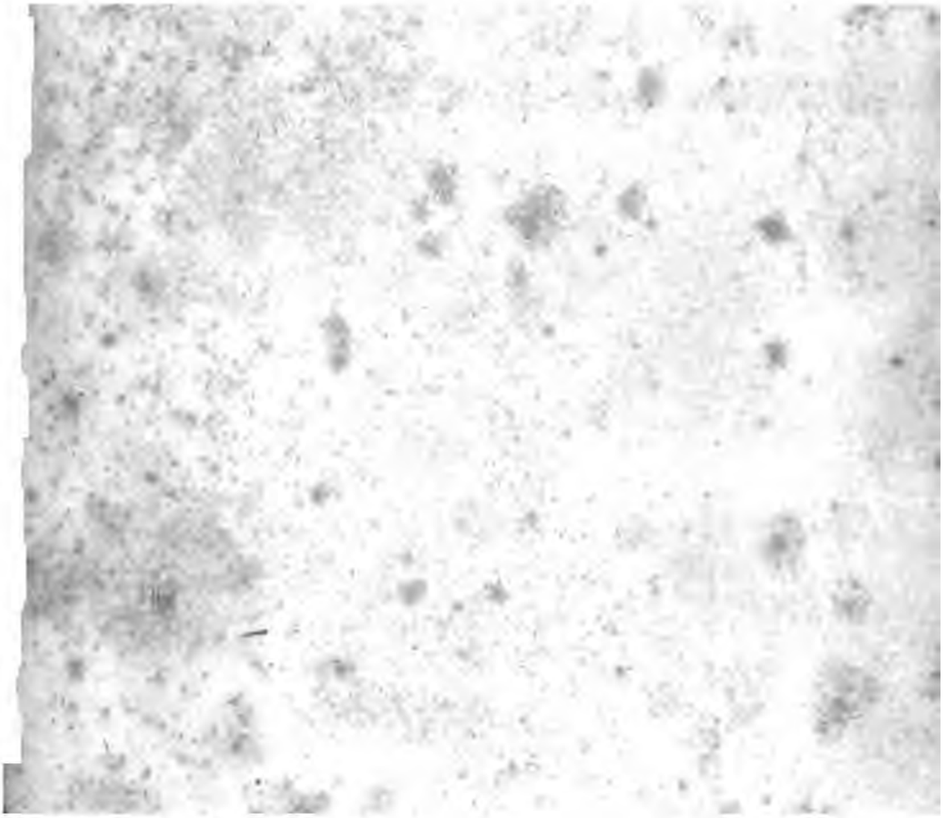
“ 331 “ 6 from top, *for* shoared *read* shored.

“ 394 “ 15 from bottom, *for* $\overline{\underline{\,|\,1\frac{1}{4}\,|\,1\,|\,2\,|\,}}$ *read* $\overline{\underline{\,|\,1\frac{1}{4}\,|\,1\frac{1}{2}\,|\,2\,|\,}}$.

“ 414 “ from bottom, *for* 01,274 *read* 0,1274.

“ 422 “ 15 from bottom, *for* $h = r\, t, =$ *read* $h = r, t =$.

“ 426 “ 12 from bottom, *for* 63,29 *read* 63,89.



Stanford University Libraries
3 6105 015 347 391

TA
350
W442
v.1

Stanford University Libraries
Stanford, California

Return this book on or before date due.

JUN 26 1977